Pricing with Constrained Supply

Outline

- Pricing with a Supply Constraint
- Opportunity Cost
- Variable Pricing
- Variable Prices in Practice
Supply Constraint Examples

1. UTD’s Cohort (full-time) MBA program has 50 spots
2. Eismann Center performance hall has 1563 seats
3. Dallas Cowboys Stadium can accommodate up to 111,000 people
4. Royal Caribbean’s cruise ship Voyager of the Seas departs from Galveston.
   - Passengers: 3114 passengers
   - Inside Cabins: 618
   - Outside Cabins: 939
   - Balcony cabins: 757
   - Suites: 119
5. Airbus A380 can seat 550 passengers
6. A sit-com
   - lasts 30 mins = for 22 min pure sit-com + 3*2.5 min advertisement + 0.5 min
   - accommodates 3 advertisement pods (breaks) of approximately about 6th, 16th and 26th minutes of the show.
     - Each pod lasts 150 seconds=2.5 minutes.
     - During 150 seconds 8-12 commercials can be shown
7. Yahoo.com has three vertical panels: “My favorites”, “Today-news”, “Advertisements”
   - Advertisements panel can accommodate 1 big and 1 small advertisement
   - Total advertisement area: 10 cm wide & 10 cm long; it can be split into smaller pieces
Supply Constraint: $b$

- Constrained pricing problem is

$$\max \quad \Pi(p) = d(p)(p - c)$$
$$st \quad d(p) \leq b$$
$$p \geq 0$$

1. Unconstrained price solution

2. Demand with unconstrained solution

3. Sell entire capacity $b$

4. Constrained price solution $p = d^{-1}(b)$

$d(.)$ maps price to demand.

$d^{-1}(.)$ maps demand to price.

If $d(p) = D - mp$ then $d^{-1}(b) = \frac{D - b}{m}$.

What is $d(d^{-1}(b))$?
Examples of Constrained Supply

- Ex: Suppose that $d(p) = 200 - 10p$ if $0 \leq p \leq 20$. Find total margin maximizing price when the cost is $c = 10$ and the capacity constraint is $b = 20$.
  - From the derivative of $(p - c)d'(p) + d(p) = 0$, we have $-20p + 300 = 0$.
  - The root of this equation is 15. The derivative of the total margin is positive from 0 to 15 and negative from 15 to 20. The price $p = 15$ solves the optimality equation.
  - The price of $p = 15$ yields the demand of $50 > b = 20$.
  - To sell all of the capacity, we price so that $20 = 200 - 10p$. This yields $p = 18$.

Aside: To compute $d^{-1}(b)$, set $b = 200 - 10p$ and solve for $p$. $p = 20 - b/10$, so $d^{-1}(b) = 20 - b/10$.

- Ex: Suppose that $d(p) = (10,000 - p)^2/2,000$ if $0 \leq p \leq 10,000$ for a baseball playoff final game. Find total margin maximizing price when the cost is $c = 0$ and the capacity constraint is $b = 1600$.
  - The profit is $(10^8p - 2 \times 10^4p + p^3)/2000$. Setting derivative $= 0 \Rightarrow 10^8 - 4 \times 10^4p + 3p^2 = 0$.
  - The root of this quadratic equation strictly in $(0,10000)$ is price of $3,333$ from $p = \frac{40,000 \pm \sqrt{40,000^2 - 4(10,000)3}}{2(3)} = \frac{40,000 \pm 20,000}{2(3)}$.
  - The price of $p = 3,333$ yields the demand of $22,224 > b = 1600$.
  - To sell all of the capacity, we price so that $1,600 = (10,000 - p)^2/2,000$ or $p = 10,000 - \sqrt{3.2 \times 10^6} = 10,000 - 1,788 = 8,212$. This yields a profit of $13,130,988 = 8,212 \times 1,599$.
  - Above example is motivated by an article on Chicago-Cleveland Baseball final in 2016. “One reason for the high prices is [in] Chicago is very limited supply of tickets. Cub fans with tickets who have waited their whole lives for this chance aren’t willing to sell, even with the potential windfall. … There are only about 1,600 tickets a night available online.”
Opportunity Cost: Going from 20 to 40

