Binary Classification / Perceptron

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Slides adapted from David Sontag and Vibhav Gogate

Supervised Learning

- **Input:** $\left(x^{(1)}, y^{(1)}\right), \dots, (x^{(M)}, y^{(M)})$
  - $x^{(m)}$ is the m^{th} data item and $y^{(m)}$ is the m^{th} **label**
- **Goal:** find a function f such that $f\left(x^{(m)}\right)$ is a "good approximation" to $y^{(m)}$
  - Can use it to predict y values for previously unseen x values

Supervised Learning

- Hypothesis space: set of allowable functions $f: X \to Y$

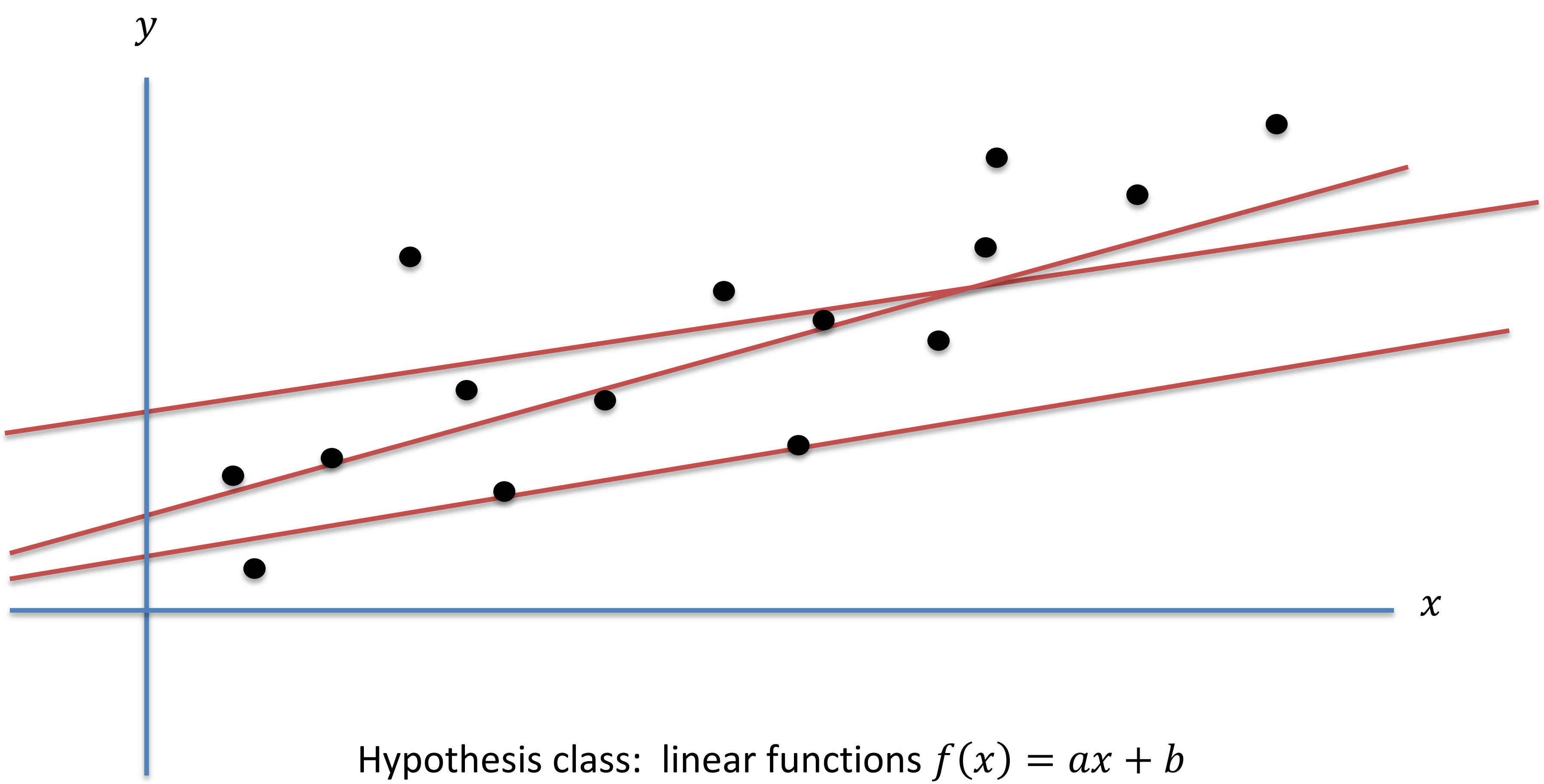
- Goal: find the "best" element of the hypothesis space

  - How do we measure the quality of f?

Examples of Supervised Learning

- Spam email detection
- Handwritten digit recognition
- Stock market prediction
- More?

Regression



Hypothesis class: linear functions $f(x) = ax + b$

How do we measure the quality of the approximation?

Linear Regression

- In typical regression applications, measure the fit using a squared loss function

$$L(f) = \frac{1}{M}\sum_{m}\left(f\left(x^{(m)}\right) - y^{(m)}\right)^2$$

- Want to minimize the average loss on the training data
- For 2-D linear regression, the learning problem is then

$$\min_{a,b} \frac{1}{M}\sum_{m}\left(ax^{(m)} + b - y^{(m)}\right)^2$$

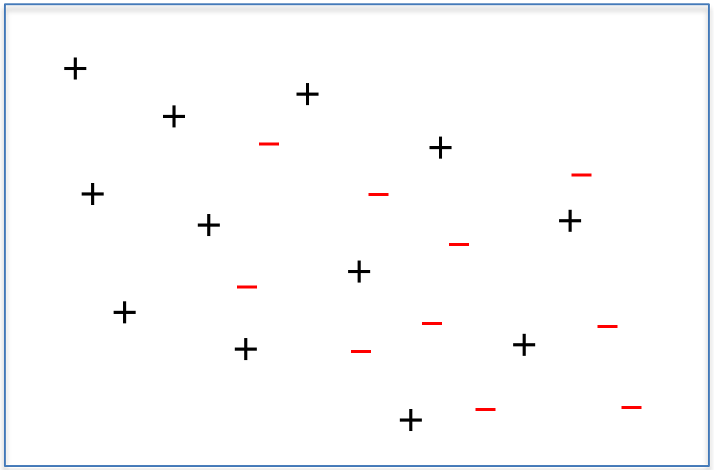
- For an unseen data point, x, the learning algorithm predicts $f(x)$

