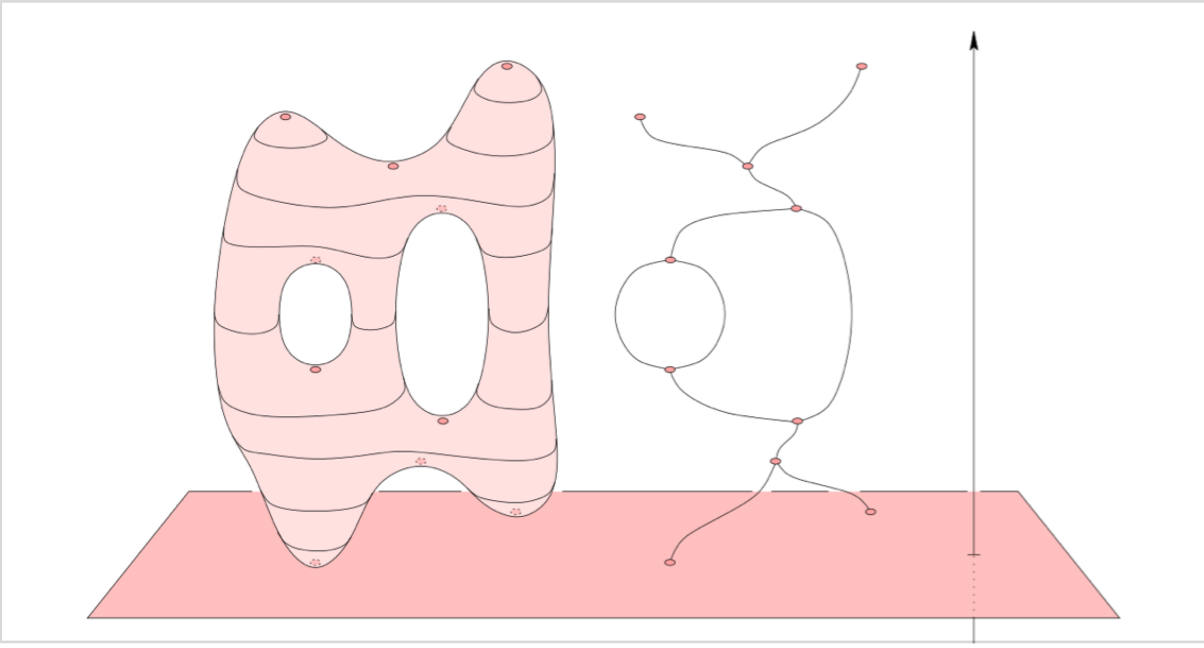
CS 7301.003.20F Lecture 29—November 25, 2020

Reeb Graphs of Morse Functions

- If X is a manifold of dimensional $d \geq 2$ and $f : X \to R$ is a Morse function, then the Reeb graph has additional structure over what we talked about last time.



- Recall, psi : X → R(f) maps each point to the equivalence class consisting of its contour. Also, pi : R(f) → R is defined so that f(x) = pi(psi(x)).
- Call u in R(f) a *node* of the Reeb graph if psi^{-1}(u) contains a critical point.
- By definition, the critical points have distinct function values, so there is a bijection between critical points of f and nodes of R(f).
- The rest of the Reeb graph is partitioned into *arcs* connecting the nodes.
- A minimum corresponds to a degree 1 node.
- An index 1 saddle that merges two contours into one corresponds to a degree 3 node.
- Symmetrically, a maximum corresponds to a degree 1 node and an index d - 1 saddle that splits a contour into two corresponds to a degree 3 node.
- However, all other critical points correspond to nodes of degree 2.
- Note that despite us talking about manifolds, the Reeb graph has no meaningful embedding in any space. It's just a (topological) graph.

Reeb Graphs of 2-Manifolds

- Let's finish by studying the Reeb graphs of a familiar setting, orientable 2-manifolds.
- Let X be an orientable 2-manifold and f : X → R Morse.
- Every saddle either merges two contours into one or splits a contour into 2.
- We'll use this observation to compute the number of loops in the Reeb graph.
- Let n_i be the number of nodes with degree i.

- Here, only n_1 and n_3 are non-zero.
- The number of arcs $e = (n_1 + 3n_3) / 2$, and the number of loops is $1 + e - (n_1 + n_3)$.
- Lemma: The Reeb graph of a Morse function on a connected, orientable 2-manifold of genus g has g loops.
    - Suppose R(f) has no loop. It is a tree with $n_1 = n_3 + 2$ degree 1 nodes. Let c_i be the number of critical points of index i so $n_1 = c_0 + c_2$ and $n_3 = c_1$. From the strong Morse inequality, we have $\chi = \beta_0 - \beta_1 + \beta_2 = c_0 - c_1 + c_2 = n_1 - n_3 = 2$. So the genus was 0.
    - Now suppose otherwise that there is at least one loop.
    - Repeatedly collapse degree 1 nodes and merge arcs across degree 2 nodes. The example above turns into two degree 3 nodes connected to each other by three arcs.
    - Both operations preserve homotopy type and therefore number of loops.
    - Let m_3 be the number of remaining degree 3 nodes. There are $(3/2)m_3$ remaining arcs. Therefore, m_3 is even.
    - Using the Euler-Poincaré formula for graphs, we see $m_3 - (3/2) m_3$ = # components - # loops. The graph is connected, so there are $m_3 / 2 + 1$ loops.
    - We had c_1 degree 3 nodes in the original graph, and for each minimum or maximum, we collapsed one degree 1 node, removing a degree 3 node in the process.
    - So now using the strong Morse inequality, we see $m_3 = c_1 - (c_0 + c_2) = 2g - 2$.
    - The number of loops is therefore $(2g - 2) / 2 + 1 = g$.
- Using a more complicated analysis, we can also say the following about non-orientable 2-manifolds.
- Lemma: The Reeb graph of a Morse function on a connected, non-orientable 2-manifold of genus g has at most $g / 2$ loops.

Finishing Up

- And it probably doesn't make sense to move on to another topic now, so let's finish up.
- I honestly hope everybody enjoyed this semester and found it interesting and useful. I'm sorry we couldn't meet in person, but hopefully you found the MS Teams lectures acceptable.
- They don't appear to be open yet, but at some point I'm sure you'll get a chance to do evaluations of this and other classes. These evaluations determine what classes enter the regular rotation and they're part of how the university evaluates instructors for raises, promotions, etc. So please fill out an honest evaluation of this and your other classes when given the chance.
- If you enjoy this style of algorithms class, Benjamin Raichel is teaching CS 6319,

Computational Geometry in Spring 2021. That class primarily covers algorithms for solving problems in low-dimensional Euclidean space, and the material has applications to graphs, motion planning, data analysis, and so on.

- The material is also more established, so the lectures should feel a bit more consistent and structured than you got this semester.
- But seriously, if you enjoyed this class at all and haven't taken the other one yet, I highly recommend signing up.
- And that's it. Take care everybody, stay safe, and good luck in your future semesters and careers!