CS 7301.003.20F Lecture 27—November 18, 2020

Relating Length to Total Curvature

- Let's finish up our discussion on cycles and curvature from last time.
- Recall, we define (for this lecture) the *Fréchet distance* between them as the infimum over all homeomorphisms the largest distance between corresponding points.
  - d_F(gamma, gamma_0) = inf_eta max_u ||gamma(u) - gamma_0(eta(u))|| where eta : S^1 ➔ S^1 is a homeomorphism and u in S^1.
- Generalized Fáry Theorem: Let gamma, gamma_0 : S^1 ➔ R^2 be two smooth cycles. Then |L(gamma) - L(gamma_0)| ≤ [K(gamma) + K(gamma_0) - 2pi] d_F(gamma, gamma_0).
- For example, if we take gamma_0 to be an arbitrarily tiny circle centered at the origin and im gamma subset B^2, then L(gamma_0) approaches 0, K(gamma_0) = 2pi, and d_F(gamma, gamma_0) approaches at most 1. We get the original Fáry Theorem.
- Surprisingly, we'll prove this theorem by using everything we know about persistent homology.
- Fix a direction u in S^1 and let h = h_u circ gamma and h_0 = h_u circ gamma_0 be the restrictions to the height functions to the curves.
- Almost all level sets of the form h^{-1}(z) consist of an even number of points decomposing gamma into the same number of arcs. Half of these live in h^{-1}(-infty, z].
- Let Chi(z) be the Euler characteristic of the sublevel set for z. It's equal to number of points in h^{-1}(z) minus the number of arcs in h^{-1}(-infty, z], which again is half that number of points. Therefore,

$$\int_{z\in\mathbb{R}} \operatorname{card}\left(h^{-1}(z)\right) \mathrm{d}z \quad = \quad 2\int_{z\in\mathbb{R}} \chi(z)\, \mathrm{d}z$$

- Recall the Cauchy-Crofton formula:

$$L(\gamma) \quad = \quad \frac{1}{4}\int_{u\in\mathbb{S}^1}\int_{z\in\mathbb{R}} \operatorname{card}\left(h^{-1}(z)\right) \mathrm{d}z\, \mathrm{d}u.$$

- If we want to study the difference of lengths, we can study the difference of integrals over card(h^{-1}) or the difference in the Euler characteristics.
- By the Euler-Poincaré formula, the Euler characteristic of a sublevel set is the number of components (beta_0) minus number of loops (beta_1).
- Therefore, Chi(z) is the number of points in Dgm_0(h) within quadrant Q_z = (-infty, z] x (z, infty] minus the number of points in Dgm_1(h) within the same quadrant.
- Let Chi_0 be the Euler characteristic for gamma_0. We can say the same things here.
- Now let's compare those persistence diagrams. Let eps = d_F(gamma, gamma_0) and eta : S^1 ➔ S^1 a homeomorphism such that the Euclidean distance between points gamma(s) and gamma_0(eta(s)) is at most eps + delta.

- The stability of persistence diagrams implies the bottleneck distance between Dgm_0(h) and Dgm_0(h_0) is at most eps + delta, and the same holds for the Dmg_1. Let psi_0 and psi_1 be the corresponding bijections between the bottleneck diagrams.
- Now observe that the contributions of points v in Dgm_k(h) and psi_k(v) in Dgm_k(h_0) cancel each other out except for values of z for which one point lies inside the quadrant Q_z and the other lies outside. For pairs of finite points, the integral over such values of z is at most 2(eps + delta).
- Note finite points in one diagram may be matched to the fake points in the diagonal of the other diagram. Each coordinate of a finite point contributes at most eps + delta to the integral.
- However, points at infinity must be matched to each other and we have exactly four; one each in the two diagrams of h and the two diagrams of h_0. The integral over values of z for which one point of a pair is in the quadrant here is at most eps + delta. So the total contribution from these finite coordinates is 2(eps + delta), which is (eps + delta) times the number of their finite coordinates minus 2.
- In other words, we get a contribution of at most eps + delta for each minimum and maximum of h and h_0, except for two.
- Taking delta to 0, we get

$$
\begin{aligned}
|L(h) - L(h_0)| &\leq \frac{1}{4} \int_{u \in \mathbb{S}^1} \int_{z \in \mathbb{R}} |\mathrm{card}\, h^{-1}(z) - \mathrm{card}\, h_0^{-1}(z)|\, \mathrm{d}z\, \mathrm{d}u \\
&\leq \frac{1}{2} \int_{u \in \mathbb{S}^1} \int_{z \in \mathbb{R}} |\chi(z) - \chi_0(z)|\, \mathrm{d}z\, \mathrm{d}u \\
&\leq \frac{\varepsilon}{2} \int_{u \in \mathbb{S}^1} [c(h) + c(h_0) - 2]\, \mathrm{d}u.
\end{aligned}
$$