Supervised Learning

- **Select a hypothesis space** (elements of the space are represented by a collection of parameters)
- **Choose a loss function** (evaluates quality of the hypothesis as a function of its parameters)
- **Minimize loss function using gradient descent** (minimization over the parameters)
- **Evaluate quality of the learned model using test data** – that is, data on which the model was not trained

Binary Classification

- Input $(x^{(1)}, y^{(1)}), \ldots, (x^{(M)}, y^{(M)})$ with $x^{(m)} \in \mathbb{R}^n$ and $y^{(m)} \in \{-1, +1\}$
- We can think of the observations as points in $\mathbb{R}^n$ with an associated sign (either +/- corresponding to 0/1)
- An example with $n = 2$



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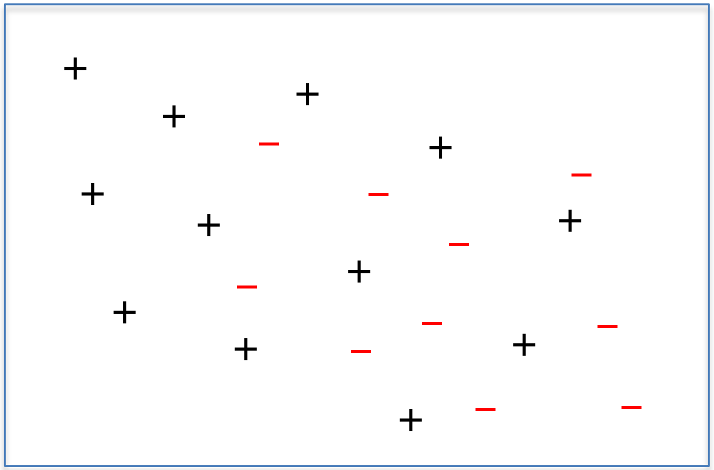

What is a good hypothesis space for this problem?

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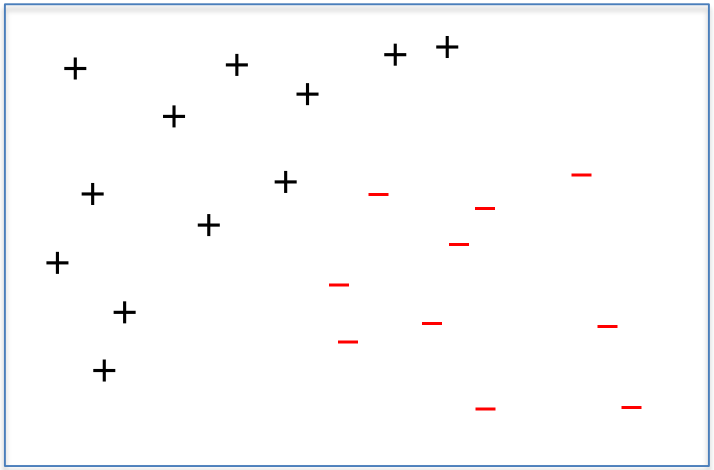

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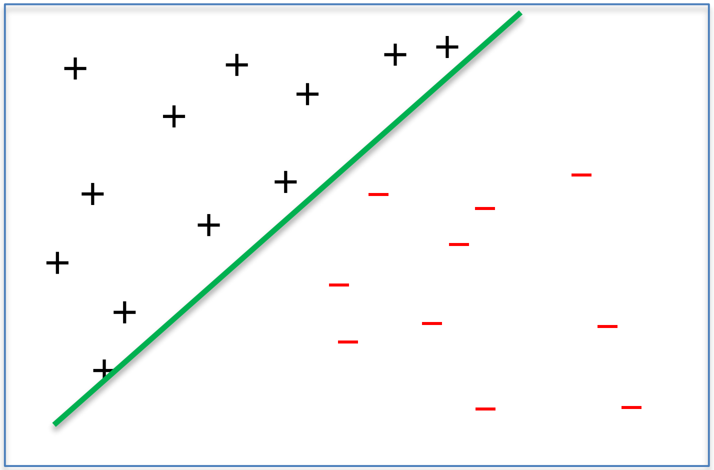

In this case, we say that the observations are linearly separable

Linear Separators

- In n dimensions, a hyperplane is a solution to the equation

$$w^T x + b = 0$$

with $w \in \mathbb{R}^n, b \in \mathbb{R}$

- Hyperplanes divide $\mathbb{R}^n$ into two distinct sets of points (called open halfspaces)

$$w^T x + b > 0$$

$$w^T x + b < 0$$

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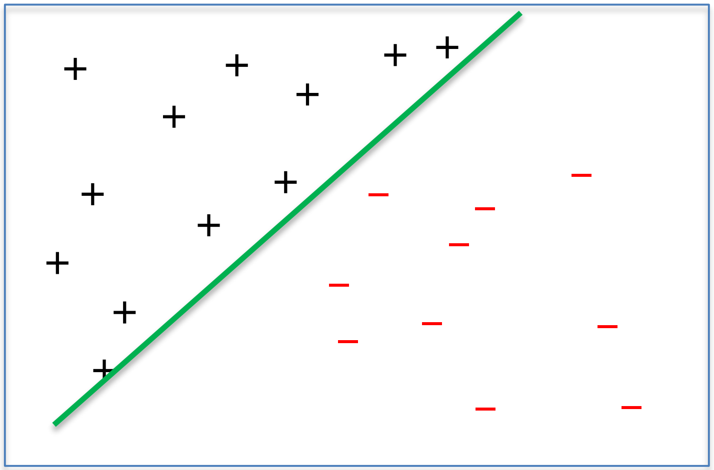

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The Linearly Separable Case

- Input $\left(x^{(1)}, y^{(1)}\right), \dots, (x^{(M)}, y^{(M)})$ with $x^{(m)} \in \mathbb{R}^n$ and $y^{(m)} \in \{-1, +1\}$
- Hypothesis space: separating hyperplanes

$$f(x) = sign\,(w^T x + b)$$

- How should we choose the loss function?

