Unions, Intersections, Independence, Conditioning, Bayes' Formula

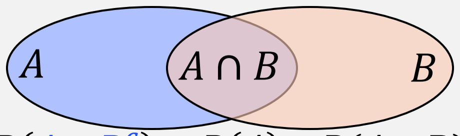
Outline

- **Unions, Intersections**
- **Independence**
- **Conditioning**
- **Bayes' Formula**

Updated in Fall 2019

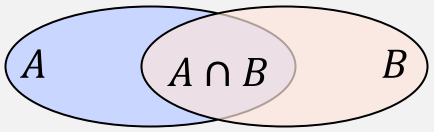
Unions, Intersections, Complements

- A probability model is a triplet $(\Omega, \mathfrak{I}, \mathrm{P})$
  - Ω: sample space
  - $\mathfrak{I}$: a σ-field (an appropriate collection of subsets of Ω)
    - » Includes Ω, closed under complement $(\cdot)^c$ and countable union operations $\cup_1^\infty$
  - P: a probability measure that maps sets in $\mathfrak{I}$ to real numbers in [0,1]
- Two events A and B can be thought as two sets $A, B \in \Omega$
  - The new event that happens "when either A or B happens" corresponds to union $A \cup B$
  - The new event that happens "when both A and B happens" corresponds to intersection $A \cap B$
    - » Note that $A \cap \mathrm{B} = (A^c \cup B^c)^c \in \mathfrak{I}$ if $A, B \in \mathfrak{I}$.
- The new event that happens "when A does but B does not happen" corresponds to $A \cap B^c$
  - $A \cap B^c = A \setminus (A \cap B)$
  - Set as a union $A = (A \cap B^c) \cup (A \cap B)$
  - Countable additivity $\mathrm{P}(A) = \mathrm{P}(A \cap B^c) + \mathrm{P}(A \cap B)$
  - $\mathrm{P}(A \cap B^c) = \mathrm{P}(A) - \mathrm{P}(A \cap B)$



$\mathrm{P}(A \cap B^c) = \mathrm{P}(A) - \mathrm{P}(A \cap \mathrm{B})$

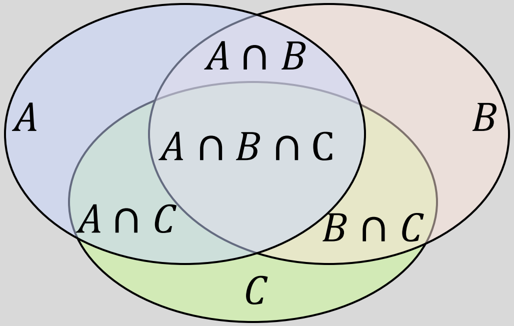
- Relating union to intersection
  - Set as a union $A \cup \mathrm{B} = (A \cap \mathrm{B}^c) \cup (A \cap B) \cup (B \cap A^c)$
  - Countable additivity $\mathrm{P}(A \cup \mathrm{B}) = \mathrm{P}(A \cap \mathrm{B}^c) + \mathrm{P}(A \cap B) + \mathrm{P}(B \cap A^c)$
  - $= \mathrm{P}(A) - \mathrm{P}(A \cap B) + \mathrm{P}(A \cap B) + \mathrm{P}(B) - \mathrm{P}(B \cap A)$
  - $= \mathrm{P}(A) - \mathrm{P}(A \cap B) + \mathrm{P}(B)$



$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap \mathrm{B})$

Unions and Intersections

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$



$$P(A \cup B \cup C \cup D) = P(A) + P(B) + P(C) + P(D) - P(A \cap B) - P(A \cap C) - P(A \cap D) - P(B \cap C) - P(B \cap D) - P(C \cap D) + P(A \cap B \cap C) + P(A \cap B \cap D) + P(A \cap C \cap D) + P(B \cap C \cap D) - P(A \cap B \cap C \cap D)$$

- Inclusion-exclusion identity for 2, 3, 4, …, *n* sets
  - Proof for *n* sets is by induction – a matter of getting indices right, see the notes

- Ex: $P(A^c) = 1 - P(A)$, from
  - $1 = P(\Omega) = P(A) + P(A^c) - P(A \cap A^c) = P(A) + P(A^c)$
- Ex: $A \subseteq B$ implies $P(A) \le P(B)$, from
  - $P(B) = P\big(A \cup (B \cap A^c)\big) = P(A) + P(B \cap A^c) \ge P(A)$

