ACTS 4301

FORMULA SUMMARY

Lesson 1: Probability Review

1. \( \text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 \)

2. \( \text{Var}(aX + bY) = a^2 \text{Var}(X) + 2ab \text{Cov}(X,Y) + b^2 \text{Var}(Y) \)

3. \( \text{Var}(\bar{X}) = \frac{\text{Var}(X)}{n} \)

4. \( \mathbb{E}_X[X] = \mathbb{E}_Y[\mathbb{E}_X[X|Y]] \) Double expectation

5. \( \text{Var}_X[X] = \mathbb{E}_Y[\text{Var}_X[X|Y]] + \text{Var}_Y(\mathbb{E}_X[X|Y]) \) Conditional variance

Distribution, Density and Moments for Common Random Variables

<table>
<thead>
<tr>
<th>Continuous Distributions</th>
<th>( f(x) )</th>
<th>( F(x) )</th>
<th>( \mathbb{E}[X] )</th>
<th>( \text{Var}(X) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exponential</td>
<td>( e^{-\frac{x}{\theta}} / \theta )</td>
<td>( 1 - e^{-\frac{x}{\theta}} )</td>
<td>( \theta )</td>
<td>( \theta^2 )</td>
</tr>
<tr>
<td>Uniform</td>
<td>( \frac{1}{b-a} ), ( x \in [a,b] )</td>
<td>( \frac{x}{\theta} ), ( x \in [a,b] )</td>
<td>( \frac{a+b}{2} )</td>
<td>( \frac{(b-a)^2}{12} )</td>
</tr>
<tr>
<td>Gamma</td>
<td>( x^{\alpha-1}e^{-\frac{x}{\theta}} / \Gamma(\alpha)\theta^\alpha )</td>
<td>( \int_0^x f(t)dt )</td>
<td>( \alpha\theta )</td>
<td>( \alpha\theta^2 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Discrete Distributions</th>
<th>( f(x) )</th>
<th>( \mathbb{E}[X] )</th>
<th>( \text{Var}(X) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Poisson</td>
<td>( e^{-\lambda} \lambda^x / x! )</td>
<td>( \lambda )</td>
<td>( \lambda )</td>
</tr>
<tr>
<td>Binomial</td>
<td>( \binom{m}{x}p^x(1-p)^{m-x} )</td>
<td>( mp )</td>
<td>( mp(1-p) )</td>
</tr>
</tbody>
</table>
Lesson 2: Survival Distributions: Probability Functions, Life Tables

1. Actuarial Probability Functions
   - $t p_x$ - probability that $(x)$ survives $t$ years
   - $t q_x$ - probability that $(x)$ dies within $t$ years
   - $t | u q_x$ - probability that $(x)$ survives $t$ years and then dies in the next $u$ years.

   $$
t + u p_x = t p_x \cdot p_{x+t}
   \quad t | u q_x = t p_x \cdot q_{x+t} = t p_x \cdot t + u p_x = t + u q_x - t q_x
   $$

2. Life Table Functions

   $$d_x = l_x - l_{x+1}
   \quad u d_x = l_x - l_{x+n}
   \quad q_x = 1 - p_x = \frac{d_x}{l_x}
   \quad t p_x = \frac{l_{x+t}}{l_x}
   \quad t | u q_x = \frac{l_{x+t} - l_{x+t+u}}{l_x}
   $$

3. Mathematical Probability Functions

   $$t p_x = S_{T(x)}(t)
   \quad t q_x = F_{T(x)}(t)
   \quad t | u q_x = Pr(t < T(x) \leq t + u) = F_{T(x)}(t + u) - F_{T(x)}(t) = Pr(x + t < X \leq x + t + u | X > x) = \frac{F_X(x + t + u) - F_X(x + t)}{s_X(x)}
   \quad S_{x+u}(t) = \frac{S_x(t + u)}{S_x(u)}
   \quad S_x(t) = \frac{S_0(x + t)}{S_0(x)}
   \quad F_x(t) = \frac{F_0(x + t) - F_0(x)}{1 - F_0(x)}
   $$
Lesson 3: Survival Distributions: Force of Mortality

\[ \mu_x = \frac{f_0(x)}{S_0(x)} = -\frac{d}{dx} \ln S_0(x) \]

\[ \mu_{x+t} = \frac{f_x(t)}{S_x(t)} = -\frac{d}{dx} \ln S_x(t) \]

\[ S_x(t) = t p_x = \exp \left( -\int_0^t \mu_{x+s} \, ds \right) = \exp \left( -\int_0^t \mu_x(s) \, ds \right) = \exp \left( -\int_x^{x+t} \mu_s \, ds \right) \]

\[ \mu_x(t) = -\frac{d_t p_x/dt}{t p_x} = -\frac{d \ln t p_x}{dt} \]

\[ f_T(x) = f_x(t) = t p_x \cdot \mu_x(t) \]

\[ t \cdot u q_x = \int_t^{t+u} s p_x \cdot \mu_x(s) \, ds \]

\[ t q_x = \int_0^t s p_x \cdot \mu_x(s) \, ds \]

If \( \mu_x^*(t) = \mu_x(t) + k \) for all \( t \), then \( s p_x^* = s p_x e^{-kt} \)

If \( \mu_x(t) = \mu_x(t) + \tilde{\mu}_x(t) \) for all \( t \), then \( s p_x = s \tilde{p}_x s \tilde{p}_x \)

If \( \mu_x^*(t) = k \mu_x(t) \) for all \( t \), then \( s p_x^* = (s p_x)^k \)
Lesson 4: Survival Distributions: Mortality Laws

Exponential distribution or Constant Force of Mortality

$$\mu_x(t) = \mu$$  
$$tq_x = e^{-\mu t}$$

Uniform distribution or De Moivre’s law

$$\mu_x(t) = \frac{1}{\omega - x - t}, \; 0 \leq t \leq \omega - x$$
$$tp_x = \frac{\omega - x - t}{\omega - x}, \; 0 \leq t \leq \omega - x$$
$$tq_x = \frac{t}{\omega - x}, \; 0 \leq t \leq \omega - x$$
$$t u q_x = \frac{u}{\omega - x}, \; 0 \leq t + u \leq \omega - x$$

