CS 7301.003.20F Lecture 22—November 2, 2020

Main topics are #simplicial_homology.

Homology

- A few weeks ago, we discussed the minimum cut problem in surface embedded graphs.
- To solve it quickly, I introduced a tool from algebraic topology called *homology*, albeit in the restricted case of homology in a surface embedding with coefficients in Z_2.
- Today, we're going to generalize the idea. As before, it's used to learn about 'cycles' in topological spaces, kind of like homotopy. But while two homotopic cycles are always homologous, homology is a bit courser. It also works on things we wouldn't normally call cycles like even subgraphs.
- And it's a lot easier to compute things with homology as we'll see.

Oriented Simplices, Chains, and Boundary Maps

- Let X be a finite simplicial complex (a proper Delta-complex). These ideas extend to generalizations like polytopal complexes and these things called CW-complexes that I didn't have time to discuss last Monday.
- To define homology, we're going to need to assign an arbitrary *orientation* to each simplex in X. An orientation of a simplex is one of the two equivalence classes of permutations of its vertices, where two permutations are equivalent if they have an even number of inversions compared to one another.
- We can write an oriented simplex as a sequence of its vertices sigma := [x_0, x_1, …, x_k]. Let -sigma denote the same simplex in the opposite orientation. For example, an orientation of the triangle with vertices x, y, and z can be denoted in a few different ways:
  - [x, y, z] = -[x, z, y] = [z, x, y] = -[z, y, x] = [y, z, x] = -[y, x, z]
- Now, fix a non-negative integer k. Let X_k denote the oriented k-dimensional simplices in X and n_k := |X_k|.
- A *k-chain* over X is a function alpha : X_k → Z such that alpha(-sigma) = -alpha(sigma). You could also think of it as a vector of n_k integers, one per k-simplex in X.
- For the sake of notation, we may also represent a k-chain as a formal sum sum_i alpha_i Delta_{k, i} where each alpha_i is an integer and each Delta_{k, i} is the ith oriented k-simplex in X.
- The set of all k-chains form an abelian (commutative) group under addition, isomorphic to Z^{n_k}, called the *kth chain group* C_k(X).
- There's nothing particular special about us using the integers, and any coefficient ring R will do. For example, we already played with R = Z_2, making k-chains 0-1 vectors, or

equivalently, a subset of the k-simplices. If R is a field, then the group C_k(X) is a *vector space*. Otherwise, like with R = Z, it is merely an *R-module* which doesn't have scalar devision.

- The *boundary* partial sigma of an oriented k-simplex sigma is a (k - 1)-chain, defined as a weighted sum of facets of sigma:

  partial[x_0, x_1, …, x_k] := sum_{i = 0}^d (-1)^i [x_0, x_1, …, hat{x_i}, …, x_k]

  where [x_0, x_1, …, hat{x_i}, …, x_k] is the facet opposite of vertex x_i. So for example,

$$\begin{aligned} \partial[w,x,y,z] &:= [x,y,z]-[w,y,z]+[w,x,z]-[w,x,y] \\ \partial[x,y,z] &:= [y,z]-[x,z]+[x,y] \\ \partial[x,y] &:= [y]-[x] \\ \partial[x] &:= [] \end{aligned}$$

- The boundary function is antisymmetric: partial(-sigma) = -partial sigma.
- And we can extend the boundary function linearly to k-chains, giving us the *kth boundary homomorphism* partial_k : C_k(X) ➜ C_{k - 1}(X).
- Because partial_k is linear, we can represent it using an n_{k - 1} x n_k integer matrix with entries in {-1, 0, 1}. Hit the matrix on the vector for a k-chain to gets its boundary's vector.
- Something interesting happens when you take the boundary of a boundary.
- Lemma: partial_{k - 1} circ partial_k = 0 for all k ≥ 2
  - Boundary functions are linear, so it suffices to show the boundary of the boundary of a k-simplex is the empty (k - 2)-chain.