- With constraint $b$, we find
  \[ \Pi(b) = \max\{d(p)(p-c) : d(p) \leq b, p \geq 0\} \]
- The benefit of more capacity, say by going from $b=20$ to $b=40$, is
  \[ \Pi(b=40) - \Pi(b=20) \]
- Under $b=20$ in the last example, we obtain a profit of $(18-10)(20)=160$.
- Under $b=40$, we still have that the demand of 50 with $p=15$ violates the capacity.
  - We solve for $p = d^{-1}(b = 40) = 20 - b/10 = 16$.
  - The profit then is $(16-10)(40)=240$.
- Opportunity cost of having the capacity of 20 as opposed to 40 is 80:
  \[ 80 = 240 - 160 = \Pi(b = 40) - \Pi(b = 20) \]
Opportunity Cost: Going from 40 to 50

- Under $b=40$ in the last example, we obtain a profit of $(16-10)(40)=240$.
- Under $b=50$, the demand of 50 with $p=15$ is within the capacity so the profit is $(15-10)(50)=250$.
- Opportunity cost of having the capacity of 40 as opposed to 60 is 10:
  \[ \Pi(b = 50) - \Pi(b = 40) = 250 - 240 \]

- Under $b=60$, the demand of 50 with $p=15$ is within the capacity so the profit is $(15-10)(50)=250$.
- Opportunity cost of having the capacity of 50 as opposed to 60 is 0:
  \[ \Pi(b = 60) - \Pi(b = 50) = 250 - 250 \]
Putting Various Capacities Together

Opportunity Cost

\[
\Pi(b = 0) = 0; \quad \Pi(b = 20) = 160; \quad \Pi(b = 40) = 240; \quad \Pi(b = 50) = 250; \quad \Pi(b \geq 60) = 250.
\]

- The profit curve between the pairs of (capacity, profit) that we have evaluated above is not linear. So the profit is drawn in dots.
- Because of nonlinearity, marginal opportunity cost is not constant.
Linear Demand Curve
Marginal Opportunity Cost

Suppose that the demand curve is linear: \( d(p) = D - mp \).

The unconstrained price \( p_0 \) that maximizes profit is found from the derivative of \( \Pi(p) = (p-c)(D-mp) \). The derivative is

\[
D + mc - 2mp = 0 \quad \text{which yields}
\]

\[
p_0 = \frac{D + mc}{2m}.
\]

The demand under this price is

\[
d(p_0) = D - m\left(\frac{D + mc}{2m}\right) = \frac{D - mc}{2}.
\]

Marginal opportunity cost of the capacity is zero when the capacity is more than the demand under the price \( p_0 \).

\[
\frac{d}{db} \Pi(b) = 0 \quad \text{for} \quad b \geq d(p_0)
\]

See the opportunity cost in the last example for capacity higher than 50.
Linear Demand Curve
Marginal Opportunity Cost with Insufficient Capacity

- Suppose that the capacity is insufficient so $b < d(p_0)$.
- Set the price equal to $d^{-1}(b)$ to sell the entire capacity.
- The marginal opportunity cost is

$$\frac{d}{db} \Pi(b) = \frac{d}{db} b(d^{-1}(b) - c) \quad \text{for linear demand curve } \frac{d^{-1}(b)}{m} = \frac{D - b}{m}$$

$$= \frac{d}{db} \left\{ b \left( \frac{D}{m} - \frac{b}{m} - c \right) \right\}$$

$$= \frac{D}{m} - c - \frac{2b}{m}$$

As an exercise check $\frac{d}{db} \Pi(b) \bigg|_{b=d(p_0)} = 0$, so the marginal cost is continuous.

When $b=d(p_0)$, $b=(D-mc)/2$. 
Example: Linear Demand Curve
Marginal Opportunity Cost

- Suppose that the demand is linear \( d(p) = D - mp = 200 - 10p \) and \( c = 10 \).
- We already computed \( p_0 = 15 \) and \( d(p_0) = 50 \).
- Then the marginal opportunity cost is

\[
\frac{d}{db} \Pi(b) = \frac{D}{m} - c - \frac{2b}{m} = \frac{200}{10} - 10 - \frac{2b}{10} = 10 - \frac{b}{5} \quad \text{for} \ b \leq 50.
\]

\[
\frac{d}{db} \Pi(b) \bigg|_{b=50} = 0.
\]
Summary of Pricing a Single Product under Capacity Constraint