Beta distribution or Generalized De Moivre’s law

$$\mu_x(t) = \frac{\alpha}{\omega - x - t}$$
$$tp_x = \left(\frac{\omega - x - t}{\omega - x}\right)^\alpha, \; 0 \leq t \leq \omega - x$$

Gompertz’s law:

$$\mu_x = B c^x, \; c > 1$$
$$tp_x = \exp\left(-\frac{B c^x (c^t - 1)}{\ln c}\right)$$

Makeham’s law:

$$\mu_x = A + B c^x, \; c > 1$$
$$tp_x = \exp\left(-A t - \frac{B c^x (c^t - 1)}{\ln c}\right)$$

Weibull Distribution

$$\mu_x = k x^n$$
$$S_0(x) = e^{-k x^{n+1}/(n+1)}$$
Lesson 5. Survival Distributions: Moments

Complete Future Lifetime

\[
\hat{e}_x = \int_0^\infty t_1 p_x \mu_x + t \, dt \\
\hat{e}_x = \int_0^\infty t p_x \, dt \\
\hat{e}_x = \frac{1}{\mu} \text{ for constant force of mortality} \\
\hat{e}_x = \frac{\omega - x}{2} \text{ for uniform (de Moivre) distribution} \\
\hat{e}_x = \frac{\omega - x}{\alpha + 1} \text{ for generalized uniform (de Moivre) distribution} \\
\mathbb{E} \left[ (T(x))^2 \right] = 2 \int_0^\infty t_1 p_x \, dt \]

\[
\text{Var} (T(x)) = \frac{1}{\mu^2} \text{ for constant force of mortality} \\
\text{Var} (T(x)) = \frac{(\omega - x)^2}{12} \text{ for uniform (de Moivre) distribution} \\
\text{Var} (T(x)) = \frac{(\omega - x)^2}{(\alpha + 1)^2(\alpha + 2)} \text{ for generalized uniform (de Moivre) distribution} \\
\]

\(n\)-year Temporary Complete Future Lifetime

\[
\hat{e}_{x:n} = \int_0^n t_1 p_x \mu_x + t \, dt + n_n p_x \\
\hat{e}_{x:n} = \int_0^n t p_x \, dt \\
\hat{e}_{x:n} = n \, p_x (n) + n \, q_x (n/2) \text{ for uniform (de Moivre) distribution} \\
\hat{e}_{x:n} = p_x + 0.5 q_x \text{ for uniform (de Moivre) distribution} \\
\hat{e}_{x:n} = \frac{1 - e^{-\mu n}}{\mu} \text{ for constant force of mortality} \\
\mathbb{E} \left[ (T(x) \wedge n)^2 \right] = 2 \int_0^n t_1 p_x \, dt \\
\]

Curtate Future Lifetime

\[
e_x = \sum_{k=1}^{\infty} k_k q_x \\
e_x = \sum_{k=1}^{\infty} k p_x \\
e_x = \hat{e}_x - 0.5 \text{ for uniform (de Moivre) distribution} \\
\mathbb{E} \left[ (K(x))^2 \right] = \sum_{k=1}^{\infty} (2k - 1) k p_x \\
\text{Var} (K(x)) = \text{Var} (T(x)) - \frac{1}{12} \text{ for uniform (de Moivre) distribution} \]
$n$-year Temporary Curtate Future Lifetime

\[ e_{\bar{x}:\eta} = \sum_{k=1}^{n-1} k|q_x + n_p x \]

\[ e_{\bar{x}:\eta} = \sum_{k=1}^{n} k p x \]

\[ e_{\bar{x}:\eta} = \bar{e}_{\bar{x}:\eta} - 0.5n q_x \] for uniform (de Moivre) distribution

\[ E \left[ (K(x) \land n)^2 \right] = \sum_{k=1}^{n} (2k - 1) k p x \]
Lesson 6: Survival Distributions: Percentiles, Recursions, and Life Table Concepts

Recursive formulas

\[
\begin{align*}
\hat{e}_x &= \hat{e}_{x:n} + np_x \hat{e}_{x+n} \\
\hat{e}_{x:m} &= \hat{e}_{x:n} + m p_x \hat{e}_{x+m:n-m} \quad m < n \\
e_x &= e_{x:n} + np_x e_{x+n} = e_{x:n-1} + np_x (1 + e_{x+n}) \\
e_x &= p_x + p_x e_{x+1} = p_x (1 + e_{x+1}) \\
e_{x:m} &= e_{x:n} + m p_x e_{x+m:n-m} = e_{x:m-1} + m p_x (1 + e_{x+m:n-m}) \quad m < n \\
e_{x:m} &= p_x + p_x e_{x+1:n-1} = p_x \left(1 + e_{x+1:n-1}\right)
\end{align*}
\]

Life Table Concepts

\[
\begin{align*}
T_x &= \int_0^\infty l_{x+t} dt, \text{ total future lifetime of a cohort of } l_x \text{ individuals} \\
nL_x &= \int_0^n l_{x+t} dt, \text{ total future lifetime of a cohort of } l_x \text{ individuals over the next } n \text{ years} \\
Y_x &= \int_0^\infty T_{x+t} dt \\
\hat{e}_x &= \frac{T_x}{l_x} \\
\hat{e}_{x:n} &= \frac{nL_x}{l_x}
\end{align*}
\]

Central death rate and related concepts

\[
\begin{align*}
nm_x &= \frac{n d_x}{nL_x} \\
m_x &= \frac{q_x}{1 - 0.5q_x} \text{ for uniform (de Moivre) distribution} \\
nm_x &= \mu_x \text{ for constant force of mortality} \\
a(x) &= \frac{L_x - l_{x+1}}{d_x} \text{ the fraction of the year lived by those dying during the year} \\
a(x) &= \frac{1}{2} \text{ for uniform (de Moivre) distribution}
\end{align*}
\]
Lesson 7: Survival Distributions: Fractional Ages