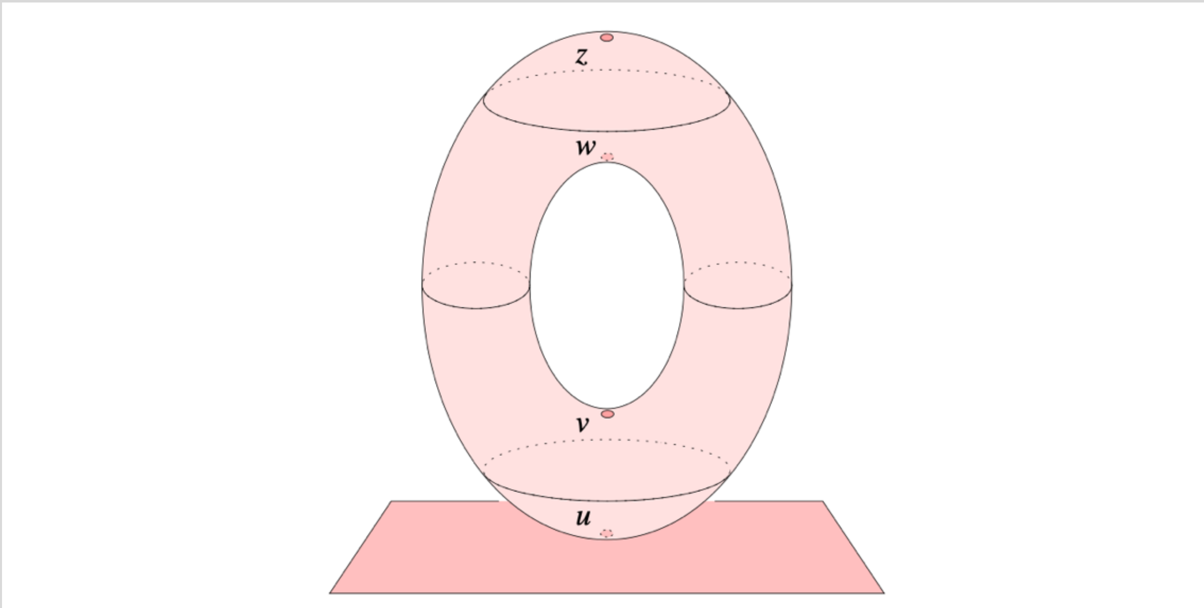
- Last time, we saw

$$
K(\gamma) = \frac{1}{2} \int_{u \in \mathbb{S}^1} c(h_u)\, \mathrm{d}u.
$$

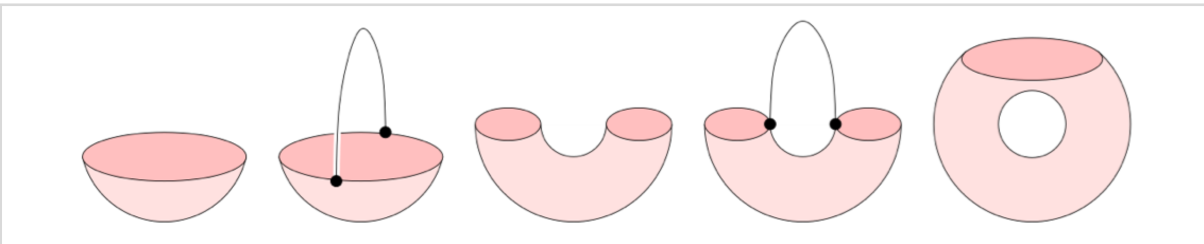
- So we can reduce the right hand side to eps[K(gamma) + K(gamma_0) - 2pi].

Height on a Torus

- For the rest of the semester(?) we're going to lessen our focus on homology and instead discuss functions in topological spaces and how to visualize them.
- To get us in the right mindset, let X be the two-dimensional torus and f(x) the height of the point x in X above a horizontal plane on which the torus rests.