In the general case of demand $d(p)$

- Underlying profit maximization problem is max\{$d(p)(p - c): d(p) \leq b, p \geq 0$\}
- Let $p_0$ be the maximizer of the unconstrained problem max\{$d(p)(p - c): p \geq 0$\}
- If $d(p_0) \leq b$, then capacity constraint is non-binding and the optimal price is $p^* = p_0$
- Else, i.e., $d(p_0) > b$, then the capacity constraint is binding and the optimal price is $p^* = d^{-1}(b)$

In the special case of linear demand $d(p) = D - mp$

- Underlying profit maximization problem max\{$(D - mp)(p - c): p \geq \frac{D-b}{m}, p \geq 0$\}
- $p_0 = \frac{D+mc}{2m}$
- If $\frac{D-mc}{2} \leq b$, then capacity constraint is non-binding and the optimal price is $p^* = p_0$
- Else, i.e., $\frac{D-mc}{2} > b$, then the capacity constraint is binding and the optimal price is $p^* = \frac{D-b}{m}$
Market Segmentation with a Capacity Constraint

- Consider accepting MBA students with and without scholarships where the capacity is 50 for full-time MBA.
  - Scholarships segment the market.
- Consider selling tickets for Eismann Center performances to senior citizens and non-seniors. Seniority segments the market for the performance hall whose capacity is 1563 seats.
- Consider selling tickets for Berkeley and Stanford football game to be held at a stadium with 60000 seating capacity.
- Demand curves for
  - general public $d_g(p_g) = (120,000 - 3,000p)^+$
  - students are given by $d_s(p_s) = (20,000 - 1,250p)^+$
Market Segmentation with a Capacity Constraint
Single Price

- When a single price is charged to both general and student attendees, we have

\[ R(p) = p(120,000 - 3,000p)^+ + p(20,000 - 1,250p)^+ \]

whose derivative is

\[
R'(p) = \begin{cases} 
0 & \text{if } p > 40 \\
120000 - 6000p & \text{if } 16 < p \leq 40 \\
120000 - 6000p + 20000 - 2500p & \text{if } p \leq 16 
\end{cases}
\]

The only feasible prices that can be solutions are \( p = 20 \) and \( p = 16 \).

- For \( 16 < 0 \leq 40 \), the candidate for optimal is \( p = \frac{120,000}{6,000} = 20 \).
- For \( p \leq 16 \), the candidate for optimal is \( p = \frac{140,000}{8,500} = 16.47 > 16 \). So \( p = 16 \).

- Since the stadium is already sold out at the price of $20, the optimal price is $20.
- The optimal revenue under single-price is 20(60,000)=1,200,000.
Market Segmentation with a Capacity Constraint Formulation with Multiple Prices

When separate prices are charged to general and student attendees, we have

\[
\max p_g (120,000 - 3,000 p_g) + p_s (20,000 - 1,250 p_s)
\]
\[
\text{st} \quad 120,000 - 3,000 p_g + 20,000 - 1,250 p_s \leq 60,000
\]
\[
(\text{or equivalently } 3 p_g + 1.25 p_s \geq 80)
\]
\[
p_g \leq 40, \quad p_s \leq 16
\]

This objective defines a ellipse in terms of prices
The maximization objective above is equivalent to the following objectives:

\[
\begin{align*}
\text{max} &\quad 120,000 p_g - 3,000 p_g^2 + 20,000 p_s - 1,250 p_s^2 \\
\text{min} &\quad 3 p_g^2 - 120 p_g + 1.25 p_s^2 - 20 p_s \\
&\quad 1000 \text{ min } 3 p_g^2 - 120 p_g + 3(400) - 3(400) + 1.25 p_s^2 - 20 p_s + 1.25(64) - 1.25(64) \\
&\quad 1000 \text{ min } 3(p_g - 20)^2 - 3(400) + 1.25(p_s - 8)^2 - 1.25(64) \\
&\quad -3(400) - 1.25(64) + 1000 \text{ min } 3(p_g - 20)^2 + 1.25(p_s - 8)^2
\end{align*}
\]

At the optimal prices, the capacity constraint is tangent to the ellipse.
Market Segmentation with a Capacity Constraint
Algebraic Solution with Multiple Prices