<table>
<thead>
<tr>
<th>Function</th>
<th>Uniform Distribution of Deaths</th>
<th>Constant Force of Mortality</th>
<th>Hyperbolic Assumption</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l_{x+s}$</td>
<td>$l_x - sd_x$</td>
<td>$l_x p_x^s$</td>
<td>$l_{x+1}/(p_x + sq_x)$</td>
</tr>
<tr>
<td>$sq_x$</td>
<td>$sq_x$</td>
<td>$1 - p_x^s$</td>
<td>$sq_x/(1 - (1 - s)q_x)$</td>
</tr>
<tr>
<td>$sp_x$</td>
<td>$1 - sq_x$</td>
<td>$p_x^s$</td>
<td>$p_x/(1 - (1 - s)q_x)$</td>
</tr>
<tr>
<td>$sq_{x+t}$</td>
<td>$sq_x/(1 - tq_x)$, $0 \leq s + t \leq 1$</td>
<td>$1 - p_x^s$</td>
<td>$sq_x/(1 - (1 - s - t)q_x)$</td>
</tr>
<tr>
<td>$\mu_{x+s}$</td>
<td>$q_x/(1 - sq_x)$</td>
<td>$- \ln p_x$</td>
<td>$q_x/(1 - (1 - s)q_x)$</td>
</tr>
<tr>
<td>$sp_x\mu_{x+s}$</td>
<td>$q_x$</td>
<td>$-p_x^s(\ln p_x)$</td>
<td>$p_xq_x/(1 - (1 - s)q_x)^2$</td>
</tr>
<tr>
<td>$m_x$</td>
<td>$q_x/(1 - 0.5q_x)$</td>
<td>$- \ln p_x$</td>
<td>$q_x^2/(p_x \ln p_x)$</td>
</tr>
<tr>
<td>$L_x$</td>
<td>$l_x - 0.5d_x$</td>
<td>$-d_x/\ln p_x$</td>
<td>$-l_{x+1} \ln p_x/q_x$</td>
</tr>
<tr>
<td>$\hat{e}_x$</td>
<td>$e_x + 0.5$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\hat{\hat{e}}_{x:\hat{m}}$</td>
<td>$e_{x:\hat{m}} + 0.5nq_x$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\hat{\hat{e}}_{x:\hat{\eta}}$</td>
<td>$p_x + 0.5q_x$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Lesson 8: Survival Distributions: Select Mortality

When mortality depends on the initial age as well as duration, it is known as select mortality, since the person is selected at that age. Suppose $q_x$ is a non-select or aggregate mortality and $q_{[x]+t}, t = 0, \cdots, n - 1$ is select mortality with selection period $n$. Then for all $t \geq n, q_{[x]+t} = q_{x+t}$. 
Lesson 9: Insurance: Payable at Moment of Death - Moments - Part 1

\[ \bar{A}_x = \int_0^\infty e^{-\delta t} p_x \mu_x(t) \, dt \]

Actuarial notation for standard types of insurance

<table>
<thead>
<tr>
<th>Name</th>
<th>Present value random variable</th>
<th>Symbol for actuarial present value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Whole life insurance</td>
<td>( v^T )</td>
<td>( \bar{A}_x )</td>
</tr>
<tr>
<td>Term life insurance</td>
<td>( v^T ) ( T \leq n ) 0 ( T &gt; n )</td>
<td>( \bar{A}^1_{x:n} )</td>
</tr>
<tr>
<td>Deferred life insurance</td>
<td>0 ( v^T ) ( T \leq n ) 0 ( T &gt; n )</td>
<td>( n</td>
</tr>
<tr>
<td>Deferred term insurance</td>
<td>0 ( v^T ) ( T \leq n ) 0 ( T &gt; n ) ( n &lt; T \leq n + m )</td>
<td>( n</td>
</tr>
<tr>
<td>Pure endowment</td>
<td>0 ( v^n ) ( T \leq n ) 0 ( T &gt; n )</td>
<td>( A^1_{x:m} ) or ( nE_x )</td>
</tr>
<tr>
<td>Endowment insurance</td>
<td>( v^T ) ( T \leq n ) ( v^n ) ( T &gt; n )</td>
<td>( \bar{A}_{x:m} )</td>
</tr>
</tbody>
</table>
Lesson 10: Insurance: Payable at Moment of Death - Moments - Part 2

Actuarial present value under constant force and uniform (de Moivre) mortality for insurances payable at the moment of death

<table>
<thead>
<tr>
<th>Type of insurance</th>
<th>APV under constant force</th>
<th>APV under uniform (de Moivre)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Whole life</td>
<td>(\frac{\mu}{\mu+\delta})</td>
<td>(\frac{\delta_{\omega-x}}{\omega-x})</td>
</tr>
<tr>
<td>(n)-year term</td>
<td>(\frac{\mu}{\mu+\delta}(1-e^{-n(\mu+\delta)}))</td>
<td>(\frac{\delta_{\omega-x}}{\omega-x})</td>
</tr>
<tr>
<td>(n)-year deferred life</td>
<td>(\frac{\mu}{\mu+\delta}e^{-n(\mu+\delta)})</td>
<td>(\frac{e^{-\delta n(\omega-(x+n))}}{\omega-x})</td>
</tr>
<tr>
<td>(n)-year pure endowment</td>
<td>(e^{-n(\mu+\delta)})</td>
<td>(\frac{e^{-\delta n(\omega-(x+n))}}{\omega-x})</td>
</tr>
<tr>
<td>(j)-th moment</td>
<td>Multiply (\delta) by (j) in each of the above formulae</td>
<td></td>
</tr>
</tbody>
</table>

**Gamma Integrands**

\[
\int_0^\infty t^n e^{-\delta t} \, dt = \frac{n!}{\delta^{n+1}}
\]

\[
\int_0^u t e^{-\delta t} \, dt = \frac{1}{\delta^2} \left( 1 - (1+\delta u)e^{-\delta u} \right) = \frac{\bar{a}_u - u\bar{a}_u}{\delta}
\]

**Variance**

If \(Z_3 = Z_1 + Z_2\) and \(Z_1, Z_2\) are mutually exclusive, then

\[
Var(Z_3) = Var(Z_1) + Var(Z_2) - 2E[Z_1]E[Z_2]
\]
Lesson 11: Insurance: Annual and m-thly: Moments

\[ A_x = \sum_{k=0}^{\infty} k|q_x v^{k+1} = \sum_{k=0}^{\infty} kP_x q_{x+k} v^{k+1} \]

