CS 7301.003.20F Lecture 6—August 31, 2020

Main topics are #homotopy.

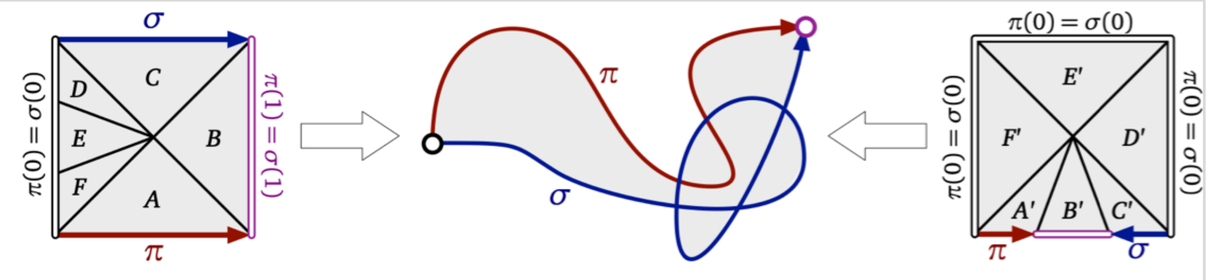
- First off, we won't be meeting Monday, because it's Labor Day. The next lecture will be next Wednesday.

Homotopy

- We're going to step back from planar graphs for a bit to discuss a fundamental topic in (computational) topology. The topic has to do with continuous transformations of objects, as opposed to continuous functions between objects.
- First, we can generalize our definition of a *path* to work in any topological space X by letting it be a continuous function from [0, 1] to X.
- A *path homotopy* in X between paths pi and pi' is a continuous function h : [0, 1] x [0, 1] to X such that
  - h(0, t) = pi(t) and h(1, t) = pi'(t) for all t in [0, 1] and
  - h(s, 0) = pi(0) = pi'(0) and h(s, 1) = pi(1) = pi'(1) for all s in [0, 1].
- So, for all t in [0, 1], the function s ➡ h(s, t) is a path from pi(0) to pi(1).
- You can imagine the path continuously morphing as s increases from 0 to 1.
- Or you could imagine the homotopy as a map from a closed topological disk (the unit square) into X that satisfies certain boundary conditions.
- Paths pi and pi' are (path) homotopic in X if there is a path homotopy between them. We use the notation pi ≃_X pi' to denote they are homotopic in X (but we may drop the X if it's clear from context).
- For example, any two paths with common endpoints in the plane are homotopic.
- Natural intuitions about paths and homotopy can take you pretty far so you don't have to worry much about the particulars of the continuous function.
- Lemma: For any two paths pi, pi': [0, 1] ➡ X and any continuous map phi : X ➡ Y, if pi ≃_X pi', then phi circ pi ≃_Y phi circ pi'.
  - Just compose the homotopy itself with phi.
- Lemma: For any path pi : [0, 1] ➡ X and any continuous map alpha : [0, 1] ➡ [0, 1] with alpha(0) = 0 and alpha(1) = 1, we have pi circ alpha ≃_X pi.
  - Use h(s, t) = pi(s alpha(t) + (1 - s) t) as the homotopy from pi to pi circ alpha.
- Given two paths pi and sigma with pi(1) = sigma(0), they can be *concatenated* to make a longer path from pi(0) to pi(1). Formally, pi dot sigma(t) :=
  - pi(2t) if t ≤ 1/2
  - sigma(2t - 1) if t ≥ 1/ 2.
- The *reversal* bar(pi) of pi is the path bar(pi)(t) := pi(1 - t). Notice bar(pi dot sigma) =

bar(sigma) dot bar(pi).

- A *loop* is a path whose endpoints coincide. The common endpoint is called the *basepoint*. Notice any homotopy between two loops must keep the basepoint fixed.
- A loop ell is *contractible* in X if it is homotopic to the constant loop t ➡ ell(0). Such a homotopy is called a *contraction* of ell.
- Lemma: Two path pi and sigma with common endpoints are homotopic if and only if the loop pi dot bar(sigma) is contractible.
  - Consider two triangulations of two copies of the unit square representing the domains for path homotopy and contraction, respectively.

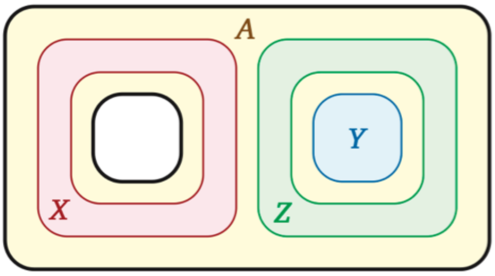


A path homotopy from π to σ is homotopic to a contraction of $\pi \cdot \overline{\sigma}$.