The solution of

\[
\max p_g (a_g - b_g p_g) + p_s (a_s - b_s p_s) \\
\text{st} \\
a_g - b_g p_g + a_s - b_s p_s \leq C \\
p_g \leq a_g / b_g, \quad p_s \leq a_s / b_s
\]

For details see the appendix to these slides and constrainedEx.pdf

If \( \frac{a_g + a_s}{2} \leq C \), capacity is sufficient or capacity constraint is non-binding at separately optimal prices (by optimizing revenues separately for each segment):

\[
p_g^* = \frac{a_g}{2b_g} \quad \text{and} \quad p_s^* = \frac{a_s}{2b_s}
\]

Check the capacity constraint with \( p_g^* = \frac{a_g}{2b_g} \) and \( p_s^* = \frac{a_s}{2b_s} \)

\[
a_g - b_g \left( \frac{a_g}{2b_g} \right) + a_s - b_s \left( \frac{a_s}{2b_s} \right) = \frac{a_g + a_s}{2} \leq C
\]

The last inequality is the condition that indicates sufficiency of capacity at the separately optimal prices.

If \( \frac{a_g + a_s}{2} > C \), i.e., capacity is insufficient or capacity constraint is binding at separately optimal prices, then

\[
p_g^* = \frac{2a_g + a_s - 2C + a_g b_s / b_g}{2(b_g + b_s)} \quad \text{and} \quad p_s^* = \frac{2a_s + a_g - 2C + a_s b_g / b_s}{2(b_s + b_g)}
\]
Market Segmentation with a Capacity Constraint
Algebraic Solution of an Insufficient Capacity Instance

The original demand curve for general public: \( 120,000 - 3,000p_g \)

The solution of
\[
\begin{align*}
\max p_g (120,000 - 3,000 p_g) + p_s (20,000 - 1,250 p_s) \\
st \quad 120,000 - 3,000 p_g + 20,000 - 1,250 p_s \leq 60,000 \\
p_g \leq 40, \quad p_s \leq 16
\end{align*}
\]

If \( \frac{a_g + a_s}{2} = \frac{120+20}{2} > 60 = C \), i.e., capacity is insufficient or capacity constraint is binding at separately optimal prices, then

\[
p_{g}^* = \frac{2a_g + a_s - 2C + a_g b_s / b_g}{2(b_g + b_s)} \quad = \frac{2(120)+20-2(60)+120(1.25)/3}{2(3+1.25)} = \frac{190}{8.5} = 22.352941 \quad \text{and}
\]

\[
p_{s}^* = \frac{2a_s + a_g - 2C + a_s b_g / b_s}{2(b_s + b_g)} \quad = \frac{2(20)+120-2(60)+20(3)/1.25}{2(1.25+3)} = \frac{88}{8.5} = 10.352941
\]

Check the capacity constraint with \( p_{g}^* = 22.352941 \) and \( p_{s}^* = 10.352941 \)

\[
\begin{align*}
120,000 - 3,000(22.352941) & = 52,941.1765 \quad \text{tickets to general public} \\
20,000 - 1,250(10.352941) & = 7,058.8235 \quad \text{tickets to students}
\end{align*}
\]

Capacity constraint is binding : 60,000 tickets to both

The corresponding revenue is $1,256,471; 4.7\%$ more than the revenue under single price.
Market Segmentation with a Capacity Constraint
Algebraic Solution of a Sufficient Capacity Instance

The reduced demand curve for general public: \(90,000 - 3,000p_g\)

The solution of

\[
\begin{align*}
\text{max } & \quad p_g(90,000 - 3,000p_g) + p_s(20,000 - 1,250p_s) \\
\text{st } & \\ & 90,000 - 3,000p_g + 20,000 - 1,250p_s \leq 60,000 \\
& p_g \leq 30, \quad p_s \leq 16
\end{align*}
\]

If \(\frac{a_g + a_s}{2} = \frac{90 + 20}{2} \leq 60 = C\), capacity is sufficient or capacity constraint is non-binding at separately optimal prices

\[
p_g^* = \frac{a_g}{2b_g} = \frac{90}{2(3)} = 15 \quad \text{and} \quad p_s^* = \frac{a_s}{2b_s} = \frac{20}{2(1.25)} = 8
\]