Limit of a Measure vs Measure of the Limit

- Let $S = \{(x, y): 0 \le x, y \le 1\}$ and $T = \{(x, y): 0 \le x, y \le 2, x + y \le \sqrt{2}\}$

- $A_n = 1_{n\ is\ odd} S + 1_{n\ is\ even} T$, i.e., an alternating sequence of ***S***quares and ***T***riangles
  - Limit of a Measure: $\lim_{n\to\infty} Area(A_n) = Area(S) = Area(T) = 1$
  - Measure of a Limit: $Area\left(\lim_{n\to\infty} A_n\right)$ does not exist

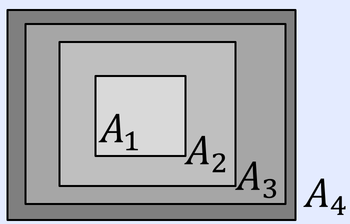
- Measure and Limit operations are not interchangeable!

Continuity of Probability Measure

- Ex: Exchangeability of limit & probability ≡ Probability measure P is continuous, only for specific events
  - i) Inner (from left): For increasing sequence of events A_n, $\mathrm{P}\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} \mathrm{P}(A_n)$.
  - ii) Outer (from right): For decreasing sequence of events A_n, $\mathrm{P}\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} \mathrm{P}(A_n)$.

Limits of Numbers → Understanding the Limits of Sets

Increasing sequence of sets

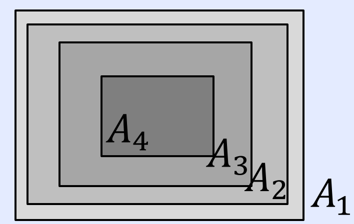


$A_4 = A_1 \cup A_2 \cup A_3 \cup A_4$

$\mathrm{P}\left(\lim_{n\to 4} A_n\right) = \mathrm{P}(\cup_{i=1}^{4} A_i)$

$\mathrm{P}\left(\lim_{n\to \infty} A_n\right) = \mathrm{P}(\cup_{i=1}^{\infty} A_i)$

Decreasing sequence of sets



$A_4 = A_1 \cap A_2 \cap A_3 \cap A_4$

$\mathrm{P}\left(\lim_{n\to 4} A_n\right) = \mathrm{P}(\cap_{i=1}^{4} A_i)$

$\mathrm{P}\left(\lim_{n\to \infty} A_n\right) = \mathrm{P}(\cap_{i=1}^{\infty} A_i)$

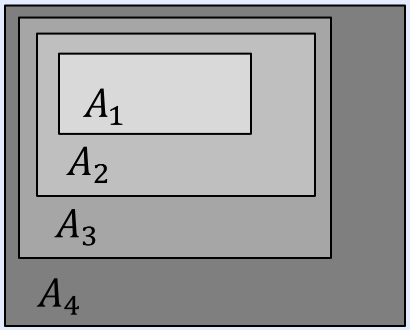
Union of finite number of closed sets is closed.

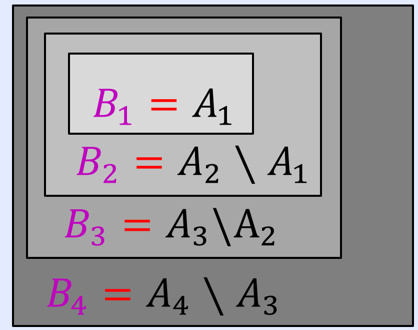
From topology: A closed set includes its limiting points

Intersection of any number of closed sets is closed.

Inner Continuity of Probability Measure

- For increasing sequence of events A_n, $\mathrm{P}\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} \mathrm{P}(A_n)$.





