Inventory
Aggregation and Discounting

Matching Supply and Demand
Outline

- Joint fixed costs for multiple products
- Long term quantity discounts
Example: Lot Sizing with Multiple Products

- Shipping multiple products over the same route to the same retailer
- Demand per year
  - $R_L = 12,000; R_M = 1,200; R_H = 120$
- Common transportation cost per delivery,
  - $S = $4,000$
- Product specific order cost per product in each delivery
  - $s_L = $1,000; s_M = $1,000; s_H = $1,000$
- Holding cost,
  - $h = 0.2$
- Unit cost
  - $C_L = $500; C_M = $500; C_H = $500$
Delivery Options

◆ No Aggregation:
  – Each product ordered separately

◆ Complete Aggregation:
  – All products delivered on each truck

◆ Tailored Aggregation:
  – Selected subsets of products for each truck
## No Aggregation:
Order each product independently

<table>
<thead>
<tr>
<th></th>
<th>Litepro</th>
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<th>Heavypro</th>
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</thead>
<tbody>
<tr>
<td>Demand per year</td>
<td>12,000</td>
<td>1,200</td>
<td>120</td>
</tr>
<tr>
<td>Fixed cost / order</td>
<td>$5,000</td>
<td>$5,000</td>
<td>$5,000</td>
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<tr>
<td>Optimal order size</td>
<td>1,095</td>
<td>346</td>
<td>110</td>
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<tr>
<td>Order frequency</td>
<td>11.0 / year</td>
<td>3.5 / year</td>
<td>1.1 / year</td>
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<tr>
<td>Annual cost</td>
<td>$109,544</td>
<td>$34,642</td>
<td>$0,954</td>
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**Total cost = $155,140**
Complete Aggregation: Order jointly All Products in All Trucks

- Total ordering cost \( S^* = S + s_L + s_M + s_H = $7,000 \)
- \( n \): common ordering frequency
- Annual ordering cost = \( n \cdot S^* \)
- Total holding cost:
  \[
  \frac{R_L}{2n} hC_L + \frac{R_M}{2n} hC_M + \frac{R_H}{2n} hC_H
  \]
- Total cost:
  \[
  TC(n) = S^* n + \frac{h}{2n} \left( R_L C_L + R_M C_M + R_H C_H \right)
  \]
  \[
  n^* = \sqrt{\frac{h \left( R_L C_L + R_M C_M + R_H C_H \right)}{2S^*}}
  \]
Complete Aggregation: Order all products jointly

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<td>12,000</td>
<td>1,200</td>
<td>120</td>
</tr>
<tr>
<td>Order frequency</td>
<td>9.75/year</td>
<td>9.75/year</td>
<td>9.75/year</td>
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<tr>
<td>Optimal order size</td>
<td>1,230</td>
<td>123</td>
<td>12.3</td>
</tr>
<tr>
<td>Annual holding cost</td>
<td>$61,512</td>
<td>$6,151</td>
<td>$615</td>
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Annual order cost = 9.75 × $7,000 = $68,250
Annual total cost = $136,528

Ordering high and low volume items at the same frequency cannot be a good idea.
Tailored Aggregation: Ordering Selected Subsets

- Example: Orders may look like \((L,M); (L,H); (L,M); (L,H)\).
- Most frequently ordered product: \(L\)
- \(M\) and \(H\) are ordered in every other delivery.
- We can associate fixed order cost \(S\) with product \(L\) because it is ordered every time there is an order.
- Products other than \(L\), the rest are associated only with their incremental order costs (s values).

An Algorithm:

Step 1: Identify most frequently ordered product
Step 2: Identify frequency of other products as a relative multiple
Step 3: Recalculate ordering frequency of most frequently ordered product
Step 4: Identify ordering frequency of all products
Tailored Aggregation: Ordering Selected Subsets

- $i$ is the generic index for items, $i$ is L, M or H.
- **Step 1:** Find **most frequently ordered item**:

$$n_i = \sqrt{\frac{hC_i R_i}{2(S + s_i)}}$$

$$n = \max \{n_i\}$$

The frequency of the most frequently ordered item will be modified later. This is an approximate computation.

- **Step 2:** Relative order frequency of other items, $m_i$

$$\bar{n}_i = \sqrt{\frac{hC_i R_i}{2s_i}}$$

$$m_i = \left\lceil \frac{n}{\bar{n}_i} \right\rceil$$

$m_i$ are relative order frequencies, they must be integers.