Check the capacity constraint with \(p_g^* = 15\) and \(p_s^* = 8\)

\[
90,000 - 3,000(15) + 20,000 - 1.25(8) = 45,000 + 10,000
\]

\[
= \frac{90,000 + 20,000}{2} = \frac{a_g + a_s}{2} \leq C = 60,000
\]
Market Segmentation with a Capacity Constraint
Continuity of Prices in Capacity

Under insufficient capacity \( C < \frac{a_g + a_s}{2} \), consider increasing capacity and check prices

\[
\lim_{C \to (a_g + a_s) / 2} p_g^*(C) = \lim_{C \to (a_g + a_s) / 2} \frac{2a_g + a_s - 2C + \frac{a_g b_s}{b_g}}{2(b_g + b_s)} = \lim_{C \to (a_g + a_s) / 2} \frac{a_g(1 + \frac{b_s}{b_g}) + a_g + a_s - 2C}{2b_g\left(1 + \frac{b_s}{b_g}\right)} = \frac{a_g(1 + \frac{b_s}{b_g})}{2b_g\left(1 + \frac{b_s}{b_g}\right)}
\]

\( = \frac{a_g}{2b_g} \).

And similarly

\[
\lim_{C \to (a_g + a_s) / 2} p_s^*(C) = \frac{a_s}{2b_s}.
\]

As capacity \( C \) increases optimal prices decrease towards the separately optimal prices and become separately optimal prices at \( C = (a_g + a_s) / 2 \).
Variable Pricing

◆ The prices for
- Theme park tickets vary over a week
- Movie theater rickets vary over a week
- Sport events vary over a season
- Airline tickets vary over a season
- Electricity vary over time of the day
- Phone calls vary over time of the day
- Disney Theme Parks announced investigation of variable pricing in Oct 2015
Variable Pricing
Single Price

- Index periods (time, day, week etc.) by $i$
- Demand in period $i$ is $D_i - m_ip$
- The same price $p$ everyday. The only decision variable is the price $p$.
- Capacity is limited at $C$.

As a consequence of this variable, we obtain sales for period $i$, which is

\[ x_i = \text{Sales in period } i = \min\{\text{Demand, Capacity}\} \text{ in period } i = \min\{D_i - m_ip, C\} \]

\[ x_i \leq \min\{D_i - m_ip, C\} \iff x_i \leq D_i - m_ip \text{ and } x_i \leq C \]

\[
\max p \sum_i x_i \\
\text{st} \\
x_i \leq D_i - m_ip \text{ for period } i \\
x_i \leq C \quad \text{for period } i \\
p \geq 0
\]

How do we know that sales is not strictly less, i.e., $x_i < D_i - m_ip$ and $x_i < C$?
Variable Pricing
Multiple Prices

- Price charged in period $i$ is denoted by $p_i$
- If $p_i < p_j$, some customers switch consumption from period $j$ to period $i$
- How many customers switch?
  - First suppose that the number of switching customers does not depend on either $i$ or $j$
  - Suppose that $s$ customers switch for each dollar of price difference
    » The number of customers switching from period $j$ to period $i$ is $s(p_j - p_i)$
  - This number $s$ is difficult to find out
- Given a price for each period, the demand for period becomes
  $$d_i = D_i - m_i p_i + s \sum_j (p_j - p_i) \quad \text{for period } i$$
- Then the price optimization problem is
  $$\max \sum p_i x_i$$
  $$\text{st}$$
  $$x_i \leq D_i - m_i p_i + s \sum_j (p_j - p_i) \quad \text{for period } i$$
  $$x_i \leq C \quad \text{for period } i$$
  $$p_i \geq 0$$
Variable Pricing
Revenue of Multiple Prices > Revenue of Single Price

- The multiple price formulation can be reduced to the single price formulation by inserting the following set of constraints:

\[ p_i = p \] for period \( i \)

- Inserting this constraint makes the objective value worse.
- In other words, revenue can be made higher with multiple prices.
Sporting Events
Variable Prices in Practice

Soccer leagues
- Big-4 English Premier League: Liverpool (shipyard workers), Chelsea (bourgeoisie), Arsenal (alternative), Manchester United (middle-class, higher Brits)
- Big-4 in Turkish Super League: Trabzon (seamen, fishermen), Fener (bourgeoisie), Beşiktaş (alternative; read “The View from the Stands” by E. Batuman, The New Yorker, March 7, 2011), Galata (middle-class, higher Turks)
- Generally, the championship is won by one of the four teams.
- Derbies are games among big four and their prices are substantially higher. UEFA Champions league is highest.