$A_4 = A_1 \cup A_2 \cup A_3 \cup A_4$

$$\mathrm{P}\left(\lim_{n\to 4} A_n\right) = \mathrm{P}(\cup_{i=1}^{4} A_i)$$

$$\mathrm{P}\left(\lim_{n\to\infty} A_n\right) = \mathrm{P}(\cup_{i=1}^{\infty} A_i)$$

$A_4 = B_1 \cup B_2 \cup B_3 \cup B_4$ and B_i's disjoint

$$\mathrm{P}(\cup_{i=1}^{\infty} A_i) = \mathrm{P}(\cup_{i=1}^{\infty} B_i)$$

$$\mathrm{P}\left(\lim_{n\to\infty} A_n\right) = \mathrm{P}(\cup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} \mathrm{P}(B_i)$$

$$\mathrm{P}\left(\lim_{n\to\infty} A_n\right) = \sum_{i=1}^{\infty} \mathrm{P}(B_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \mathrm{P}(B_i) = \lim_{n\to\infty} \mathrm{P}(\cup_{i=1}^{n} B_i) = \lim_{n\to\infty} \mathrm{P}(\cup_{i=1}^{n} A_i) = \lim_{n\to\infty} \mathrm{P}(A_n)$$

Insight: New sets B_i's are disjoint and allow us to use countable additivity of the probability measure. This takes the limit outside of the probability measure.

- For decreasing sequence of events A_n, $\mathrm{P}\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} \mathrm{P}(A_n)$. Use the previous result

$$\mathrm{P}\left(\lim_{n\to\infty} A_n\right) = \mathrm{P}(\cap_{i=1}^{\infty} A_i) = 1 - \mathrm{P}(\cup_{i=1}^{\infty} A_i^c) = 1 - \mathrm{P}\left(\lim_{n\to\infty} A_n^c\right) = 1 - \lim_{n\to\infty} \mathrm{P}(A_n^c) = 1 - \left(1 - \lim_{n\to\infty} \mathrm{P}(A_n)\right)$$

$$= \lim_{n\to\infty} \mathrm{P}(A_n).$$

Independence

- Events A and B are independent if $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$.

- Ex: If A and B are independent, so are their complements: $\mathrm{P}(A^c \cap B^c) = \mathrm{P}(A^c)\mathrm{P}(B^c)$ if $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$

$$\begin{aligned}\mathrm{P}(A^c \cap B^c) &= \mathrm{P}\big((A \cup B)^c\big) = 1 - \mathrm{P}(A \cup B) = 1 - \mathrm{P}(A) - \mathrm{P}(B) + \mathrm{P}(A \cap B)\\ &= 1 - \mathrm{P}(A) - \mathrm{P}(B) - \mathrm{P}(A)\mathrm{P}(B) = (1 - \mathrm{P}(A))(1 - \mathrm{P}(B))\\ &= \mathrm{P}(A^c)\mathrm{P}(B^c)\end{aligned}$$

- In repeated experiments, each experiment is often independent. Such repeated experiments include dice rolls, coin tosses, picking a number from $\{0,1,2,\dots,9\}$ with repetition.

- Independence is generally assumed
  - Sometimes with statistical justification. See independence hypothesis tests.
  - Other times with a verbal, intuitive, managerial argument
  - In a few times, for convenience without any justification
    - » Independence really holds
    - » Independence fails
      - Probability of each one of two engines of a plane to simultaneously hit a bird
      - Probability of a hurricane & the electricity grid failure at the same location and hour
      - Probability of tuberculosis cases in New York City
      - Probability of two bank bankruptcies: Lehman Brother and Bear Stearns
        - Bear Stearns → JP Morgan Chase

Independence: Urn Example

- An urn contains 3 White and 2 Black balls. 3 balls are drawn without replacement one after another.
- Let A_i be the event that ball i is White for i=1,2,3. Are A_1, A_2, A_3 independent?

This experiment creates sequences of ball colors of the form WWW, WWB, etc. Since 3 balls are drawn and each ball can potentially take 2 colors, the sample space has $8=2^3$ elements. Note BBB is considered as an outcome with no probability. Table below shows all the outcomes and their probabilities.