  - There is a piecewise-linear homeomorphism phi from the left square to the right one that fixes (0, 0) and takes each letter x to x′. If h : [0, 1]^2 ➡ X is a path homotopy from pi to sigma, then h circ phi : [0, 1]^2 ➡ X is a contraction of pi dot bar(sigma). (Note how the top side of the right square represents the constant path thanks to the mapping of E to E′. Also the mapping to F′ and D′ keep the basepoint fixed during the contraction.)
  - Likewise, if h′ : [0, 1]^2 ➡ X is a contraction, then h′ circ phi^{-1} is a path homotopy.
- Homotopy is an equivalence relation, meaning it is reflexive, symmetric, and transitive. It partitions paths into equivalence classes called *homotopy classes*. We write [pi] to denote the homotopy class of pi.
- Homotopy is closed under reversing both paths or concatenating pairs of paths.
- In particular, we can define reversal and concatenation of homotopy classes as [pi] dot [sigma] := [pi dot sigma] and bar([pi)] := [bar(pi)].
- Other natural things that are true, given three paths pi, sigma, tau : [0, 1] ➡ X.
  - Identity: If sigma is a contractible loop, pi dot sigma ≃_X pi.
  - Inverse: Loop pi dot bar(pi) is contractible.
  - Associativity: (pi dot sigma) dot tau ≃ pi dot (sigma dot tau)
- You may recognize these as the three group axioms.
- For any point x in X, the homotopy classes of loops with basepoint x define a group under concatenation with contractible loops being the identity element. It is called the *fundamental group* of X with basepoint x, denoted pi_1(X, x).
- We can't concatenate arbitrary paths, so their homotopy classes don't form a group. Instead, they form something called the fundamental groupoid.

Retractions

- Let's discuss a few related concepts. Near the end of the lecture, we'll start discussing an actual computation problem on homotopy that we should be able to solve by the end of Wednesday.
- A *retraction* from space X to subspace Y subseteq X is a continuous function r : X → Y whose restriction to Y is the identity map id_Y.
- A retraction r is called a *deformation retraction* if there is a continuous function R : [0, 1] x X → X such that R(0, dot) is the identity map on X and R(1, dot) = r. In other words, it's like a homotopy to the identity map.
- If a (deformation) retraction exists, we call Y a *(deformation) retract* of X.



X is a deformation retract of *A*; *Y* is a retract of *A*; and *Z* is neither.

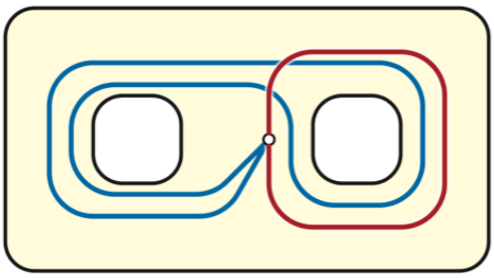
- Retractions are closely related to contractions.
- Lemma: For any retract Y of X, any loop ell in Y is contractible if and only if ell is contractible in X.
  - only if is trivial, just use the same contraction
  - for if, let r : X → Y be the retraction. r circ h is the contraction in Y
- Lemma: Let r be a deformation retraction from X to Y. Any loop ell in X is contractible if and only if loop r circ ell is contractible in Y.
  - If h : [0, 1] ^2 → X is a contraction of ell, then r circ h :[0, 1]^2 → Y is a contraction of r circ ell.
  - Suppose h' : [0, 1]^2 → Y is a contraction of r circ ell. Fix a map R : [0, 1] x X → X such that R(0, dot) = id_X and R(1, dot) = r. Map h : [0, 1]^2 → X defined by setting h(s, t) := R(s, h'(s, t)) is a contraction of ell.

Cycles and Free Homotopy

- A *cycle* in X is a continuous function from the circle S^1 to X.
- We can define a circle (and index into it) as the quotient space [0, 1] / (0 ~ 1) obtained by identifying the endpoints of the unit interval.
- Now any loop ell : [0, 1] → X can be transformed into an equivalent circle ell^circ : S^1 → X or vice versa by setting ell^circ(t) = ell(t).
- A *free homotopy* between cycles alpha and beta is a continuous function h : [0, 1] x S^1 →

X such that h(0, theta) = alpha(theta) and h(1, theta) = beta(theta) for all theta in S^1.

- Two cycles are *(freely) homotopic* if there is a free homotopy between them.
- A cycle is (freely) contractible if it is homotopic to the constant cycle; the homotopy is called a *free contraction*.
- Lemma: A loop ell is path-contractible if and only if the corresponding cycle ell^circ is freely contractible.
  - If h : [0, 1]^2 → X is a path-contraction of ell, then h^circ : [0, 1] x S^1 → X defined by h^circ(s, t) = h(s, t) is a free contraction of ell^circ.
  - Now, let h^circ : [0, 1] x S^1 → X be a free contraction of ell^circ. Define h : [0, 1]^2 → X as h(s, t) :=
    - h(2s, 1/2 + (2t - 1) / (2s - 1)) if s < t < 1 - s,
    - h(max{2 - 2s, 2 - 2t, 2t}, 0) otherwise
  - I'll skip the details, but this homotopy h keeps the basepoint ell(0) fixed but connected to the evolving cycle by a "tail" that consists of a path and its reversal. After the cycle contracts to a point, the homotopy contracts the tail back to the basepoint.
- We can also prove a path homotopy between loops alpha and beta shows a free homotopy between their cycles alpha^circ and beta^circ.
- Surprisingly though, two loops sharing a basepoint can be freely homotopy but not path-homotopic in case the free homotopy can't keep the basepoint fixed.



Freely homotopic loops that are not path-homotopic