Actuarial present value under constant force and uniform (de Moivre) mortality for insurances payable at the end of the year of death

<table>
<thead>
<tr>
<th>Type of insurance</th>
<th>APV under constant force</th>
<th>APV under uniform (de Moivre)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Whole life</td>
<td>( \frac{q}{q+i} )</td>
<td>( \frac{a_{\omega-x}}{\omega-x} )</td>
</tr>
<tr>
<td>( n )-year term</td>
<td>( \frac{q}{q+i} (1 - (vp)^n) )</td>
<td>( \frac{a_{\omega-x}}{\omega-x} )</td>
</tr>
<tr>
<td>( n )-year deferred life</td>
<td>( \frac{q}{q+i} (vp)^n )</td>
<td>( \frac{v^n a_{\omega-(x+n)}}{\omega-x} )</td>
</tr>
<tr>
<td>( n )-year pure endowment</td>
<td>( (vp)^n )</td>
<td>( \frac{v^n (\omega-(x+n))}{\omega-x} )</td>
</tr>
</tbody>
</table>
Lesson 12: Insurance: Probabilities and Percentiles

Summary of Probability and Percentile Concepts

- To calculate $Pr(Z \leq z)$ for continuous $Z$, draw a graph of $Z$ as a function of $T$. Identify parts of the graph that are below the horizontal line $Z = z$, and the corresponding $t$'s. Then calculate the probability of $T$ being in the range of those $t$'s.

  Note that

  $$Pr(Z < z^*) = Pr(v^t < z^*) = Pr\left(t > -\frac{\ln z^*}{\delta}\right) = t^* p_x,$$

  where $t^* = -\frac{\ln z^*}{\delta} = -\frac{\ln z^*}{\ln(1+i)}$

The following table shows the relationship between the type of insurance coverage and corresponding probability:

<table>
<thead>
<tr>
<th>Type of insurance</th>
<th>$Pr(Z &lt; z^*)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Whole life</td>
<td>$t^* p_x$</td>
</tr>
<tr>
<td>$n$-year term</td>
<td>$\begin{cases} np_x &amp; z^* \leq v^n \ t^* p_x &amp; z^* &gt; v^n \end{cases}$</td>
</tr>
<tr>
<td>$n$-year deferred life</td>
<td>$\begin{cases} nq_x + t^* p_x &amp; z^* &lt; v^n \ 1 - z^* &amp; z^* \geq v^n \end{cases}$</td>
</tr>
<tr>
<td>$n$-year endowment</td>
<td>$\begin{cases} 0 &amp; z^* &lt; v^n \ t^* p_x &amp; z^* \geq v^n \end{cases}$</td>
</tr>
<tr>
<td>$n$-year deferred $m$-year term</td>
<td>$\begin{cases} nq_x + n+m p_x &amp; z^* &lt; v^{m+n} \ nq_x + t^* p_x &amp; v^{m+n} \leq z^* &lt; v^n \ 1 &amp; z^* \geq v^n \end{cases}$</td>
</tr>
</tbody>
</table>

- In the case of constant force of mortality $\mu$ and interest $\delta$, $Pr(Z \leq z) = z^{\mu/\delta}$
- For discrete $Z$, identify $T$ and then identify $K + 1$ corresponding to those $T$. In other words, $Pr(Z > z^*) = Pr(T < t^*) = k q_x$, where $k = \lfloor t^* \rfloor$ - the greatest integer smaller than $t^*$
  
  $Pr(Z < z^*) = Pr(T > t^*) = k + 1 p_x$, where $k = \lceil t^* \rceil$ - the greatest integer smaller or equal to $t^*$

- To calculate percentiles of continuous $Z$, draw a graph of $Z$ as a function of $T$. Identify where the lower parts of the graph are, and how they vary as a function of $T$.

  For example, for whole life, higher $T$ lead to lower $Z$.

  For $n$-year deferred whole life, both $T < n$ and higher $T$ lead to lower $Z$.

  Write an equation for the probability $Z$ less than $z$ in terms of mortality probabilities expressed in terms of $t$. Set it equal to the desired percentile, and solve for $t$ or for $e^{kt}$ for any $k$. Then solve for $z$ (which is often $v^t$)
Lesson 13: Insurance: Recursive Formulas, Varying Insurances

Recursive Formulas

\[ A_x = vq_x + vp_x A_{x+1} \]
\[ A_x = vq_x + v^2 p_x q_{x+1} + v^2 p_x A_{x+2} \]
\[ A_{x:n} = vq_x + vp_x A_{x+1:n-1} \]
\[ A_{x:n}^1 = vq_x + vp_x A_{x+1:n-1}^1 \]
\[ n|A_x = vp_{x:n-1} A_{x+1} \]
\[ A_{x:n}^1 = vp_x A_{x+1:n-1}^1 \]

Increasing and Decreasing Insurance

\[ \int_0^\infty t^n e^{-\delta t} dt = \frac{n}{\delta^{n+1}} \]
\[ \int_0^u t e^{-\delta t} dt = \frac{1}{\delta^2} \left( 1 - (1 + \delta u) e^{-\delta u} \right) = \frac{\bar{a} - uv - \bar{x}}{\delta} \]

\[ (\bar{I}A)_x = \frac{\mu}{(\mu + \delta)^2} \] for constant force

\[ \mathbb{E}[Z^2] = \frac{2\mu}{(\mu + 2\delta)^3} \] for \( Z \) a continuously increasing continuous insurance, constant force.

\[ (\bar{I}A)^1_{x:n} + (\bar{D}A)^1_{x:n} = n\bar{A}^1_{x:n} \]
\[ (I\bar{A})^1_{x:n} + (D\bar{A})^1_{x:n} = (n+1)\bar{A}^1_{x:n} \]
\[ (I\bar{A})^1_{x:n} + (D\bar{A})^1_{x:n} = (n+1)\bar{A}^1_{x:n} \]

