CS 7301.003.20F Lecture 28—November 23, 2020

Morse Functions

- Let X be a smooth d-manifold, meaning for each point there is a small neighborhood with a homeomorphism to an open ball in R^d which is smooth in both directions.
- Denote the tangent space at a point x in X by TM_x. It is the d-dimensional space consisting of all tangent vectors of M at x.
- Let f : X - > R be a smooth map from X to the reals. The derivatives Df_x : TM ➔ R are real-valued linear maps on the tangent spaces, so their images consist of either the entire real line or just zero.
- We call x in X a *regular point* of f if the image of Df_x is the entire line and a *critical point* otherwise.
- Consider a local coordinate system (x_1, x_2, …, x_d) in a neighborhood of x. Then x is critical if and only if all its partial derivatives vanish,

$$\frac{\partial f}{\partial x_1}(x) \; = \; \frac{\partial f}{\partial x_2}(x) \; = \; \ldots \; = \; \frac{\partial f}{\partial x_d}(x) \; = \; 0.$$

- The image of a critical point f(x) is called a *critical value* of f.
- Now, there are different kinds of critical points.
- The *Hessian* of f at the point x is the matrix of second derivatives,

$$H(x) \; = \; \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) & \ldots & \frac{\partial^2 f}{\partial x_1 \partial x_d}(x) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_2 \partial x_2}(x) & \ldots & \frac{\partial^2 f}{\partial x_2 \partial x_d}(x) \\ \vdots & \ddots & \vdots & \\ \frac{\partial^2 f}{\partial x_d \partial x_1}(x) & \frac{\partial^2 f}{\partial x_d \partial x_2}(x) & \ldots & \frac{\partial^2 f}{\partial x_d \partial x_d}(x) \end{bmatrix}.$$

- A critical point x is *non-degenerate* if the Hessian is non-singular, i.e., det H(x) ≠ 0.
- The points u, v, w, and z from that torus are non-degenerate critical points.
- So by definition, all partial derivatives vanish at a critical point, implying a local Taylor expansion has no linear terms.
- But if it's non-degenerate, then the behavior of the function in a small neighborhood is dominated by the quadratic terms. It turns out there is local coordinate system such that these quadratic terms suffice.
- Morse Lemma: Let u be a non-degenerate critical point of f : X ➔ R. There are local coordinates with u = (0, 0, …, 0) such that

$$f(x) \; = \; f(u) - x_1^2 - \ldots - x_p^2 + x_{p+1}^2 + \ldots + x_d^2$$

  for every point x = (x_1, x_2, …, x_d) in a small neighborhood of u.
- The number of minus signs in the quadratic polynomial is the *index* of the critical point, index(u) = p.

- The index classifies the non-degenerate critical points into d + 1 basic types.
- For a 2-manifold for example, there are three types:
  - *minima* with index 0
  - *saddles* with index 1
  - *maxima* with index 2
- A consequence of the Morse Lemma is that non-degenerate critical points are isolated and therefore, there are a finite number of critical points on a compact manifold.
- We say f is a *Morse function* if
  1. all critical points are non-degenerate and
  2. the critical points have distinct function values.
- Finally, we have the following *Morse Inequalities*. Let c_p denote the number of critical points of index p.
- Weak inequality: c_p ≥ beta_p(X) for all p.
- Strong inequality: sum_{p = 0}^j (-1)^{j - p} c_p ≥ sum_{p = 0}^j (-1)^{j - p} beta_p(X) for all j.
- The strong Morse inequality for j = d is an equality. Also, the strong ones imply the weak ones:

$$\begin{aligned} \sum_{p=0}^{j}(-1)^{j-p}c_p &\geq \beta_j(\mathbb{M}) - \sum_{p=0}^{j-1}(-1)^{j-p-1}\beta_p(\mathbb{M}) \\ &\geq \beta_j(\mathbb{M}) - \sum_{p=0}^{j-1}(-1)^{j-p-1}c_p. \end{aligned}$$

  (except use X instead of M)
- Removing the common terms from both sides leaves c_j ≥ beta_j(X).

