CS 7301.003.20F Lecture 4—August 26, 2020

Main topics are #planar_graphs.

Correspondences

- Last time, we discussed abstract and topological graphs and how they work together with rotation systems to describe planar and spherical embeddings.
- Today, we'll begin with the reason I really like thinking of the rotation system primarily instead of the continuous function embedding.
- Many fundamental structures in a connected planar graph G correspond directly to other structures in the dual graph G^*, and you can verify some of these claims by working directly with the rotation system.

primal G	**dual G^***	**primal G**	**dual G^***
vertex v	face v^*	empty loop	spur
dart d	dart d^*	loop	bridge
edge e	edge e^*	cycle	bond
face f	vertex f^*	even subgraph	edge cut
$tail(d)$	$left(d^*)$	spanning tree	complement of spanning tree
$head(d)$	$right(d^*)$	$G \setminus e$	G^* / e^*
$succ$	$rev \circ succ$	G / e	$G^* \setminus e^*$
clockwise	counterclockwise	minor $G \setminus X / Y$	minor $G^* \setminus Y^* / X^*$

Correspondences between features of primal and dual planar maps

- Here are some definitions that may not be obvious. Say G has n vertices and m edges.
  - a *spur* is an edge incident to a vertex of degree 1
  - a *bridge* is a single edge whose removal separates a graph into two components
  - an *even subgraph* is a subgraph in which every vertex has positive even degree. It's also the union of one or more edge-disjoint cycles.
  - A *cut* of G is a partition of its vertices into two non-empty subsets. Edges *cross* the cut from one side to the other. Edges crossing a cut form its *boundary*. An *edge cut* is the boundary of some cut.
  - An edge cut C is called a *bond* if no proper subset of C is the boundary of a cut.
  - G \ e denotes deleting edge e from G leaving n vertices and m - 1 edges.
  - G / e denotes *contracting* e, meaning we merge the endpoints of e and destroy e, leaving n - 1 vertices and m - 1 edges. If there are edges parallel with e, they become loops. By convention, we normally don't allow contraction of loops (but we can extend the definition to allow for contracting loops, maybe in the homework?)
  - A *minor* of G is any graph obtained by contracting edges of a subgraph of G. A *proper minor* is any minor that is not G itself. Note every minor of a planar graph G is also planar.

    - Given a subgraph X of G where every vertex has positive degree, I'll use X^* to denote its dual
- Some of these duality relations are pretty easy to see, but some are less obvious.
- Lemma: Fix a connected plane graph G. For any edge e of G that is not a loop, (G / e)^ * = G^ * \ e^ *. Symmetrically, for any edge e of G that is not a bridge, (G \ e)^ * = G^ * / e^ *.
    - Suppose e is not a loop. G / e is a connected plane graph with every dart in G except e^+ and e^-. Let succ / e and next / e be the primal and dual rotation system of G / e.
    - Recall, next(d) := rev(succ(d))
    - A contraction means as we go over darts entering one endpoint of e in counterclockwise order, we should move to the other endpoint when we'd normally encounter e, so
    - For any dart d of G / e, we have (succ / e)(d) =
        - succ(e^-) if succ(d) = e^+,
        - succ(e^+) if succ(d) = e^-
        - succ(d) otherwise
    - which implies (next / e)(d) = rev circ (succ / e)(d) =
        - rev circ succ(e^-) = next(e^-) if next(d) = e^+,
        - rev circ succ(e^+) = next(e^+) if next(d) = e^-,
        - next(d) otherwise
    - But if we delete e^* from G^*, then we should just skip over e^ * in the rotation around both of its dual endpoints, which is what happened here.
    - The other direction is the same argument but symmetric.
- Lemma: Fix a connected plane graph G. A subgraph H is an even subgraph of G if and only if H^* is an edge cut of G.
    - Let H be an even subgraph of G. We can decompose H into edge-disjoint cycles C_1, C_2, …, C_k. By Jordan Curve theorem, each cycle has a subset of faces on the inside and a subset on the outside.
    - Color each vertex f^* of G^* black if f lies in the interior of an odd number of cycles C_i and white otherwise. Each edge of H lies between a black and white face, so each edge of H^* lies between a back and white vertex. Therefore, H^* is an edge cut.
    - Conversely, let H^* be an edge cut of G^*. Let S^ * be the vertices on one side of the cut. Color a face of G black if its dual vertex lies in S^ * and white otherwise. Primal subgraph H has precisely the edges with one white and one black shore. And if you look at the incident faces around any vertex of G, you'll switch between black and white faces an even number of times. H is an even subgraph of G.
- As a corollary, we see any cycle (a minimal even subgraph) is dual to a bond, a minimal edge cut.
- Theorem: Fix a connected plane graph G. A subgraph T is a spanning tree of G if and only if G^* \ T^* is a spanning tree of G^*