Recursive Formulas for Increasing and Decreasing Insurance

\[ (IA)^1_{x:n} = A^1_{x:n} + vp_x (IA)^1_{x+1:n-1} \]
\[ (IA)^1_{x:n} = A^1_{x:n} + vp_x (I\bar{A})^1_{x+1:n-1} \]
\[ (DA)^1_{x:n} = nA^1_{x:n} + vp_x (DA)^1_{x+1:n-1} \]
\[ (DA)^1_{x:n} = A^1_{x:n} + (DA)^1_{x:n} \]
Lesson 14: Relationships between Insurance Payable at Moment of Death and Payable at End of Year

Summary of formulas relating insurances payable at moment of death to insurances payable at the end of the year of death assuming uniform distribution of deaths

\[
\bar{A}_x = \frac{i}{\delta} A_x
\]

\[
\bar{A}_x^{1.5} = \frac{i}{\delta} A_x^{1.5}
\]

\[
\overline{nA}_x = \frac{i}{\delta} n \overline{A}_x
\]

\[
(IA)_x^{1.5} = \frac{i}{\delta} (IA)_x^{1.5}
\]

\[
(ID)_x^{1.5} = \frac{i}{\delta} (ID)_x^{1.5}
\]

\[
\bar{A}_x = \frac{i}{\delta} A_x^{1.5} + A_x^{1.5}
\]

\[
A_x^{(m)} = \frac{i}{m} A_x
\]

\[
\bar{A}_x^2 = \frac{2i + i^2}{2\delta} 2A_x
\]

\[
(IA)_x^{1.5} = (IA)_x^{1.5} - \bar{A}_x^{1.5} \left( \frac{1}{\Delta} - 1 \right)
\]

Summary of formulas relating insurances payable at moment of death to insurances payable at the end of the year of death using claims acceleration approach

\[
\bar{A}_x = (1 + i)^{0.5} A_x
\]

\[
\bar{A}_x^{1.5} = (1 + i)^{0.5} A_x^{1.5}
\]

\[
\overline{nA}_x = (1 + i)^{0.5} n \overline{A}_x
\]

\[
\bar{A}_x^{1.5} = (1 + i)^{0.5} A_x^{1.5} + A_x^{1.5}
\]

\[
A_x^{(m)} = (1 + i)^{(m-1)/2m} A_x
\]

\[
\bar{A}_x^2 = (1 + i)^2 A_x
\]
Lesson 15: Annuities: Continuous, Expectation

Actuarial notation for standard types of annuity

<table>
<thead>
<tr>
<th>Name</th>
<th>Payment per annum at time $t$</th>
<th>Present value random variable</th>
<th>Symbol for actuarial present value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Whole life annuity</td>
<td>$1\ t \leq T$</td>
<td>$\bar{a}_T$</td>
<td>$\bar{a}_x$</td>
</tr>
<tr>
<td>Temporary life annuity</td>
<td>$1\ t \leq \min(T, n)$</td>
<td>$\bar{a}_T\ T \leq n$</td>
<td>$\bar{a}_{x:T}$</td>
</tr>
<tr>
<td></td>
<td>$0\ t &gt; \min(T, n)$</td>
<td>$\bar{a}_n\ T &gt; n$</td>
<td></td>
</tr>
<tr>
<td>Deferred life annuity</td>
<td>$0\ t \leq n$ or $t &gt; T$</td>
<td>$0\ T \leq n$</td>
<td>$n</td>
</tr>
<tr>
<td></td>
<td>$1\ n &lt; t \leq T$</td>
<td>$\bar{a}_T - \bar{a}_n\ T &gt; n$</td>
<td></td>
</tr>
<tr>
<td>Deferred temporary life annuity</td>
<td>$0\ t \leq n$ or $t &gt; T$</td>
<td>$0\ T \leq n$</td>
<td>$n</td>
</tr>
<tr>
<td></td>
<td>$1\ n &lt; t \leq n + m$ and $t \leq T$</td>
<td>$\bar{a}_T - \bar{a}_n\ T &gt; n$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$0\ T &gt; n + m$</td>
<td>$\bar{a}_n + m\ T &gt; n$</td>
<td></td>
</tr>
<tr>
<td>Certain-and-life</td>
<td>$1\ t \leq \max(T, n)$</td>
<td>$\bar{a}_T\ T \leq n$</td>
<td>$\bar{a}_{x:T}$</td>
</tr>
<tr>
<td></td>
<td>$0\ t &gt; \max(T, n)$</td>
<td>$\bar{a}_T\ T &gt; n$</td>
<td></td>
</tr>
</tbody>
</table>

Relationships between insurances and annuities

\[
\bar{a}_x = \frac{1 - \bar{A}_x}{\delta} \\
\bar{A}_x = 1 - \delta \bar{a}_x \\
\bar{a}_{x:n} = \frac{1 - \bar{A}_{x:n}}{\delta} \\
\bar{A}_{x:n} = 1 - \delta \bar{a}_{x:n} \\
n|\bar{a}_x = \frac{\bar{A}_{x:n} - \bar{A}_x}{\delta}
\]

General formulas for expected value

\[
\bar{a}_x = \int_0^\infty \bar{a}_x p_x \mu_{x+t} \ dt \\
\bar{a}_x = \int_0^\infty v^t p_x \ dt \\
\bar{a}_{x:n} = \int_0^n v^t p_x \ dt \\
n|\bar{a}_x = \int_n^\infty v^t p_x \ dt
\]

Formulas under constant force of mortality
\[
\bar{a}_x = \frac{1}{\mu + \delta} \\
\bar{a}_{x: \overline{\alpha}} = \frac{1 - e^{-(\mu+\delta)n}}{\mu + \delta} \\
\overline{n|\bar{a}_x} = \frac{e^{-(\mu+\delta)n}}{\mu + \delta}
\]

Relationships between annuities

\[
\bar{a}_x = \bar{a}_{x: \overline{\alpha}} + n \bar{E}_x \bar{a}_{x+n} \\
\bar{a}_{x: \overline{\alpha}} = \bar{\alpha} + n \overline{|\bar{a}_x} 
\]
Lesson 16: Annuities: Discrete, Expectation