National Basketball League (NBA)
- What are the premier teams: LA Lakers, Boston Celtics, perhaps? Premier teams are not too easy to identify
- Not much of a derby game concept: Except for Lakers – Celtics rivalry.

Baseball leagues
- Outdoor games in the spring (cold), summer (nice), fall (cold). Charge higher for summer games.
- No dynamic pricing until 2010.

Football league
- Dallas Cowboys – no variable prices
- Cheapest ticket at $75 vs.
  - Atlanta Falcons on Oct 25;
  - Washington Redskins on Nov 22;
  - Philly Eagles on Jan 3

Limited variable pricing in sports
Baseball League - Update in 2010
Variable Prices at San Francisco Giants

♦ San Francisco Giants adopted dynamic pricing in 2009 season
  – Implemented in 2010 season; revenue up by 6% with similar attendance figures

♦ Ticket price depends on
  – Data from the secondary ticket market (ticket agencies)
  – Status of the pennant race
  – Success of Giants on the field
  – Opposing team, higher prices with
    Historic rivalry: Giants versus Dodgers
    Pennant (e.g., National League West title) contenders
    Rarely seen inter-league teams (such as Yankees)
  – Pitching match-ups
  – Day of the week
  – Weather forecast

♦ Example ticket prices for Giants vs. San Diego Padres game on Oct 1
  – Left-field upper deck stand: $5 at the start of the season; $5.75 on Aug 1, 2010; $20 after Giants and Padres become contenders of NL West title
  – Field Club behind home plate: $68 at the start of the season; $92 on Aug 1, 2010; $121 on Aug 9, 2010; $145 on Sep 4, 2010; $175 before the game.
  – Prices went up as Giants were competing to advance to playoffs first time since 2003.
    – See Page 4 of ORMS Today, October 2010 Issue, Vol.37, No.5.
Passenger Airlines
Variable Prices in Practice

- **Southwest (discount) airlines**
  - Limited customer segmentation
  - If demand for the next weekend’s morning flight to Chicago-Midway is high, increase price.

- **Other airlines (full-service) airlines**
  - Customer segmentation
  - Take coach class and split into fare classes
  - Some classes are more expensive than others
  - If demand is high, close low-fare classes
Electric Power
Variable Prices in Practice

- Peak electricity demand at 6 pm
- Off-peak electricity demand at 4 am
- Power generation from coal, nuclear, etc.
  - Big generators are on throughout the day, smaller ones are turned on during peak periods
  - Smaller generators are not efficient so their electricity is more costly
  - A 5% reduction in electricity production during peak period reduces marginal cost by 55%
- Reduce the demand during the peak period by increasing the price
- Adjusting price with demand is not fully implemented
  - Historical reasons: Electric utilities were monopolies in local markets
  - Political reasons: Power companies generated little electricity to keep prices high in California. Then governor Davis was recalled and Schwarzenegger replaced him.
  - US electricity grid is not integrated – east, west and Texas. ERCOT: Electric Reliability Council of Texas manages the flow of electric to 22 million Texas customers.
- Isolated applications in US: Recallable capacity (Industrial Demand Response) in Ercot, NYISO and PJM markets.
Television Advertising
Variable Prices in Practice

- During a 30 min program, 3 pods of advertisements are shown. A pod lasts 150 sec.
- Pods during prime time are more demanded than those during insomnia time
- Advertisement buyers require a certain number of viewers for pods
  - Program ratings are important
- Upfront market
  - For Fall, it lasts 1-2 weeks in May or June after the announcement of the Fall schedule
  - Broadcaster guarantee of a number of viewers
  - To fulfill the guarantee, broadcasters run additional advertisements (makeups)
  - Broadcaster do not charge extra when there are more viewers than guaranteed
  - How to fulfill the guarantee without overshooting it?
    » Issue: Number of viewers is not known in advance

- Scatter (spot) market
  - Scatter market for Fall happens in Fall
  - Scatter sales are cheaper and have no guarantees for the number of viewers

- Constraints:
  - No competing advertisements in the same pod;
  - Not more than a certain number of advertisements for the same product in a single day.