    - Let T be a spanning tree of G, and let $C^* = G^* \setminus T^*$ be the complementary dual subgraph of T.
    - Every cycle of G excludes at least one edge in T, and every bond of G includes at least one edge of T.
    - Cycle-bond duality implies every bond of G^* contains at least one edge of C^*, and every cycle of G^* excludes at least one edge in C^*.
    - But that implies C^* is connected and acyclic, meaning it is a spanning tree of G^*.
- The trees T and C^* are sometimes called the *interdigitating spanning trees* of G and G^*. We'll see later in the semester that being able to partition the edges into a pair of trees makes it possible to implement some algorithms very quickly.

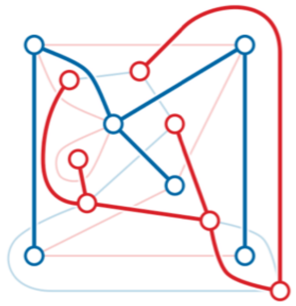

Figure 2.7. Interdigitating spanning trees of a planar map and its dual.

Euler's Formula

- There's a well-known fact about plane graphs that proves incredibly useful in many algorithms.
- Euler's formula: For any connected plane graph G with n vertices, m edges, and f faces, we have $n - m + f = 2$.
- I'll give the standard induction proof and then a really slick one based on interdigitating trees.
- Proof 1:
    - If G has no edges, it has one vertex and one face. Otherwise, let e by any edge in G; there are two overlapping cases.
    - If e is not a loop, then contracting e yields a a connected plane graph G / e with $n - 1$ vertices, $m - 1$ edges, and f faces. By induction, $(n - 1) - (m - 1) + f = 2$.
    - If e is not a bridge, then deleting e yields a connected plane graph $G \setminus e$ with n vertices, $m - 1$ edges, and $f - 1$ faces. By induction, $n - (m - 1) + (f - 1) = 2$.
    - Either way, $n - m + f = 2$.
- Proof 2 (von Staudt 1847):
    - Let T be a spanning tree of G.
    - T has n vertices and $n - 1$ edges.
    - The complementary dual subgraph $C^* = (G \setminus T)^*$ is a spanning tree of G^*.

    - C^* has f vertices and f - 1 edges.
    - Every edge is in exactly one of T or C^* so $m = (n - 1) + (f - 1)$.
- Euler's formula has a very useful corollary: Every simple planar graph G with $n \geq 3$ has at most $3n - 6$ edges and at most $2n - 4$ faces, with equality if some embedding of G is a triangulation.
    - Let G be such a graph.
    - Every face has degree at least 3 and every edge contributes exactly 2 to the total degree of all faces. Therefore, $3f \leq 2m$
    - So, $f \leq (2/3) m$, implying $n - m + (2/3)m \geq 2 \Rightarrow m \leq 3n - 6$.
    - Also, $m \geq (3/2) f$, implying $n - (3/2) f + f \geq 2 \Rightarrow f \leq 2n - 4$.
- In particular, every simple planar graph has a vertex with degree strictly less than 6.