The Linearly Separable Case

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- Hypothesis space: separating hyperplanes

$$f(x) = sign\,(w^T x + b)$$

- How should we choose the loss function?
  - Count the number of misclassifications

$$loss = \sum_m \left| y^{(m)} - sign(w^T x^{(m)} + b) \right|$$

    - Tough to optimize, gradient contains no information

The Linearly Separable Case

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- Hypothesis space: separating hyperplanes

$$f(x) = sign\ (w^T x + b)$$

- How should we choose the loss function?
  - Penalize misclassification linearly by the size of the violation

$$perceptron\ loss = \sum_m \max\{0, -y^{(m)}(w^T x^{(m)} + b)\}$$

    - Modified hinge loss (this loss is convex, but not differentiable)

The Perceptron Algorithm

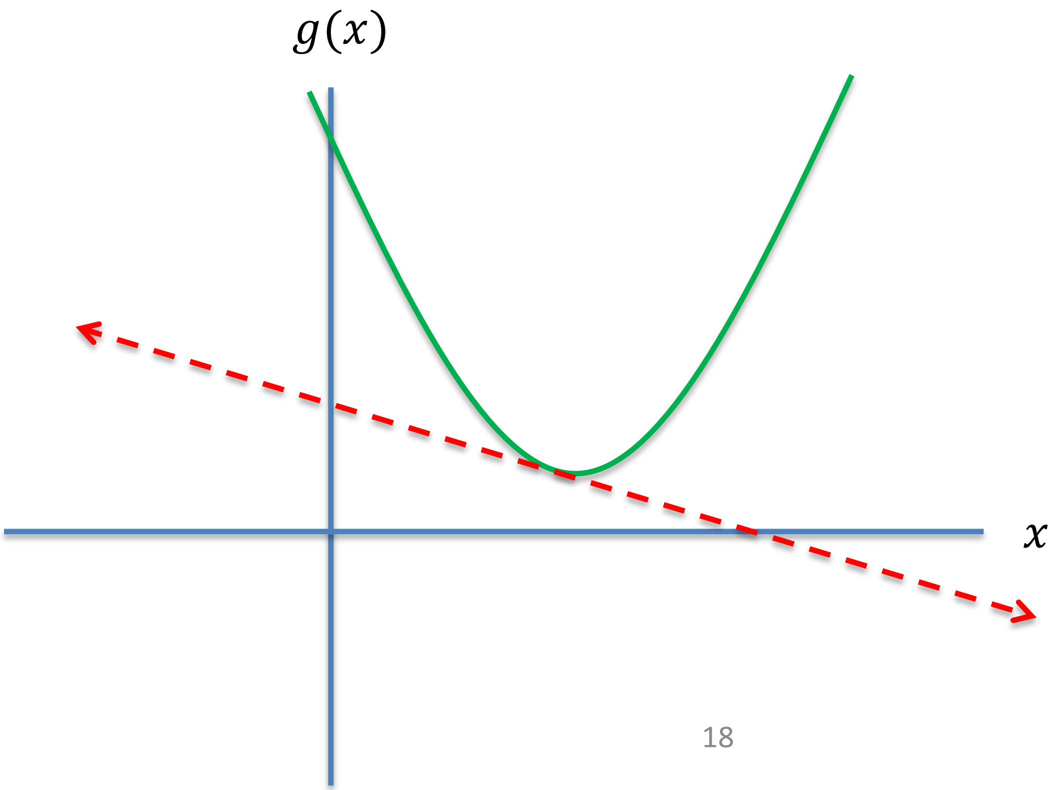
- Try to minimize the perceptron loss using gradient descent

  - The perceptron loss isn't differentiable, how can we apply gradient descent?

  - Need a generalization of what it means to be the gradient of a **convex** function

Gradients of Convex Functions

- For a differentiable convex function $g(x)$ its gradients are **linear underestimators**