Relationships between insurances and annuities
\[ \ddot{a}_x = \frac{1 - A_x}{d} \]
\[ A_x = 1 - d\ddot{a}_x \]
\[ \ddot{a}_x:q = \frac{1 - A_x:q}{d} \]
\[ A_x:q = 1 - d\ddot{a}_x:q \]
\[ A_x:q^1 = v\ddot{a}_x:q - a_x:q^1 \]
\[ (1 + i)A_x + ia_x = 1 \]

Relationships between annuities
\[ \ddot{a}_x:q^m = \ddot{a}_x + n |\ddot{a}_x \]
\[ a_x:q^m = \ddot{a}_x - n E_x\ddot{a}_x+n \]
\[ n|\ddot{a}_x = n E_x \ddot{a}_x+n \]
\[ \ddot{a}_x = \ddot{a}_x:q + n |\ddot{a}_x \]
\[ a_x = \ddot{a}_x - 1 \]
\[ \ddot{a}_x:q^m = a_x:q^m + 1 - n E_x = a_x:q^{m-1} + 1 \]

Other annuity equations
\[ \ddot{a}_x:q^m = \sum_{k=1}^{n-1} \ddot{a}_x^{q^k}p_{x+k-1} + \ddot{a}_x^{q^k-1}p_x \]
\[ \ddot{a}_x:q^m = \sum_{k=0}^{n-1} v^k p_x, \quad a_x:q^m = \sum_{k=1}^{n} v^k p_x \]
\[ \ddot{a}_x = \frac{1 + i}{q + i}, \text{ if } q_x \text{ is constant} \]

Accumulated value
\[ \ddot{s}_x:q^m = \frac{\ddot{a}_x:q^m}{n E_x} \]
\[ s_x:q^m = \ddot{s}_x:q^{m-1} + 1 \]
\[ s_x:q^m = s_x:q^{m-1} + \frac{1}{n-1 E_x+1} \]
\[ s_x:q^m = \ddot{s}_x:q^m + 1 - \frac{1}{n E_x} \]

mthly annuities
\[ \ddot{a}_x^{(m)} = \sum_{k=0}^{\infty} \frac{1}{m} v^k p_x \]
Lesson 17: Annuities: Variance

General formulas for second moments

\[
E[\bar{Y}^2_x] = \int_0^\infty \bar{a}^2_t p_x \mu_{x+t} \, dt
\]

\[
E[\ddot{Y}^2_x] = \sum_{k=1}^{\infty} \ddot{a}^2_{k-1} q_x
\]

\[
E[\ddot{Y}^2_{x:n}] = \sum_{k=1}^{n} \ddot{a}^2_{k-1} q_x + np_x \ddot{a}^2_n = \sum_{k=1}^{n-1} \ddot{a}^2_{k-1} q_x + n-1p_x \ddot{a}^2_n
\]

Special formulas for variance of whole life annuities and temporary life annuities

\[
Var(\bar{Y}_x) = \frac{2 \bar{A}_x - (\bar{A}_x)^2}{\delta^2} = \frac{2(\bar{a}_x - 2 \bar{\bar{a}}_x)}{\delta} - (\bar{\bar{a}}_x)^2
\]

\[
Var(\bar{Y}_{x:n}) = \frac{2 \bar{A}_{x:n} - (\bar{A}_{x:n})^2}{\delta^2} = \frac{2(\bar{a}_{x:n} - 2 \bar{\bar{a}}_{x:n})}{\delta} - (\bar{\bar{a}}_{x:n})^2
\]

\[
Var(\bar{Y}_x) = Var(\ddot{Y}_x) = \frac{2 A_x - (A_x)^2}{d^2} = \frac{2(\bar{a}_x - 2 \bar{\bar{a}}_x)}{d^2} + 2 \bar{\bar{a}}_x - (\bar{\bar{a}}_x)^2
\]

\[
Var(\bar{Y}_{x:n}) = Var(\ddot{Y}_{x:n-1}) = \frac{2 A_{x:n} - (A_{x:n})^2}{d^2}
\]

Lesson 18: Annuities: Probabilities and Percentiles

- To calculate a probability for an annuity, calculate the \( t \) for which \( \bar{a}_t \) has the desired property. Then calculate the probability \( t \) is in that range.
- To calculate a percentile of an annuity, calculate the percentile of \( T \), then calculate \( \bar{a}_T \).
- Some adjustments may be needed for discrete annuities or non-whole-life annuities.
- If forces of mortality and interest are constant, then the probability that the present value of payments on a continuous whole life annuity will be greater than its actuarial present value is

\[
Pr(\bar{a}_{T[x]} > \bar{a}_x) = \left( \frac{\mu}{\mu + \delta} \right)^{\mu/\delta}
\]
Lesson 19: Annuities: Varying Annuities, Recursive Calculations

Increasing/Decreasing Annuities

\[(\bar{I}a)_x = \frac{1}{(\mu + \delta)^2}, \text{ if } \mu \text{ is constant}\]

\[(\bar{I}a)_{x:n} + (\bar{D}a)_{x:n} = n\bar{a}_{x:n}\]

\[(Ia)_{x:n} + (Da)_{x:n} = (n + 1)a_{x:n}\]

Recursive Formulas

\[\bar{a}_x = v p_x \bar{a}_{x+1} + 1\]

\[a_x = v p_x a_{x+1} + v p_x\]

\[\bar{a}_x = v p_x a_{x+1} + \bar{a}_{x:n}\]

\[\bar{a}_x;n = v p_x \bar{a}_{x+1:n-n} + 1\]

\[a_x;n = v p_x a_{x+1:n-n} + v p_x\]

\[\bar{a}_x;n = v p_x \bar{a}_{x+1:n-n} + \bar{a}_{x:n}\]

\[n \bar{a}_x = v p_{x:n-1} \bar{a}_{x+1}\]

\[n a_x = v p_{x:n-1} a_{x+1}\]

\[n \bar{a}_x = v p_{x:n-1} \bar{a}_{x+1}\]

\[\bar{a}_x;n = 1 + v p_x a_{n-n} + v p_x \bar{a}_{x+1:n-n}\]

\[a_x;n = v + v p_x a_{n-n} + v p_x a_x^{x+1:n-n}\]

\[\bar{a}_x;n = \bar{a}_{n-n} + v p_x a_{n-n} + v p_x \bar{a}_x^{x+1:n-n}\]
Lesson 20: Annuities: m-thly Payments