ω with ≥ 2 W	P(ω)	ω with ≤ 1 W	P(ω)
WWW	(3/5)(2/4)(1/3)=0.1	WBB	(3/5)(2/4)(1/3)=0.1
WWB	(3/5)(2/4)(2/3)=0.2	BWB	(2/5)(3/4)(1/3)=0.1
WBW	(3/5)(2/4)(2/3)=0.2	BBW	(2/5)(1/4)(3/3)=0.1
BWW	(2/5)(3/4)(2/3)=0.2	BBB	(2/5)(1/4)(0/3)=0.0

$$\mathrm{P}(A_1) = \mathrm{P}(WWW \text{ or } WWB \text{ or } WBW \text{ or } WBB) = 0.1 + 0.2 + 0.2 + 0.1 = 0.6.$$
$$\mathrm{P}(A_2) = \mathrm{P}(WWW \text{ or } WWB \text{ or } BWW \text{ or } BWB) = 0.6.$$
$$\mathrm{P}(A_3) = P(WWW \text{ or } WBW \text{ or } BWW \text{ or } BBW) = 0.6.$$

$$\mathrm{P}(A_1 \cap A_2) = \mathrm{P}(WWW \text{ or } WWB) = 0.1 + 0.2 = 0.3.$$
$$\mathrm{P}(A_1 \cap A_3) = P(WWW \text{ or } WBW) = 0.3.$$
$$\mathrm{P}(A_2 \cap A_3) = P(WWW \text{ or } BWW) = 0.3.$$

$\mathrm{P}(A_1 \cap A_2) = 0.3 \neq 0.36 = P(A_1)P(A_2)$. Events are not independent.

$\mathrm{P}(A_2 \cap A_3) = 0.3 \neq 0.36 = P(A_2)P(A_3)$. Events are not independent.

$\mathrm{P}(A_1 \cap A_3) = 0.3 \neq 0.36 = P(A_1)P(A_3)$. Events are not independent.

Replacements can induce independence

- An urn has 3 White and 2 Black balls. 3 balls are drawn ~~without~~ with replacement one after another.
- Let B_i be the event that ball i is White for i=1,2,3. Are B_1, B_2, B_3 independent?

This experiment creates sequences of ball colors WWW, WWB, etc. The sample space has $8=2^3$ elements. Table below shows all the outcomes and their probabilities.

ω with ≥ 2 W	P(ω)	ω with ≤ 1 W	P(ω)
WWW	(3/5)(3/5)(3/5)=27/125	WBB	(3/5)(2/5)(2/5)=12/125
WWB	(3/5)(3/5)(2/5)=18/125	BWB	(2/5)(3/5)(2/5)=12/125
WBW	(3/5)(2/5)(3/5)=18/125	BBW	(2/5)(2/5)(3/5)=12/125
BWW	(2/5)(3/5)(3/5)=18/125	BBB	(2/5)(2/5)(2/5)=8/125

$$\mathrm{P}(B_1) = \mathrm{P}(WWW \text{ or } WWB \text{ or } WBW \text{ or } WBB) = \frac{27+18+18+12}{125} = \frac{3}{5}.$$
$$\mathrm{P}(B_2) = \mathrm{P}(WWW \text{ or } WWB \text{ or } BWW \text{ or } BWB) = 3/5.$$
$$\mathrm{P}(B_3) = P(WWW \text{ or } WBW \text{ or } BWW \text{ or } BBW) = 3/5$$

$$\mathrm{P}(B_1 \cap B_2) = \mathrm{P}(WWW \text{ or } WWB) = \frac{27+18}{125} = \frac{9}{25}.$$
$$\mathrm{P}(B_1 \cap B_3) = P(WWW \text{ or } WBW) = 9/25.$$
$$\mathrm{P}(B_2 \cap B_3) = P(WWW \text{ or } BWW) = 9/25.$$
$$\mathrm{P}(B_1 \cap B_2 \cap B_3) = \mathrm{P}(WWW) = \frac{27}{125}.$$

$\mathrm{P}(B_1 \cap B_2) = \mathrm{P}(B_2 \cap B_3) = \mathrm{P}(B_1 \cap B_3) = 9/25 = P(B_1)P(B_2) = P(B_2)P(B_3) = P(B_1)P(B_3)$.
Events are pairwise independent even when they are not disjoint.

$\mathrm{P}(B_1 \cap B_2 \cap B_3) = 27/125 = P(B_1)P(B_2)P(B_3)$. Three events are independent.