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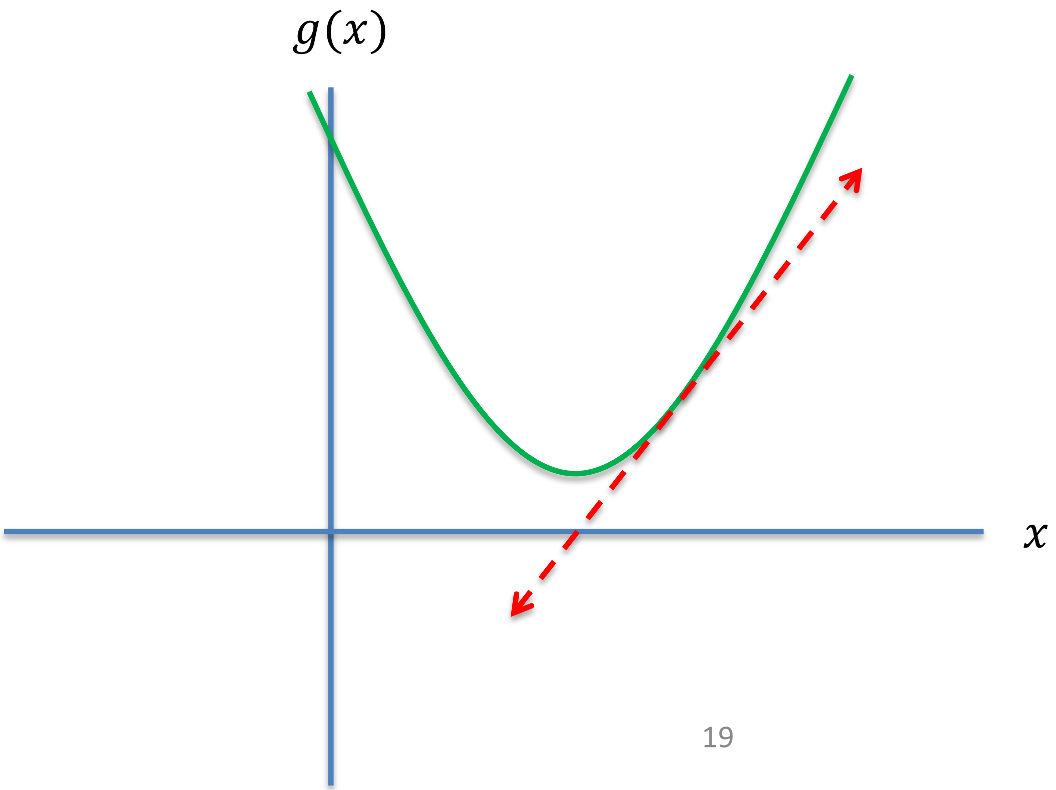
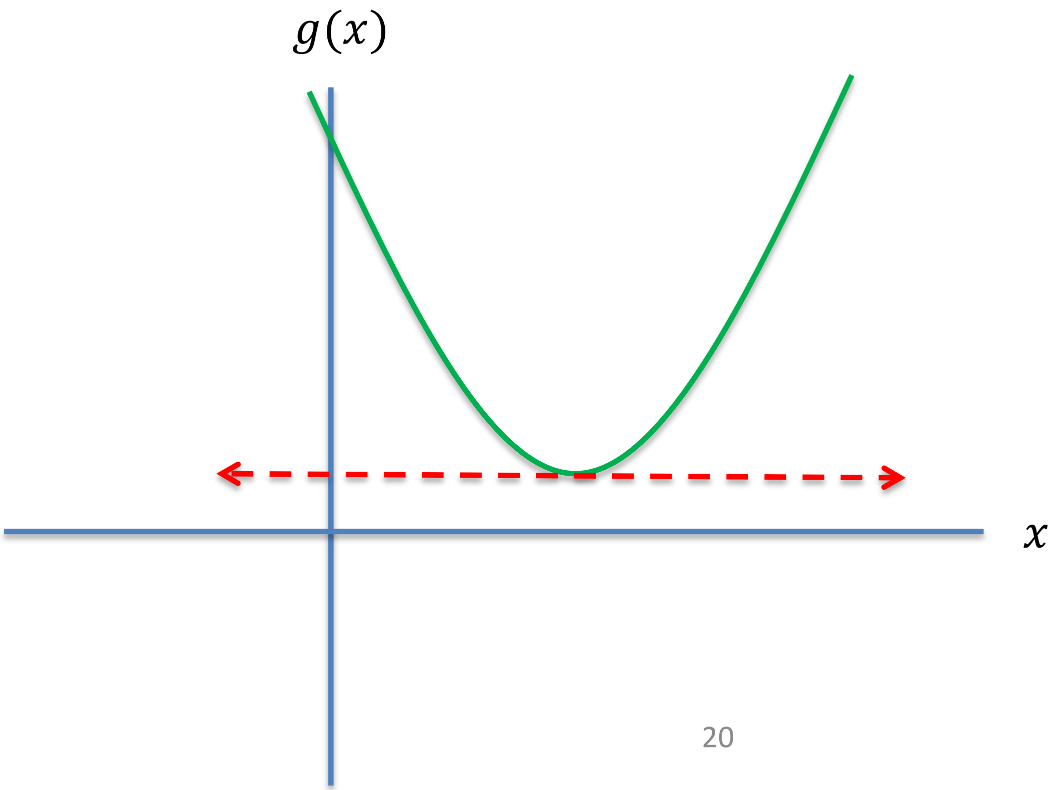