They do not change in the remainder.
Tailored Aggregation: Ordering Selected Subsets

- **Step 3:** Recompute the frequency of the most frequently ordered item. This item is ordered in every order whereas others are ordered in every $m_i$ orders. The average fixed ordering cost is:

  \[ S + \sum_i \frac{s_i}{m_i} \]

  Annual ordering cost = \(n(S + \sum_i \frac{s_i}{m_i})\)

  Annual holding cost = \(\sum_i \frac{R_i}{2n / m_i} hC_i\)

  \[ n^* = \sqrt{\frac{\sum_i R_i m_i hC_i}{2 \left( S + \sum_i \frac{s_i}{m_i} \right)}} \text{ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ formula (10.9) on p.274 of the textbook} \]
Tailored Aggregation: Ordering Selected Subsets

◆ Step 4: Recompute the ordering frequency $n_i$ of other products:

$$n_i = \frac{n}{m_i}$$

◆ Total Annual ordering cost: $nS + n_Hs_H + n_Ms_M + n_Ls_L$

- $n$ (the frequency of the most frequently ordered product) is one of the following values $n_H, n_M, n_L$

◆ Total Holding cost:

$$\frac{R_L}{2n_L} hC_L + \frac{R_M}{2n_M} hC_M + \frac{R_H}{2n_H} hC_H$$
Tailored Aggregation: Ordering Selected Subsets

◆ Step 1:
\[ \bar{n}_L = \sqrt{\frac{hC_L R_L}{2(S + s_L)}} = 11, \quad \bar{n}_M = 3.5, \quad \bar{n}_H = 1.1 \quad n = \max\{\bar{n}_i\} = 11 \]

◆ Step 2:
\[ \bar{n}_M = \sqrt{\frac{hC_M R_M}{2s_M}} = 7.7, \quad \bar{n}_H = 2.4; \quad m_M = \left\lceil \frac{n}{\bar{n}_M} \right\rceil = 2, \quad m_H = 5 \]

Item L is ordered most frequently.
Every other L order contains one M order.
Every 5 L orders contain one H order.
At this step we only now relative frequencies, not the actual frequencies.
Tailored Aggregation: Ordering Selected Subsets

- Step 3: 
  \[ n^* = \frac{\sqrt{\sum_i hC_i R_i m_i}}{2 \left( S + \sum_i \frac{s_i}{m_i} \right)} = \sqrt{\frac{(0.2)500(12000 \times 1 + 1200 \times 2 + 120 \times 5)}{2(4000 + 1000/1 + 1000/2 + 1000/5)}} = 11.47 \]

- Step 4:
  \[ n_M = \frac{n^*}{m_M} = 5.73 \quad \text{and} \quad n_H = \frac{n^*}{m_H} = 2.29 \]

- Total ordering cost:
  \[ nS + n_H s_H + n_M s_M + n_L s_L = 11.47(4000) + 11.47(1000) + 5.73(1000) + 2.29(1000) \]
  \[ = 45,880 + 11,470 + 5,730 + 2,290 = 65,370 \]

- Total holding cost
  \[ \frac{R_L}{2n_L} hC_L + \frac{R_M}{2n_M} hC_M + \frac{R_H}{2n_H} hC_H \]
  \[ = \frac{12000}{2(11.47)} (0.2)500 + \frac{1200}{2(5.73)} (0.2)500 + \frac{120}{2(2.29)} (0.2)500 \]
  \[ = (528.1 + 104.71 + 26.2)100 \]
Tailored Aggregation: Order selected subsets

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<td>11.47/year</td>
<td>5.73/year</td>
<td>2.29/year</td>
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<tr>
<td>Optimal order size</td>
<td>1046.2</td>
<td>209.4</td>
<td>42.4</td>
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<tr>
<td>Annual holding cost</td>
<td>$52,810</td>
<td>$10,470</td>
<td>$2,630</td>
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Annual order cost = $65,370
Total annual cost = $130,650

Compare with $136K of total aggregation and with $155K of no aggregation
Lessons From Aggregation

- Aggregation allows a firm to lower lot size without increasing cost
  - Order frequencies without aggregation and with tailored aggregation
    » (11; 3.5; 1.1) vs. (11.47; 5.73; 2.29)
    » More frequent ordering implies smaller order sizes

- Tailored aggregation is effective if product specific fixed cost is a large fraction of joint fixed cost

- Complete aggregation is effective if product specific fixed cost is a small fraction of joint fixed cost

- Information technology can decrease product specific ordering costs.
The word of the moment: Retail

- **Retail**: The sale of goods in small quantities directly to the customer. Opposite of the word *wholesale*.

- **Retail** is a very flexible word. It can be used as a
  - *Noun*: I work in retail.
  - *Verb*: Albertson *retails* various groceries.
  - *Adjective*: Retail margins are too narrow.
  - *Adverb*: Wal-mart sells everything retail.