Iso-surface Extraction

- Now that we've established some vocabulary and basic results on functions, consider the following scenario.
- We're given three-dimensional density data as a continuous function f : [0, 1]^3 ➡ R.
- An *iso-surface* is a level set f^{-1}(t).
- If f is smooth and t is a regular value, then the level set is a 2-manifold, possibly with boundary.
- Suppose we want to extract an iso-surface. Let's assume f is piecewise linear. If we start at a point x with f(x) = t, we can essentially do a graph search to find all the other triangles in x's component.
- But what if there are other components? We'd have to essentially check every tet to see if any of them contain bits of the iso-surface.
- Today, we're going to discuss a structure that concisely describes all the components,

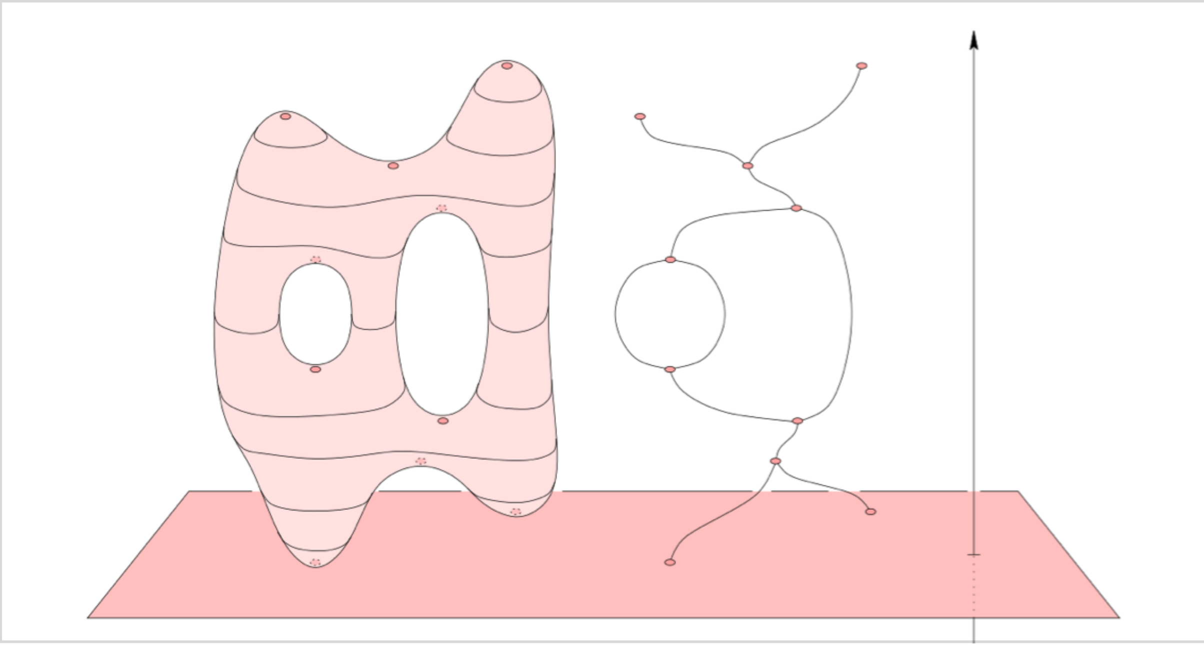
helping us to find them quickly and discover how the iso-surfaces connect topologically.