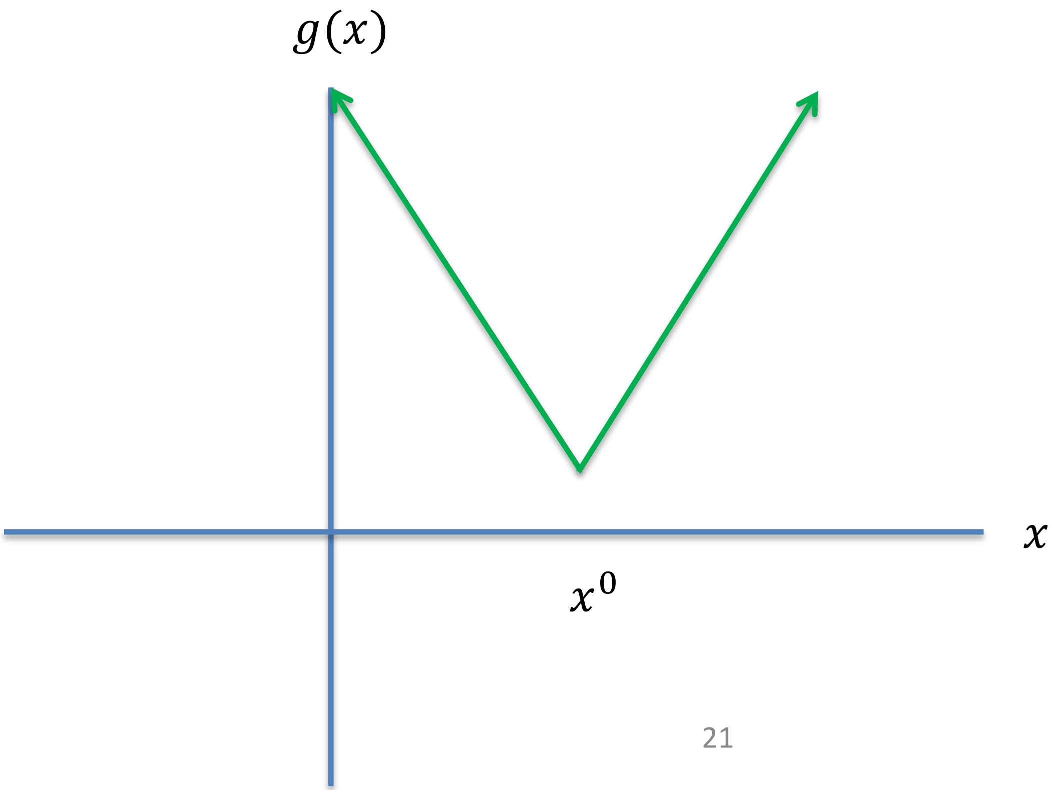
Gradients of Convex Functions

- For a differentiable convex function $g(x)$ its gradients are **linear underestimators**: zero gradient corresponds to a global optimum



Subgradients

- For a convex function $g(x)$, a subgradient at a point x^0 is given by any line, l, such that $l(x^0) = g(x^0)$ and $l(x) \leq g(x)$ for all x, i.e., it is a linear underestimator



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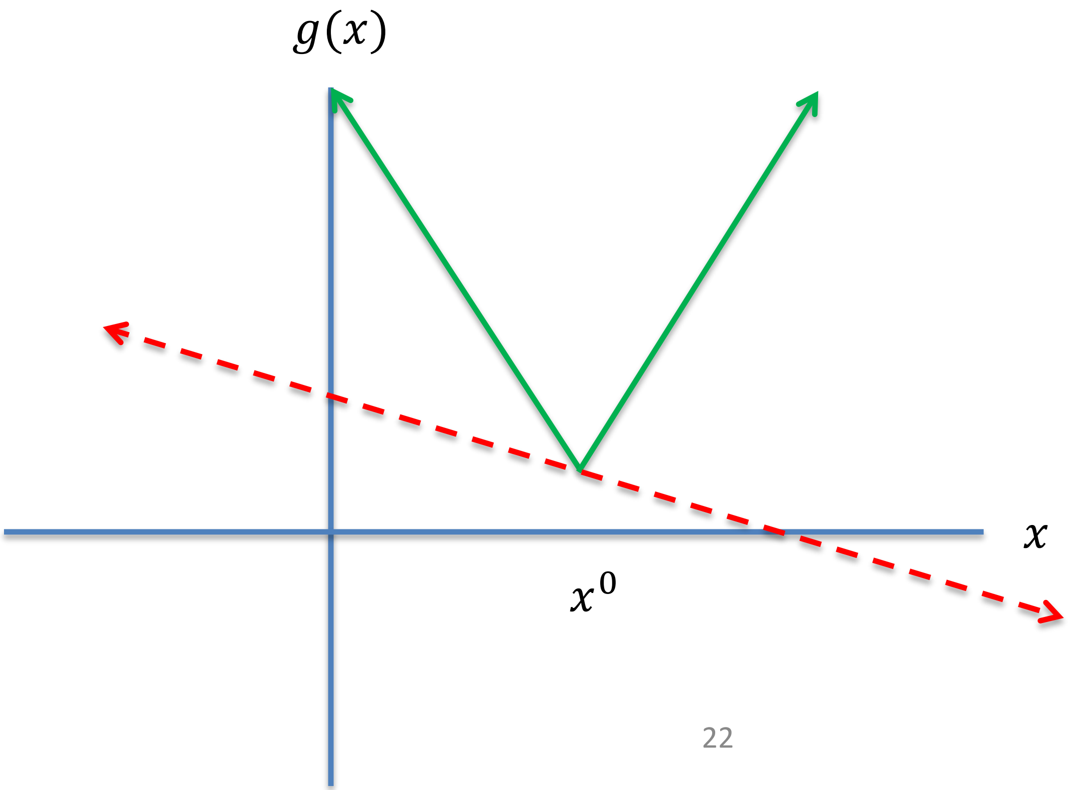