Independence of 3 or more events

- For independence of 3 events A_1, A_2, A_3, we need to check

  Pairwise independence:

  $\mathrm{P}(A_1 \cap A_2) = \mathrm{P}(A_1)\mathrm{P}(A_2), \mathrm{P}(A_1 \cap A_3) = \mathrm{P}(A_1)\mathrm{P}(A_3), \mathrm{P}(A_2 \cap A_3) = \mathrm{P}(A_2)\mathrm{P}(A_3).$

  As well as triplet-wise independence:

  $\mathrm{P}(A_1 \cap A_2 \cap A_3) = \mathrm{P}(A_1)\mathrm{P}(A_2)\mathrm{P}(A_3).$

- Pairwise independence of 3 or more events does not imply independence of all of them.
- Ex: An urn contains 4 balls numbered as 1,2,3,4, and a ball is drawn randomly. Let A_2 be the event that the drawn ball is either 1 or 2, so $A_2 = \{1,2\}$. Similarly $A_3 = \{1,3\}$ and $A_4 = \{1,4\}$.
  - $\mathrm{P}(A_2) = \frac{2}{4} = \mathrm{P}(A_3) = \mathrm{P}(A_4)$
  - $\mathrm{P}(A_2 \cap A_3)$ =1/4= $\mathrm{P}(A_2)\mathrm{P}(A_3)$ and $\mathrm{P}(A_2 \cap A_4)$ =1/4= $\mathrm{P}(A_2)\mathrm{P}(A_4)$ and $\mathrm{P}(A_3 \cap A_4)$ =1/4= $\mathrm{P}(A_3)\mathrm{P}(A_4)$.
  - However, $P(A_2 \cap A_3 \cap A_4)$ =1/4≠1/8= $P(A_2)P(A_3)P(A_4)$.

- Ex: $|\mathrm{P}(A \cap B) - \mathrm{P}(A)\mathrm{P}(B)| \le 1/4$ for any two events A and B. This difference is zero for independent events and 1/4 for complementary/equally likely events. The difference is a measure of dependence between two events.

$$\mathrm{P}(A \cap B) - \mathrm{P}(A)\mathrm{P}(B) = \big(\mathrm{P}(A) + \mathrm{P}(A^c)\big)\mathrm{P}(A \cap \mathrm{B}) - \mathrm{P}(A)(\mathrm{P}(A \cap B) + \mathrm{P}(A^c \cap B))$$

$$= \mathrm{P}(A^c)\,\mathrm{P}(A \cap \mathrm{B}) - \mathrm{P}(A)\,\mathrm{P}(A^c \cap B)$$

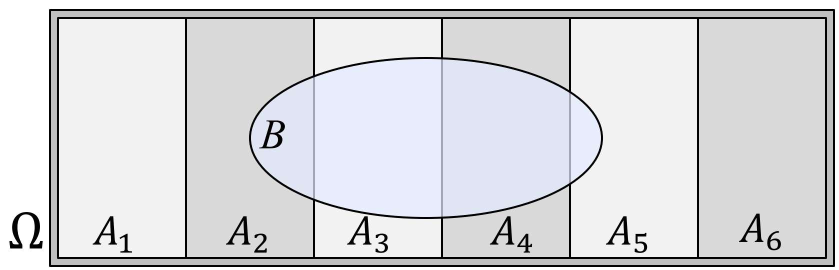
Let $q = P(A^c)$ then $\mathrm{P}(A \cap B) \le \mathrm{P}(A) = 1 - q$.
$\mathrm{P}(A^c)\,\mathrm{P}(A \cap \mathrm{B}) \le q(1-q) \le$1/4 for $0 \le q \le 1$.
Similarly, $\mathrm{P}(A)\,\mathrm{P}(A^c \cap B) \le$1/4.

Desired quantity is the difference of two nonnegative numbers, each is ≤ 1/4.
Absolute value of the difference ≤ 1/4.

Conditioning

- The conditional probability of B given A is $\mathrm{P}(B|A) = \frac{\mathrm{P}(B \cap A)}{\mathrm{P}(A)}$ for $\mathrm{P}(A) > 0$.

- Total probability formula: For a partition A_i of Ω,



$$\mathrm{P}(B) = \sum_{i=1}^{\infty} \mathrm{P}(B \cap A_i) = \sum_{i=1}^{\infty} \mathrm{P}(B|A_i)\mathrm{P}(A_i)$$ by countable additivity.