Reeb Graphs

- First, let's formally define a construction of topological spaces we've informally used a few times already.
- Let ~ be an equivalence relation on a topological space X.
- Let X_~ be the set of equivalence classes, and let psi : X → X_~ map each point x to its equivalence class.
- The *quotient topology* of X_~ consists of all subsets U subseteq X_~ whose preimages, psi^{-1}(U) are open in X. The *quotient space* is X_~ coupled with the quotient topology.
- Now, let f : X → R be a continuous function. We call a component of a level set a *contour*.
- Say two points x, y in X are equivalent if they belong to the same contour in f^{-1}(t). The *Reeb graph* of f, denoted R(f) = X_~, is the quotient space defined by this equivalence relation.
- It has a point for each contour provided by the map psi : X → R(f).
- Define pi : R(f) → R so that f(x) = pi(psi(x)). Using pi, we can construct a level set by going from the real line, to the Reeb graph, to the topological space. Specifically, for t in R, we get pi^{-1}(t), a collection of points in R(f) and psi^{-1}(pi^{-1}(t)), the collection of contours for the level set of t.
- The Reeb graph loses a lot of the original topological structure of X, but we can still use it to learn about X or the function f.
- psi is a continuous surjection mapping components of X to components of R.
- A loop in X that maps to a loop in R(f) is not contractible.
- Two loops in X that map to different loops in R(f) are not homologous.
- Therefore, the number of components is preserved and the number of loops cannot increase…
    - beta_0(R(f)) = beta_0(X)
    - beta_1(R(f)) ≤ beta_1(X)

Morse Functions

- We can say more if X is a manifold of dimensional d ≥ 2 and f : X → R is a Morse function like in the figure below.



- Call u in R(f) a *node* of the Reeb graph if $\psi^{-1}(u)$ contains a critical point.
- By definition, the critical points have distinct function values, so there is a bijection between critical points of f and nodes of R(f).
- The rest of the Reeb graph is partitioned into *arcs* connecting the nodes.
- A minimum corresponds to a degree 1 node.
- An index 1 saddle that merges two contours into one corresponds to a degree 3 node.
- Symmetrically, a maximum corresponds to a degree 1 node and an index d - 1 saddle that splits a contour into two corresponds to a degree 3 node.
- However, all other critical points correspond to nodes of degree 2.
- Note that despite us talking about manifolds, the Reeb graph has no meaningful embedding in any space. It's just a (topological) graph.

Reeb Graphs of 2-Manifolds

- Let's finish by studying the Reeb graphs of a familiar setting, orientable 2-manifolds.
- Let X be an orientable 2-manifold and $f : X \to R$ Morse.
- Every saddle either merges two contours into one or splits a contour into 2.
- We'll use this observation to compute the number of loops in the Reeb graph.
- Let n_i be the number of nodes with degree i.
- Here, only n_1 and n_3 are non-zero.
- The number of arcs $e = (n_1 + 3n_3) / 2$, and the number of loops is $1 + e - (n_1 + n_3)$.
- Lemma: The Reeb graph of a Morse function on a connected, orientable 2-manifold of genus g has g loops.
  - Suppose R(f) has no loop. It is a tree with $n_1 = n_3 + 2$ degree 1 nodes. Let c_i be the number of critical points of index i so $n_1 = c_0 + c_2$ and $n_3 = c_1$. From the strong Morse inequality, we have $\chi = \beta_0 - \beta_1 + \beta_2 = c_0 - c_1 + c_2 = n_1 -$

$n_3 = 2$.

  - Now suppose otherwise that there is at least one loop.
  - Repeatedly collapse degree 1 nodes and merge arcs across degree 2 nodes. The example above turns into two degree 3 nodes connected to each other by three arcs.
  - Both operations preserve homotopy type and therefore number of loops.
  - Let m_3 be the number of remaining degree 3 nodes. There are $(3/2)m_3$ remaining arcs. Therefore, m_3 is even.
  - Using the Euler-Poincaré formula for graphs, we see $m_3 - (3/2) m_3$ = # components - # loops. The graph is connected, so there are $m_3 / 2 + 1$ loops.
  - We had c_1 degree 3 nodes in the original graph, and for each minimum or maximum, we collapsed one degree 1 node, removing a degree 3 node in the process.
  - So now using the strong Morse inequality, we see $m_3 = c_1 - (c_0 + c_2) = 2g - 2$.
  - The number of loops is therefore $(2g - 2) / 2 + 1 = g$.
- Using a more complicated analysis, we can also say the following about non-orientable 2-manifolds.
- Lemma: The Reeb graph of a Morse function on a connected, non-orientable 2-manifold of genus g has at most $g / 2$ loops.