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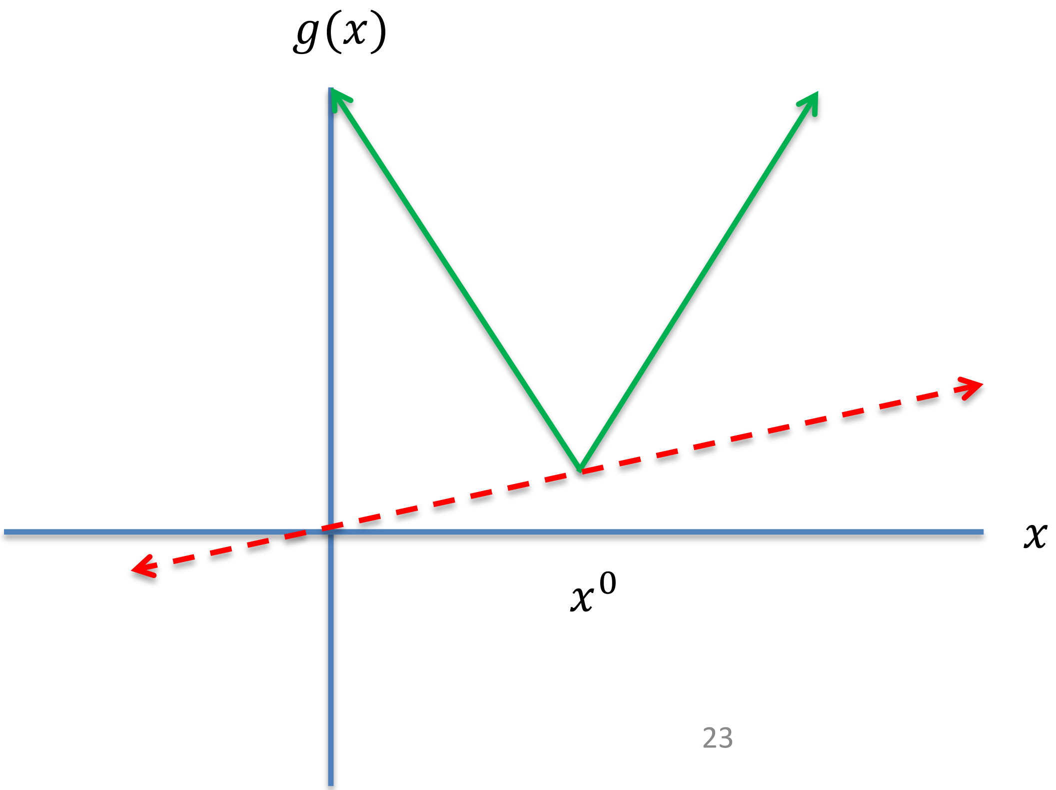

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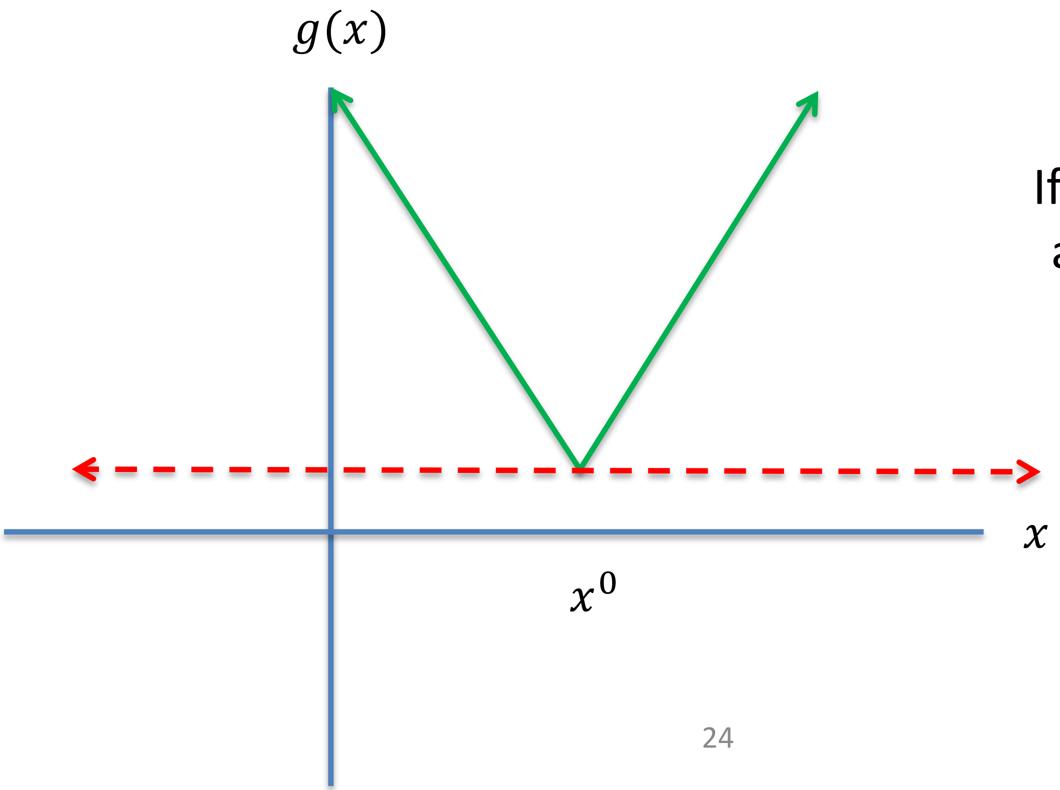

If $\vec{0}$ is a subgradient at x^0, then x^0 is a global minimum

Subgradients

- If a convex function is differentiable at a point x, then it has a unique subgradient at the point x given by the gradient

- If a convex function is not differentiable at a point x, it can have many subgradients

    - E.g., the set of subgradients of the convex function $|x|$ at the point $x = 0$ is given by the set of slopes $[-1,1]$

- Subgradients only guaranteed to exist for convex functions

The Perceptron Algorithm

- Try to minimize the perceptron loss using (sub)gradient descent

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$$\nabla_w(perceptron\ loss) = -\sum_{m=1}^{M}\left(y^{(m)}x^{(m)} \cdot 1_{-y^{(m)}f_{w,b}(x^{(m)})\geq 0}\right)$$

$$\nabla_b(perceptron\ loss) = -\sum_{m=1}^{M}\left(y^{(m)} \cdot 1_{-y^{(m)}f_{w,b}(x^{(m)})\geq 0}\right)$$

The Perceptron Algorithm

- Try to minimize the perceptron loss using (sub)gradient descent

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Is equal to zero if the m^{th} data point is correctly classified and one otherwise

The Perceptron Algorithm

- Try to minimize the perceptron loss using (sub)gradient descent

$$w^{(t+1)} = w^{(t)} + \gamma_t \sum_{m=1}^{M} \left(y^{(m)} x^{(m)} \cdot 1_{-y^{(m)} f_{w,b}\left(x^{(m)}\right) \geq 0} \right)$$

$$b^{(t+1)} = b^{(t)} + \gamma_t \sum_{m=1}^{M} \left(y^{(m)} \cdot 1_{-y^{(m)} f_{w,b}\left(x^{(m)}\right) \geq 0} \right)$$