In general:

\[ a_x^{(m)} = \bar{a}_x^{(m)} - \frac{1}{m} \]

\[ 1 + i = \left(1 + \frac{i^{(m)}}{m}\right)^m = \left(1 - \frac{d^{(m)}}{m}\right)^{-m} \]

\[ i^{(m)} = m \left(1 + i \frac{1}{m} - 1\right) \]

\[ d^{(m)} = m \left(1 - (1 + i) - \frac{1}{m}\right) \]

For small interest rates:

\[ a_x^{(m)} \approx \bar{a}_x - \frac{m - 1}{2m} \]

\[ a_x^{(m)} \approx a_x + \frac{m - 1}{2m} \]

Under the uniform distribution of death (UDD) assumption:

\[ a_x^{(m)} = \bar{a}_x - \frac{m - 1}{2m} \]

\[ a_x^{(m)} = a_x + \frac{m - 1}{2m} \]

\[ a_x^{(m)} = a_x \]

\[ a_x^{(m)} = \alpha(m)\bar{a}_x - \beta(m) \]

\[ a_x^{(m)} = \alpha(m)\bar{a}_x - \beta(m)(1 - n E_x) \]

\[ a_x^{(m)} = \alpha(m)\bar{a}_x - \beta(m)\]

\[ a_x^{(m)} = a_x \]

\[ a_x^{(m)} = a_x \]

\[ a_x^{(m)} = a_x \]

\[ a_x^{(m)} = a_x \]

\[ a_x^{(m)} = a_x \]

\[ a_x^{(m)} = a_x \]

Similar conversion formulae for converting the modal insurances to annuities hold for other types of insurances and annuities (only whole life version is shown).

\[ \alpha(m) = \frac{id}{i^{(m)}d^{(m)}} \]

\[ \beta(m) = \frac{i - i^{(m)}}{i^{(m)}d^{(m)}} \]

\[ i^{(\infty)} = d^{(\infty)} = \ln(1 + i) = \delta \]
Woolhouse’s formula for approximating $\bar{a}_x^{(m)}$:

$$\bar{a}_x^{(m)} \approx \bar{a}_x - \frac{m - 1}{2m} - \frac{m^2 - 1}{12m^2}(\mu_x + \delta)$$

$$\bar{a}_x \approx \bar{a}_x - \frac{1}{2} - \frac{1}{12}(\mu_x + \delta)$$

$$\bar{a}_{x,n}^{(m)} \approx \bar{a}_{x,n} - \frac{m - 1}{2m}(1 - n E_x) - \frac{m^2 - 1}{12m^2}(\mu_x + \delta - n E_x(\mu_{x+n} + \delta))$$

$$n|\bar{a}_x^{(m)} \approx n|\bar{a}_x - \frac{m - 1}{2m}n E_x - \frac{m^2 - 1}{12m^2}n E_x(\mu_{x+n} + \delta)$$

$$\bar{e}_x = e_x + \frac{1}{2} - \frac{1}{12}\mu_x$$

When the exact value of $\mu_x$ is not available, use the following approximation:

$$\mu_x \approx -\frac{1}{2}(\ln p_{x-1} + \ln p_x)$$
Lesson 21: Premiums: Fully Continuous Expectation

The equivalence principle: The actuarial present value of the benefit premiums is equal to the actuarial present value of the benefits. For instance:

whole life insurance $\bar{A}_x = \bar{P}(\bar{A}_x) \bar{a}_x$

$n$-year endowment insurance $\bar{A}_{x:n} = \bar{P}(\bar{A}_{x:n}) \bar{a}_{x:n}$

$n$-year term insurance $\bar{A}_{x:1} = \bar{P}(\bar{A}_{x:1}) \bar{a}_{x:1}$

$n$-year deferred insurance $n|\bar{A}_x = n \bar{P}(n|\bar{A}_x) \bar{a}_{x:n}$

$n$-pay whole life insurance $\bar{A}_x = n \bar{P}(\bar{A}_x) \bar{a}_{x:n}$

$n$-year deferred annuity $n|\bar{a}_x = \bar{P}(n|\bar{a}_x) \bar{a}_{x:n}$

For constant force of mortality, $\bar{P}(\bar{A}_x)$ and $\bar{P}(\bar{A}_{x:n})$ are equal to $\mu$.

Future loss formulas for whole life with face amount $b$ and premium amount $\pi$:

$$0L = bv^T - \pi a_T = bv_T - \pi \left(\frac{1 - v_T^T}{\delta}\right) = v_T^T \left(b + \frac{\pi}{\delta}\right) - \frac{\pi}{\delta}$$

$$E[0L] = b \bar{A}_x - \pi \bar{a}_x = \bar{A}_x \left(b + \frac{\pi}{\delta}\right) - \frac{\pi}{\delta}$$

Similar formulas are available for endowment insurance.
Lesson 22: Premiums: Net Premiums for Discrete Insurances Calculated from Life Tables

Assume the following notation

1. \( P_x \) is the premium for a fully discrete whole life insurance, or \( A_x/\bar{a}_x \)
2. \( P_{x:n} \) is the premium for a fully discrete \( n \)-year term insurance, or \( A_{x:n}/\bar{a}_{x:n} \)
3. \( P_{x:1} \) is the premium for a fully discrete \( n \)-year pure endowment, or \( A_{x:1}/\bar{a}_{x:1} \)
4. \( P_{x:n} \) is the premium for a fully discrete \( n \)-year endowment insurance, or \( A_{x:n}/\bar{a}_{x:n} \)
Lesson 23: Premiums: Net Premiums for Discrete Insurances Calculated from Formulas

Whole life and endowment insurance benefit premiums

\[ P_x = \frac{1}{\overline{a}_x} - d \]
\[ P_x = \frac{dA_x}{1 - A_x} \]
\[ P_{x:n} = \frac{1}{\overline{a}_{x:n}} - d \]
\[ P_{x:n} = \frac{dA_{x:n}}{1 - A_{x:n}} \]

For fully discrete whole life and term insurances:
If \( q_x \) is constant, then \( P_x = vq_x \) and \( P_{x:n} = vq_x \).