- Ex: An instructor gives 5 questions for homeworks but grades only 2 of them. A student wants to solve only the questions that will be graded so he attempts to guess 2 questions correctly. He discovers that the instructor always asks a numerical question and grades it. Subsequently, he confidently guesses 1 question out of 5 correctly. What is the probability that he guesses 2 questions correctly given that he guesses 1 correctly?
  - Let A_i be the event that he guesses i questions correctly for i=1,2.
  - We are asking for $\mathrm{P}(A_2|A_1)$. Note that $A_2 \subseteq A_1$, so $\mathrm{P}(A_2|A_1) = \frac{P(A_2)}{P(A_1)}$.
  - Inserting $\mathrm{P}(A_1) = 2/5$ and $\mathrm{P}(A_2) = \frac{1}{C_2^5} = 1/10$, we obtain $\mathrm{P}(A_2|A_1)$=1/4.

Conditioning: Specifying a Restaurant's Greeting Policy Not-Well Specified Setting

- Ex: A restaurant can have 2 waitresses to greet customers, {Young, Experienced}. Sample space for the waiter personnel $\Omega = \{YY, YE, EY, EE\}$, each outcome has equal probability.
  - When you arrive a **Y**oung lady waitress greets you and you wonder about the probability of the other waitress to be also a **Y**oung lady as opposed to an **E**xperienced lady. What is P(the other is Y | yours is Y)?

    - Intuitively, you may answer
      - 1/2 by thinking that the other waitress is either young or experienced with equal probabilities, or
      - 1/3 by computing P(YY)/(P(YY)+P(YE)+P(EY))=(1/4)/(3/4)=1/3.

- P(the other is Y | yours is Y) can be computed by P(the other is Y and yours is Y) / P(yours is Y).
  - We can see that P(the other is Y and yours is Y)=P(YY)=1/4.
  - Can we say that P(your waitress is Y)=1/2? No! This probability is actually
    - 1 if the waitresses are YY,
    - 0 if the waitresses are EE.

- P(your waitress is Y) is to be specified only under YE or EY, in which cases let q denote it.
- P(your waitress is Y)= P(yours Y under YY) +P(yours Y under YE) +P(yours Y under EY) +P(yours Y under EE)
  = 1(1/4) + q(1/4) + q(1/4) + 0(1/4).
- Hence, P(the other waitress is Y | your waitress is Y)=(1/4)/(1/4+q/2)=1/(1+2q).
  - Only with P(your waitress is Y)=1/2 or q=1/2, we have P(the other waitress is Y | your waitress is Y)=1/2.
  - If the restaurant has a policy of q=1, i.e., young waitress greets the customers if available, then
    - P(the other waitress is Y | your waitress is Y)=1/3.
  - At the other extreme of q=0, i.e., experienced waitress greets customers if available,
    - P(the other waitress is Y | your waitress is Y)=1.

- Depending on the restaurant's policy $q=1,\frac{1}{2},0$, the correct answer ranges from $\frac{1}{3}$ to $\frac{1}{2}$ then to 1.

Variation on Sample Space:
Restaurant's Greeting Policy

- Ex: A restaurant can have 2 waitresses to greet customers, {Young, Experienced}. Sample space for the waiter personnel $\Omega = \{YY, YE, EE\}$ with $P(YY) = P(EE) = P(YE)/2 = 1/4$.
  - When you arrive a **Y**oung lady waitress greets you and you wonder about the probability of the other waitress to be also a **Y**oung lady as opposed to an **E**xperienced lady. What is P(the other is Y | yours is Y)?
    - You may answer
      - 1/2 by thinking that the other waitress is either young or experienced with equal probabilities, or
      - 1/3 by computing P(YY)/(P(YY)+P(YE))=(1/4)/(3/4)=1/3.
- P(the other is Y | yours is Y) can be computed by P(the other is Y and yours is Y) / P(yours is Y).
  - We can see that P(the other is Y and yours is Y)=P(YY)=1/4.
  - Can we say that P(your waitress is Y)=1/2? No! This probability is actually
    - 1 if the waitresses are YY,
    - 0 if the waitresses are EE.
- P(your waitress is Y) is to be specified only under YE, in which cases let q denote it.
- P(your waitress is Y)= P(yours Y under YY) +P(yours Y under YE) +P(yours Y under EE)
  = 1(1/4) + q(1/2) + 0(1/4).
- Hence, P(the other waitress is Y | your waitress is Y)=(1/4)/(1/4+q/2)=1/(1+2q).
  - Only with P(your waitress is Y)=1/2 or q=1/2, we have P(the other waitress is Y | your waitress is Y)=1/2.
  - If the restaurant has a policy of q=1, i.e., young waitress greets the customers if available, then
    - P(the other waitress is Y | your waitress is Y)=1/3.
  - At the other extreme of q=0, i.e., experienced waitress greets customers if available,
    - P(the other waitress is Y | your waitress is Y)=1.
- Depending on the restaurant's policy $q = 1, \frac{1}{2}, 0$, the correct answer ranges from $\frac{1}{3}$ to $\frac{1}{2}$ then to 1.