- With step size γ_t (also called the learning rate)
- Note that, for convergence of subgradient methods, a diminishing step size, e.g., $\gamma_t = \frac{1}{1+t}$ is required

Stochastic Gradient Descent

- To make the training more practical, stochastic (sub)gradient descent is often used instead of standard gradient descent
- Approximate the gradient of a sum by sampling a few indices (as few as one) uniformly at random and averaging

$$\nabla_x \left[\sum_{m=1}^{M} g_m(x) \right] \approx \frac{1}{K} \sum_{k=1}^{K} \nabla_x g_{m_k}(x)$$

here, each m_k is sampled uniformly at random from $\{1, \dots, M\}$

- Stochastic gradient descent converges to the global optimum under certain assumptions on the step size

Stochastic Gradient Descent

- Setting $K = 1$, we pick a random observation m and perform the following update

if the $\boldsymbol{m^{th}}$ data point is misclassified:

$$w^{(t+1)} = w^{(t)} + \gamma_t y^{(m)} x^{(m)}$$

$$b^{(t+1)} = b^{(t)} + \gamma_t y^{(m)}$$

if the $\boldsymbol{m^{th}}$ data point is correctly classified:

$$w^{(t+1)} = w^{(t)}$$

$$b^{(t+1)} = b^{(t)}$$

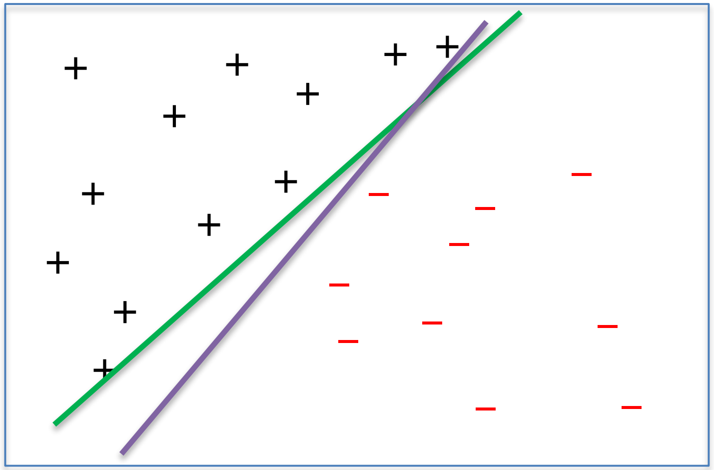
- Sometimes, you will see the perceptron algorithm specified with $\gamma_t = 1$ for all t

Applications of Perceptron

- Spam email classification
  - Represent emails as vectors of counts of certain words (e.g., sir, madam, Nigerian, prince, money, etc.)
  - Apply the perceptron algorithm to the resulting vectors
  - To predict the label of an unseen email
    - Construct its vector representation, x'
    - Check whether or not $w^T x' + b$ is positive or negative

Perceptron Learning Drawbacks

- No convergence guarantees if the observations are not linearly separable
- Can overfit
  - There can be a number of perfect classifiers, but the perceptron algorithm doesn't have any mechanism for choosing between them



What If the Data Isn‘t Separable?

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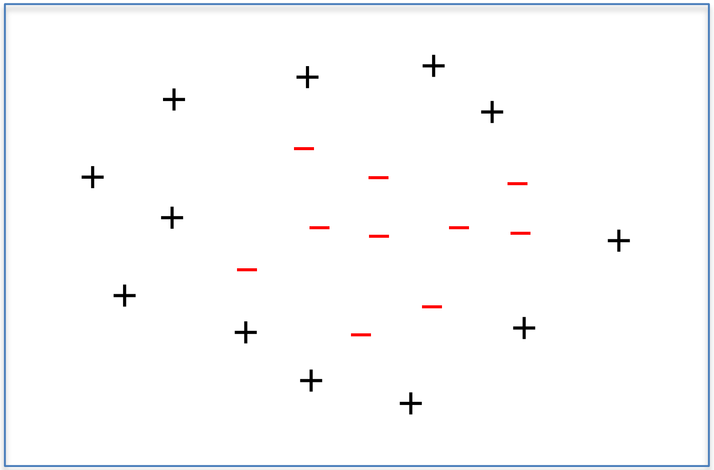

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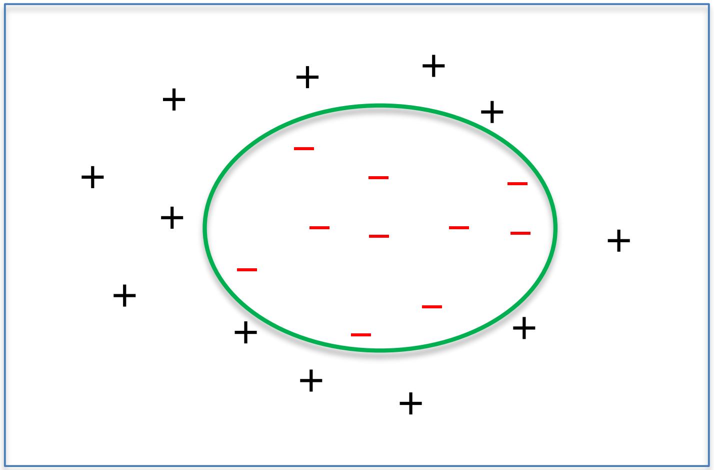
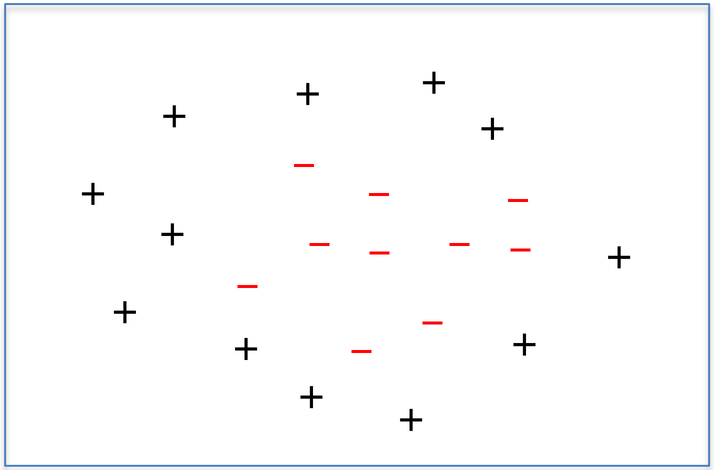


What is a good hypothesis space for this problem?