$$\begin{aligned} &\partial_{k-1}(\partial_k[x_0,\dots,x_d]) \\ &= \sum_{i=0}^{d}(-1)^i\partial_{k-1}[x_0,\dots,\hat{x}_i,\dots,x_d] \\ &= \sum_{i=0}^{d}(-1)^i\left(\sum_{j=0}^{i-1}(-1)^j[x_0,\dots,\hat{x}_j,\dots,\hat{x}_i,\dots,x_d]+\sum_{j=i+1}^{d}(-1)^{j-1}[x_0,\dots,\hat{x}_i,\dots,\hat{x}_j,\dots,x_d]\right) \\ &= \sum_{0\le j<i\le d}(-1)^{i+j}[x_0,\dots,\hat{x}_j,\dots,\hat{x}_i,\dots,x_d]-\sum_{0\le i<j\le d}(-1)^{i+j}[x_0,\dots,\hat{x}_i,\dots,\hat{x}_j,\dots,x_d] \\ &= 0 \qquad \square \end{aligned}$$

  - (except replace every d in the equations above with k)

Cycles and Homology

- There are two important subgroups of C_k(X) that we need to discuss before finally getting to homology.
- A *k-cycle* is a k-chain alpha such that partial_k alpha = 0.
- A *k-boundary* is a k-chain alpha such that alpha = partial_{k + 1} beta for some (k + 1)-chain beta.
- The previous lemma implies every k-boundary is a k-cycle, but not every cycle is a boundary.
- The k-cycles and k-boundaries form subgroups of C_k(X) called the *kth cycle group* Z_k(X)

and the *kth boundary group* B_k(X). By definition

Z_k(X) := ker partial_k and B_k(X) := im partial_{k + 1}

- And if you prefer to think in terms of linear algebra:

∂_k	linear map $C_k \to C_{k-1}$	$n_{k-1} \times n_k$ integer matrix
$B_k(X)$	$\operatorname{im} \partial_{k+1}$	row space of ∂_{k+1}
$Z_k(X)$	$\ker \partial_k$	right null space of ∂_k

Equivalent definitions of the cycle and boundary groups.

- These are also free abelian groups, meaning they are each isomorphic to Z^c for some integer c.
- Now we can finally define homology equivalence classes.
- Two k-cycles alpha and beta are *homologous* if the k-cycle alpha - beta is a k-boundary. Again, alpha lies in the *homology (equivalence) class* [alpha] and addition of homology classes is well defined as [alpha] + [beta] := [alpha + beta].
- Homology classes of k-cycles under addition form the *kth homology group* H_k(X).
- It can also be defined as a quotient group:

H_k(X) := Z_k(X) / B_k(X)

- This group is an abelian group, but it is not free in general.
- However, we can use a fact from abstract algebra: Since it is a finitely-generated abelian group, it is isomorphic to a product of cyclic groups:

$$H_k(X) \cong \mathbb{Z}^{\beta_k(X)} \oplus \bigoplus_i (\mathbb{Z}/d_i\mathbb{Z})$$

for some integers beta_k and 1 ≤ d_1 ≤ d_2 ≤ … ≤ d_m where each integer d_i is a divisor of its successor d_{i + 1}.

- The rank beta_k of the free component of H_k(X) is called the *kth Betti number* of X. The components (Z / d_i Z) are called *torsion subgroups*.
- Using just the examples we've seen so far in this class, I think I can offer the following intuition.
  - The kth Betti number tells you how many k-dimensional features the simplicial complex has. beta_0 is the number of components. beta_1 is the number of "handles" or "twists". beta_2 is the number of "cavities", and so on.
  - The existence of torsion is a sign of non-orientability.
- Like before, we can define chains and other things derived from them using other coefficient rings. We use H_k(X; R) to denote the kth homology group of X with coefficients in ring R. Different choices of R lead to different homology groups.
- But one nice thing is that if R is a field of characteristic zero like Q or R or C, we have H_k(X; R) ~= R^{beta_k}. There's no more torsion.
- As one final note, we observe the following.
- Theorem: If X and Y are homeomorphic simplicial complexes, then H_**(X) = H_**(Y).