Future loss at issue formulas for fully discrete whole life with face amount \( b \):
\[ L_0 = v^{K_x+1} \left( b + \frac{P_x}{d} \right) - \frac{P_x}{d} \]
\[ \mathbb{E}[L_0] = \ddot{A}_x \left( b + \frac{P_x}{d} \right) - \frac{P_x}{d} \]

Similar formulas are available for endowment insurances.

Refund of premium with interest
To calculate the benefit premium when premiums are refunded with interest during the deferral period, equate the premiums and the benefits at the end of the deferral period. Past premiums are accumulated at interest only.

Three premium principle formulae
\[ nP_x - P_{x:n}^1 = P_{x:n} \overline{A}_{x+n} \]
\[ P_{x:n} - nP_x = P_{x:n} \frac{1}{1 - A_{x+n}} \]
Lesson 24: Premiums: Net Premiums Paid on an m-thly Basis

If premiums are payable \( m \)thly, then calculating the annual benefit premium requires dividing by an \( m \)thly annuity. If you are working with a life table having annual information only, \( m \)thly annuities can be estimated either by assuming UDD between integral ages or by using Woolhouse’s formula (Lesson 20). The \( m \)thly premium is then a multiple of the annual premium. For example, for \( h \)-pay whole life payable at the end of the year of death,

\[
hP_x^{(m)} = \frac{A_x^{(m)}}{a_x^{(m)}} = \frac{hP_x a_x^{(m)}}{a_x^{(m)}}
\]
Lesson 25: Premiums: Gross Premiums

The gross future loss at issue $L_0^g$ is the random variable equal to the present value at issue of benefits plus expenses minus the present value at issue of gross premiums:

$$L_0^g = PV(Ben) + PV(Exp) - PV(P^g)$$

To calculate the gross premium $P^g$ by the equivalence principle, equate $P^g$ times the annuity-due for the premium payment period with the sum of

1. An insurance for the face amount plus settlement expenses
2. $P^g$ times an annuity-due for the premium payment period of renewal percent of premium expense, plus the excess of the first year percentage over the renewal percentage
3. An annuity-due for the coverage period of the renewal per-policy and per-100 expenses, plus the excess of first year over renewal expenses
Lesson 26: Premiums: Variance of Loss at Issue, Continuous

The following equations are for whole life and endowment insurance of 1. For whole life, drop $\bar{m}$.

\[ Var(L_0) = \left( 2 \bar{A}_x \bar{m} - (\bar{A}_x \bar{m})^2 \right) \left( 1 + \frac{P}{\delta} \right)^2 \]

\[ Var(L_0) = \frac{2 \bar{A}_x \bar{m} - (\bar{A}_x \bar{m})^2}{(1 - \bar{A}_x \bar{m})^2}, \text{ if equivalence principle premium is used} \]

\[ Var(L_0) = \frac{\mu}{\mu + 2\delta}. \text{ For whole life with equivalence principle and constant force of mortality only} \]

For whole life and endowment insurance with face amount $b$:

\[ Var(L_0) = \left( 2 \bar{A}_x - (\bar{A}_x)^2 \right) \left( b + \frac{P}{\delta} \right)^2 \]

\[ Var(L_0) = \left( 2 \bar{A}_x \bar{m} - (\bar{A}_x \bar{m})^2 \right) \left( b + \frac{P}{\delta} \right)^2 \]

If the benefit is $b$ instead of 1, and the premium $P$ is stated per unit, multiply the variances by $b^2$.

For two whole life or endowment insurances, one with $b'$ units and total premium $P'$ and the other with $b$ units and total premium $P$, the relative variance of loss at issue of the first to second is $\left( (b'\delta + P') / (b\delta + P) \right)^2$. 
Lesson 27: Premiums: Variance of Loss at Issue, Discrete

The following equations are for whole life and endowment insurance of 1. For whole life, drop \( \bar{m} \).

\[
Var(0L) = \left(2 A_{x:\bar{m}} - (A_{x:\bar{m}})^2\right) \left(1 + \frac{P}{d}\right)^2
\]

If equivalence principle premium is used: \( Var(0L) = \frac{2 A_{x:\bar{m}} - (A_{x:\bar{m}})^2}{(1 - A_{x:\bar{m}})^2} \)

For whole life with equivalence principle and constant force of mortality only: \( Var(0L) = \frac{q(1 - q)}{q + 2i} \)

If the benefit is \( b \) instead of 1, and the premium \( P \) is stated per unit, multiply the variances by \( b^2 \).

For two whole life or endowment insurances, one with \( b' \) units and total premium \( P' \) and the other with \( b \) units and total premium \( P \), the relative variance of loss at issue of the first to second is \( ((b'd + P') / (bd + P))^2 \).
Lesson 28: Premiums: Probabilities and Percentiles of Loss at Issue

- For level benefit or decreasing benefit insurance, the loss at issue decreases with time for whole life, endowment, and term insurances. To calculate the probability that the loss at issue is less than something, calculate the probability that survival time is greater than something.
- For level benefit or decreasing benefit deferred insurance, the loss at issue decreases during the deferral period, then jumps at the end of the deferral period and declines thereafter.
  - To calculate the probability that the loss at issue is greater than a positive number, calculate the probability that survival time is less than something minus the probability that survival time is less than the deferral period.
  - To calculate the probability that the loss at issue is greater than a negative number, calculate the probability that survival time is less than something that is less than the deferral period, and add that to the probability that survival time is less than something that is greater than the deferral period minus the probability that survival time is less than the deferral period.
- For a deferred annuity with premiums payable during the deferral period, the loss at issue decreases until the end of the deferral period and increases thereafter.
- The 100\textsuperscript{th} percentile premium is the premium for which the loss at issue is positive with probability \( p \). For fully continuous whole life, this is the loss that occurs if death occurs at the 100\textsuperscript{th} percentile of survival time.