Conditioning

Any First Bidder Advantage in Sequential Bidding?

- Suppose n suppliers bid for m projects of a buyer for $n \geq m$. The suppliers are to choose their turn to bid and are awarded projects depending on the number of suppliers and projects at the time of their bid.
  - If a supplier is awarded a project, the number of available suppliers and projects both decrease by one. Otherwise, only the number of available suppliers decrease by one.
  - Before the kth bidder, suppose there are n_k available suppliers and m_k projects remaining.
    - » Initially, $n_1 = n$ and $m_1 = m$.
  - The buyer accepts the kth bidder randomly, i.e., with probability m_k/n_k.
    - » Is $m_k > n_k$ possible?
- Is the first bidder have a higher chance of getting a project than the second or third?

Let A_k be the event that bidder k gets a project.

- $\mathrm{P}(A_1) = m/n$.
- For $\mathrm{P}(A_2)$, we use conditioning,

$$\mathrm{P}(A_2) = \mathrm{P}(A_1)\mathrm{P}(A_2|A_1) + \mathrm{P}(A_1^c)\mathrm{P}(A_2|A_1^c) = \mathrm{P}(A_1 \cap A_2) + \mathrm{P}(A_1^c \cap A_2) = \frac{m}{n}\frac{m-1}{n-1} + \left(1 - \frac{m}{n}\right)\frac{m}{n-1} = \frac{m}{n}$$

- For $\mathrm{P}(A_3)$,

$$\mathrm{P}(A_3) = \mathrm{P}(A_1 \cap A_2)\mathrm{P}(A_3|A_1 \cap A_2) + \mathrm{P}(A_1 \cap A_2^c)\mathrm{P}(A_3|A_1 \cap A_2^c) + \mathrm{P}(A_1^c \cap A_2)\mathrm{P}(A_3|A_1^c \cap A_2) + \mathrm{P}(A_1^c \cap A_2^c)\mathrm{P}(A_3|A_1^c \cap A_2^c)$$

$$= \frac{m}{n}\frac{m-1}{n-1}\frac{m-2}{n-2} + \frac{m}{n}\frac{n-m}{n-1}\frac{m-1}{n-2} + \frac{n-m}{n}\frac{m}{n-1}\frac{m-1}{n-2} + \frac{n-m}{n}\frac{n-1-m}{n-1}\frac{m}{n-2}$$

$$= \frac{m}{n}$$

- ➢ Fairness: The first bidder does not have an advantage over the others.
- ➢ Does naivety cause fairness?

Bayes' Formula

- For a partition A_j of Ω, $\quad \mathrm{P}(A_j|B) = \frac{\mathrm{P}(A_j \cap B)}{\mathrm{P}(B)} = \frac{\mathrm{P}(A_j)\mathrm{P}(B|A_j)}{\sum_{i=1}^{\infty} \mathrm{P}(A_i)\mathrm{P}(B|A_i)}$.

- A woman has recently started a friendship with a man. She is generally happy with the friendship.
  - But she expects gifts such as flowers. Common wisdom: Gifts are indication of affection.
  - Getting no gifts, she wants to assess whether the man is truly in love or possibly searching for another.
  - Events: N no gifts, L truly in love, S possibly searching.
  - She wants to know $\mathrm{P}(L|N)$.
  - Available data:
    - » $\mathrm{P}(L)=30\%$ judging from the friendship.
    - » $\mathrm{P}(N|L)=40\%$ and $\mathrm{P}(N|S)=60\%$ according to magazine surveys, best available data.

$$\mathrm{P}(L|N) = \frac{\mathrm{P}(N|L)\mathrm{P}(L)}{\mathrm{P}(N|L)\mathrm{P}(L) + \mathrm{P}(N|S)\mathrm{P}(S)} = \frac{0.4 * 0.3}{0.4 * 0.3 + 0.6 * 0.7} = \frac{12}{54} = 0.2222 < 0.3$$