- Broadcasters have advertisement rate sheets (catalog prices) but give big discounts (pocket price is much less).
Pricing with Constrained Supply

Summary

- Pricing with a Supply Constraint
- Opportunity Cost
  - With and without market segmentation
- Variable Pricing
- Variable Prices in Practice
  - Sport events; Airlines; Electricity Markets; Advertising

Based on Phillips (2005) Chapter 5
Market Segmentation with a Capacity Constraint
Graphical Depiction with Multiple Prices
Solution Method

At the optimal prices, the capacity constraint is tangent to the ellipse.

If we treat student price as y-variable and the general price as x-variable, the slope of the capacity constraint is

\[
\frac{dp_s}{dp_g} = -\frac{3}{1.25}
\]

This slope can be represented by either one of the following vectors:

\([-1.25, 3]\) or \([1.25, -3]\)
Level Curves

3\((p_g - 20)^2 + 1.25(p_s - 8)^2 = 17\)

3\((p_g - 20)^2 + 1.25(p_s - 8)^2 = 4.25\)

3\((p_g - 20)^2 + 1.25(p_s - 8)^2 = 0\)

Level curve is about to touch the constraint to yield feasible prices.

When they touch, the constraint is tangent to the curve or perpendicular to the normal vector.
Normal Vectors to Level Curves

- Gradient vector of a curve is normal (perpendicular) to the curve. Given a curve \( \Pi(p_1, ..., p_n) = \text{constant} \), the associated gradient is
  \[
  \nabla \Pi = \left[ \frac{\partial \Pi}{\partial p_1}, ..., \frac{\partial \Pi}{\partial p_n} \right]
  \]

- Given the curve for the ellipse \( \Pi(p_g, p_s) = 3(p_g - 20)^2 + 1.25(p_s - 8)^2 = 17 \)
  the associated gradient is
  \[
  \nabla \Pi = [2(2)(p_g - 20), 1.25(2)(p_s - 8)]
  \]

- For example, at \((p_g = 22, p_s = 10)\) the gradient is
  \[
  \nabla \Pi = [3(2)(22 - 20), 1.25(2)(10 - 8)] = 4[3, 1.25] \propto [3, 1.25].
  \]
At \((22, 38, 8) = (20 + \sqrt{17}/3, 8)\) the gradient is
\[
\nabla \Pi = [3(2)(22.38 - 20), 1.25(2)(8 - 8)] \propto [1, 0].
\]
At \((22, 6)\) the gradient is
\[
\nabla \Pi = [3(2)(22 - 20), 1.25(2)(6 - 8)] = 4[3, -1.25] \propto [3, -1.25].
\]
Market Segmentation with a Capacity Constraint
Obtaining the Optimality Condition with Multiple Prices

- Two vectors are parallel if they are a positive multiple of each other.
- Two vectors are perpendicular if their scalar product is zero

\[
\begin{align*}
[3, 6][0; 3] &= 18 \neq 0 \\
[3, 6][-1; 0] &= -3 \neq 0 \\
[3, 6][-2; 1] &= 0 \\
[3, 6][4; -2] &= 0 \\
[3, 1.25][1.25; -3] &= 0
\end{align*}
\]

Only the last three pairs of vectors are perpendicular above.

- We want the gradient of the objective to be perpendicular to the vector representing the slope of the capacity constraint:

\[
\begin{align*}
[3(2)(p_g - 20), 1.25(2)(p_s - 8)][1.25; -3] &= 3(2)(1.25) \left( p_g - 20 - (p_s - 8) \right) = 0 \\
\text{So } p_g &= p_s + 12.
\end{align*}
\]
We intersect two lines $3p_g + 1.25p_s = 80$ and $p_g = p_s + 12$ to find optimal prices $p_g = 22.35$ and $p_s = 10.35$. Then $\Pi(p_g = 22.35, p_s = 10.35) = 4.25(2.35)^2 = 23.47$. 

$3(p_g - 20)^2 + 1.25(p_s - 8)^2 = 23.47$