- We call f : X ➡ R a *height function*.
- Each a in R has a preimage f^{-1}(a) which we call a *level set*.
- And we already saw *sublevel sets* X_a := f^{-1}(-infty,] = {x in X | f(x) ≤ a}.
- Now consider the evolution of the sublevel sets as we increase the threshold a.
- For a < f(u), the sublevel set is empty.
- For f(u) < a < f(v), it is a disk, which has the homotopy type of a point.
- For f(v) < a < f(w), the sublevel set is a cylinder. Cylinders have the homotopy type of a circle. You might imagine gluing two ends of an interval to a disk and then shrinking the disk to a point.



- For f(w) < a < f(z), the sublevel set is a capped torus which has the homotopy type of a figure-8. Again, we could glue an interval to cylinder and then shrink it to a circle.
- Finally, for f(z) < a, we have the complete torus.
- We're going to discuss these moments of change and a bit on how to analyze them.

Morse Functions

- Let X be a smooth d-manifold, meaning for each point there is a small neighborhood with a homeomorphism to an open ball in R^d which is smooth in both directions.
- Denote the tangent space at a point x in X by TM_x. It is the d-dimensional space consisting of all tangent vectors of M at x.
- Let f : X - > R be a smooth map from X to the reals. The derivatives Df_x : TM ➡ R are real-valued linear maps on the tangent spaces, so their images consist of either the entire real

line or just zero.

- We call x in X a *regular point* of f if Df_x if its image is the entire line and a *critical point* otherwise.
- Consider a local coordinate system (x_1, x_2, …, x_d) in a neighborhood of x. Then x is critical if and only if all its partial derivatives vanish,

$$\frac{\partial f}{\partial x_1}(x) \;=\; \frac{\partial f}{\partial x_2}(x) \;=\; \ldots \;=\; \frac{\partial f}{\partial x_d}(x) \;=\; 0.$$

- The image of a critical point f(x) is called a *critical value* of f.
- Now, there are different kinds of critical points.
- The *Hessian* of f at the point x is the matrix of second derivatives,

$$H(x) \;=\; \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_d}(x) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_2 \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_d}(x) \\ \vdots & \ddots & \vdots & \\ \frac{\partial^2 f}{\partial x_d \partial x_1}(x) & \frac{\partial^2 f}{\partial x_d \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_d \partial x_d}(x) \end{bmatrix}.$$

- A critical point x is *non-degenerate* if the Hessian is non-singular, i.e., det H(x) ≠ 0.
- The points u, v, w, and z from that torus are non-degenerate critical points.
- So by definition, all partial derivatives vanish at a critical point, implying a local Taylor expansion has no linear terms.
- But if it's non-degenerate, then the behavior of the function in a small neighborhood is dominated by the quadratic terms. It turns out there is local coordinate system such that these quadratic terms suffice.
- Morse Lemma: Let u be a non-degenerate critical point of f : X ➜ R. There are local coordinates with u = (0, 0, …, 0) such that

$$f(x) \;=\; f(u) - x_1^2 - \ldots - x_p^2 + x_{p+1}^2 + \ldots + x_d^2$$

  for every point x = (x_1, x_2, …, x_d) in a small neighborhood of u.
- The number of minus signs in the quadratic polynomial is the *index* of the critical point, index(u) = p.
- The index classifies the non-degenerate critical points into d + 1 basic types.
- For a 2-manifold for example, there are three types:
    - *minima* with index 0
    - *saddles* with index 1
    - *maxima* with index 2
- A consequence of the Morse Lemma is that non-degenerate critical points are isolated and therefore, there are a finite number of critical points on a compact manifold.
- We say f is a *Morse function* if
    1. all critical points are non-degenerate and
    2. the critical points have distinct function values.

- Finally, we have the following *Morse Inequalities*. Let c_p denote the number of critical points of index p.
- Weak inequality: $c_p \geq \beta_p(X)$ for all p.
- Strong inequality: $\sum_{p = 0}^j (-1)^{j - p} c_p \geq \sum_{p = 0}^j (-1)^{j - p} \beta_p(X)$ for all j.
- The strong Morse inequality for j = d is an equality. Also, the strong ones imply the weak ones:

$$\begin{aligned} \sum_{p=0}^{j}(-1)^{j-p}c_p &\geq \beta_j(\mathbb{M}) - \sum_{p=0}^{j-1}(-1)^{j-p-1}\beta_p(\mathbb{M}) \\ &\geq \beta_j(\mathbb{M}) - \sum_{p=0}^{j-1}(-1)^{j-p-1}c_p. \end{aligned}$$

  (except use X instead of M)
- Removing the common terms from both sides leaves $c_j \geq \beta_j(X)$.