- **Etymology**: A variant of Old French *retaille* "piece cut off" from *retaillier* "to cut up" from re- "repeat" + tailler "cut." Akin to "tailor" which comes from Old French *tailleor* from *taillier* "to cut" going back to Late Latin *taliare* "cut."
Quantity Discounts

- Lot size based
  - All units
  - Marginal unit at the end of these file
- Volume based

- How should buyer react?
- What are appropriate discounting schemes?
All-Unit Quantity Discounts

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<tr>
<th>Order Quantity</th>
<th>Cost/Unit</th>
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<tr>
<td>5,000</td>
<td>$3</td>
</tr>
<tr>
<td>10,000</td>
<td>$2.96</td>
</tr>
<tr>
<td></td>
<td>$2.92</td>
</tr>
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</table>

Cost/Unit

$3
$2.96
$2.92

Order Quantity

5,000
10,000

Total Material Cost

Order Quantity

5,000
10,000
All-Unit Quantity Discounts

- \{0, q_1, q_2, \ldots\} are price break quantities
- Find EOQ for price in range \(q_i\) to \(q_{i+1}\)
  - If \(q_i \leq \text{EOQ} < q_{i+1}\),
    - » Candidate in this range is EOQ, evaluate cost of ordering EOQ
  - If \(\text{EOQ} < q_i\),
    - » Candidate in this range is \(q_i\), evaluate cost of ordering \(q_i\)
  - If \(\text{EOQ} \geq q_{i+1}\),
    - » Candidate in this range is \(q_{i+1}\), evaluate cost of ordering \(q_{i+1}\)

- Warning: Do not ignore purchase cost
  - The annual material cost of buying in lot sizes of \(q_i \leq Q < q_{i+1}\) is \(C_i R\).
- Find minimum cost over all candidates
Finding Q with all units discount

\[ Q_1 = \sqrt{\frac{2RS}{hc_1}} \]

\[ Q_2 = \sqrt{\frac{2RS}{hc_2}} \]

\[ Q_3 = \sqrt{\frac{2RS}{hc_3}} \]
Finding $Q$ with all units discount

\[ Q_1 = \sqrt{\frac{2RS}{hc_1}} \]

\[ Q_2 = \sqrt{\frac{2RS}{hc_2}} \]

\[ Q_3 = \sqrt{\frac{2RS}{hc_3}} \]
Finding Q with all units discount
Summary

- Aggregation: Joint fixed costs for multiple products
- Discounts: All unit quantity discounts over a long term
Marginal Unit Quantity Discounts

Cost/Unit

Total Material Cost

$3
$2.96
$2.92

Order Quantity

Order Quantity

$q_1$
$q_2$

$q_1$
$q_2$

5,000 10,000
5,000 10,000

$V_1$
$V_2$
Marginal Unit Quantity Discounts

\[ V_i = \text{Cost of buying exactly } q_i. \quad V_0 = 0. \]
\[ V_i = c_0(q_1 - q_0) + c_1(q_2 - q_1) + \cdots + c_{i-1}(q_i - q_{i-1}) \]
If \( q_i \leq Q \leq q_{i+1}, \)

Annual order cost = \( \frac{R}{Q} S \)

Annual holding cost = \( (V_i + (Q - q_i)c_i) \frac{h}{2} \)

Annual material cost = \( \frac{R}{Q} \left( V_i + (Q - q_i)c_i \right) \)

\[ \frac{\partial \text{Total cost}(Q)}{\partial Q} = -\frac{R}{Q^2} S + c_i \frac{h}{2} - \frac{R}{Q^2} (V_i - q_i c_i) = 0 \]

For range \( i, \) \( EOQ = \sqrt{\frac{2R(S + V_i - q_i c_i)}{hc_i}} \)
Marginal-Unit Quantity Discounts

- Find EOQ for price in range $q_i$ to $q_{i+1}$
  - If $q_i \leq \text{EOQ} < q_{i+1}$,
    » Candidate in this range is EOQ, evaluate cost of ordering EOQ
  - If EOQ < $q_i$,
    » Candidate in this range is $q_i$, evaluate cost of ordering $q_i$
  - If EOQ $\geq q_{i+1}$,
    » Candidate in this range is $q_{i+1}$, evaluate cost of ordering $q_{i+1}$

- Find minimum cost over all candidates
Marginal Unit Quantity Discounts

Compare this total cost graph with that of all unit quantity discounts. Here the cost graph is continuous whereas that of all unit quantity discounts has breaks.
Marginal Unit Quantity Discounts

Total cost vs. Lot size with various EOQs indicated for discount purposes.