  - Available data change: A searching man gifts more than others to hide his intentions
    - » $\mathrm{P}(N|L)=60\%$ and $\mathrm{P}(N|S)=40\%$ according to recent magazine surveys.

$$\mathrm{P}(L|N) = \frac{\mathrm{P}(N|L)\mathrm{P}(L)}{\mathrm{P}(N|L)\mathrm{P}(L) + \mathrm{P}(N|S)\mathrm{P}(S)} = \frac{0.6 * 0.3}{0.6 * 0.3 + 0.4 * 0.7} = \frac{18}{46} = 0.3913 > 0.3$$

Loves or not

Guilty Verdict After a Confession

- A person is accused of an offence.
  - Generally, a confession of the offence is treated as a sign that he is guilty.
  - Events: I the person is innocent, G he is guilty, V he verbally confesses.

Poor use of English: Double negatives make a positive & possibly a confession

$$\frac{P(G|V)}{P(I|V)} = \frac{P(G|V) = \frac{P(V|G)P(G)}{P(V|G)P(G)+P(V|I)P(I)}}{P(I|V) = \frac{P(V|I)P(I)}{P(V|G)P(G)+P(V|I)P(I)}} = \frac{P(V|G)}{P(V|I)}\frac{P(G)}{P(I)}$$

Prior $\frac{P(G)}{P(I)}$ —Confession→ $\frac{P(V|G)}{P(V|I)}$ $\frac{P(G)}{P(I)}$ Posterior

>1 or <1?

Can a confession decrease the chances of a guilty verdict?

What does guilty "*beyond a reasonable doubt*" mean?

Beyond a reasonable doubt holds when **no other explanation** can be derived from the facts except that the accused committed the crime, thereby overcoming the presumption that a person is innocent until proven guilty.
Source: http://legal-dictionary.thefreedictionary.com

Doubt ⇒ Uncertainty ⇒ P(other explanation) > 0

No other explanation ⇒ P(other explantion) = 0
Almost certainly no other explanation

Inconsistency in interpretation of laws?

- A person, guilty with probability 70%=P(G)=1-P(I), confesses, what is the likelihood of being guilty?
  - If innocent, confesses with 40%. If guilty, confesses with 60%.
    - P(V|I)=0.4, P(V|G)=0.6, then P(G|V)/P(I|V) = (0.6/0.4)(0.7/0.3) = 7/2.
    - Since P(G|V)+P(I|V)=1, P(G|V)=7/9=0.77 > 0.7=P(G). Guilty probability ↑ with a confession.
  - Suppose innocent confesses with 60+ϵ%>60% of a guilty confessing. Everything else is the same.
    - P(V|I)=0.6+ϵ, then P(G|V)/P(I|V) = (0.6/(0.6+ϵ))(0.7/0.3) < 7/3.
    - Hence, P(G|V)<0.7=P(G). Guilty probability ↓ with a confession.

Answering a Question Correctly by Chance

- Ex: In a multiple choice exam, the instructor provides 4 alternatives: a), b), c) and d). A student knows the true answer to a particular question with probability 0.6. If the student does not know the answer, he randomly picks one of the alternatives from a) to d).
- If this question is answered correctly by the student, what is the probability that it is answered correctly by chance?

- Let A_T be the event that the student knows the True answer.
- Let A_C be the event that the student answers the question Correctly.
- We want $\mathrm{P}(A_T^c|A_C)$.

- We know $\mathrm{P}(A_T) = 1 - \mathrm{P}(A_T^c) = 0.6$ and

$$\mathrm{P}(A_T^c|A_C) = \frac{\mathrm{P}\left(A_C\middle|A_T^c\right)\mathrm{P}(A_T^c)}{\mathrm{P}\left(A_C\middle|A_T^c\right)\mathrm{P}(A_T^c)+\mathrm{P}(A_C|A_T)\mathrm{P}(A_T)} = \frac{(0.25)(0.4)}{(0.25)(0.4)+1(0.6)} = \frac{1}{7} = 0.142$$

- The student guesses the correct answer without knowing the correct answer with 14.2% chance.

Summary

- Unions, Intersections
- Independence
- Conditioning
- Bayes' Formula