Adding Features

- Perceptron algorithm only works for linearly separable data



Can add features to make the data linearly separable in a higher dimensional space!

Essentially the same as higher order polynomials for linear regression!

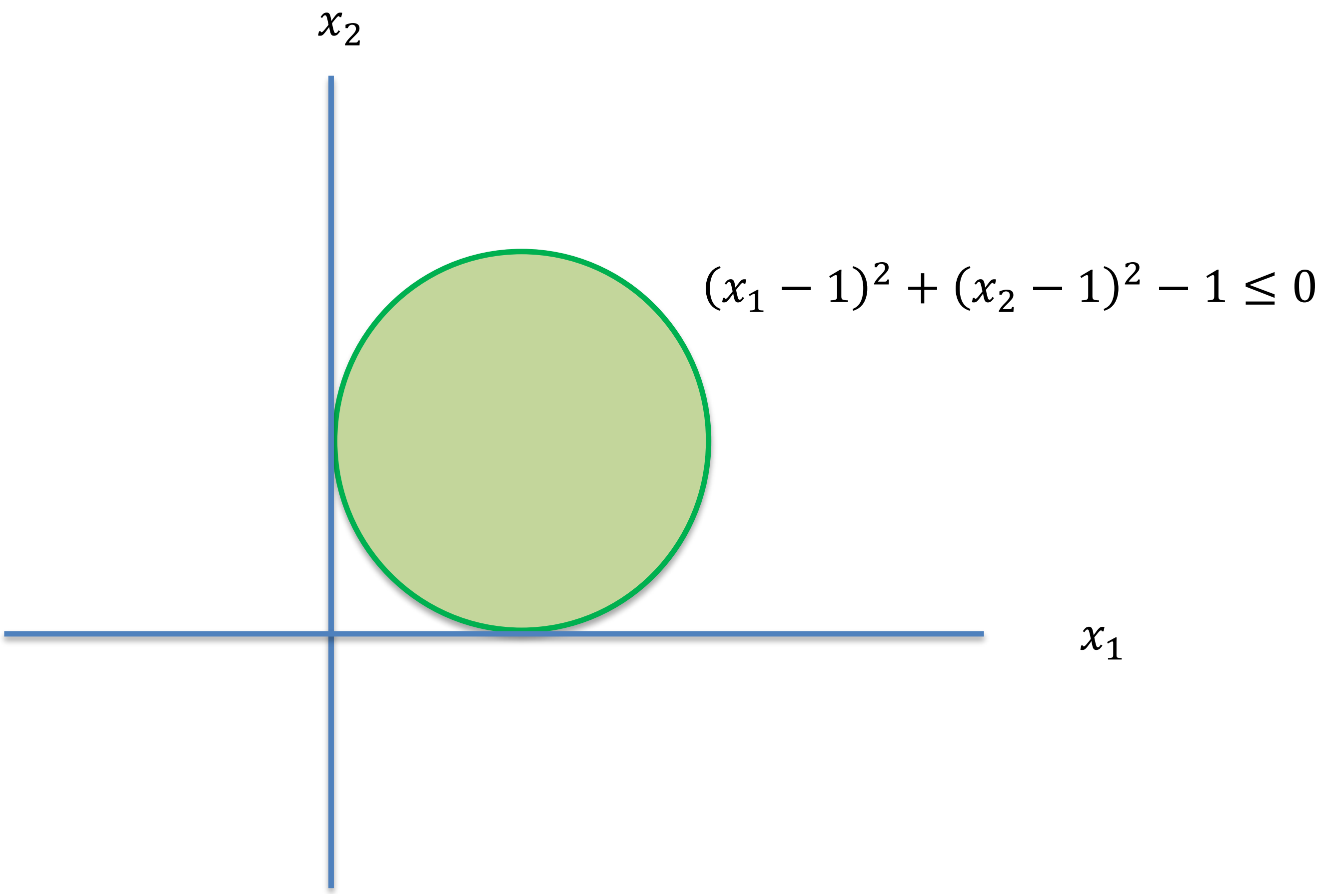
Adding Features

- The idea, choose a feature map $\phi: \mathbb{R}^n \to \mathbb{R}^k$
  - Given the observations $x^{(1)}, \ldots, x^{(M)}$, construct feature vectors $\phi(x^{(1)}), \ldots, \phi(x^{(M)})$
  - Use $\phi(x^{(1)}), \ldots, \phi(x^{(M)})$ instead of $x^{(1)}, \ldots, x^{(M)}$ in the learning algorithm
  - Goal is to choose ϕ so that $\phi(x^{(1)}), \ldots, \phi(x^{(M)})$ are linearly separable in $\mathbb{R}^k$
  - Learn linear separators of the form $w^T\phi(x)$ (instead of $w^T x$)
- Warning: more expressive features can lead to overfitting!

Adding Features: Examples

- $\phi\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
  - This is just the input data, without modification
- $\phi\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ x_1 \\ x_2 \\ x_1^2 \\ x_2^2 \end{bmatrix}$
  - This corresponds to a second degree polynomial separator, or equivalently, elliptical separators in the original space

Adding Features



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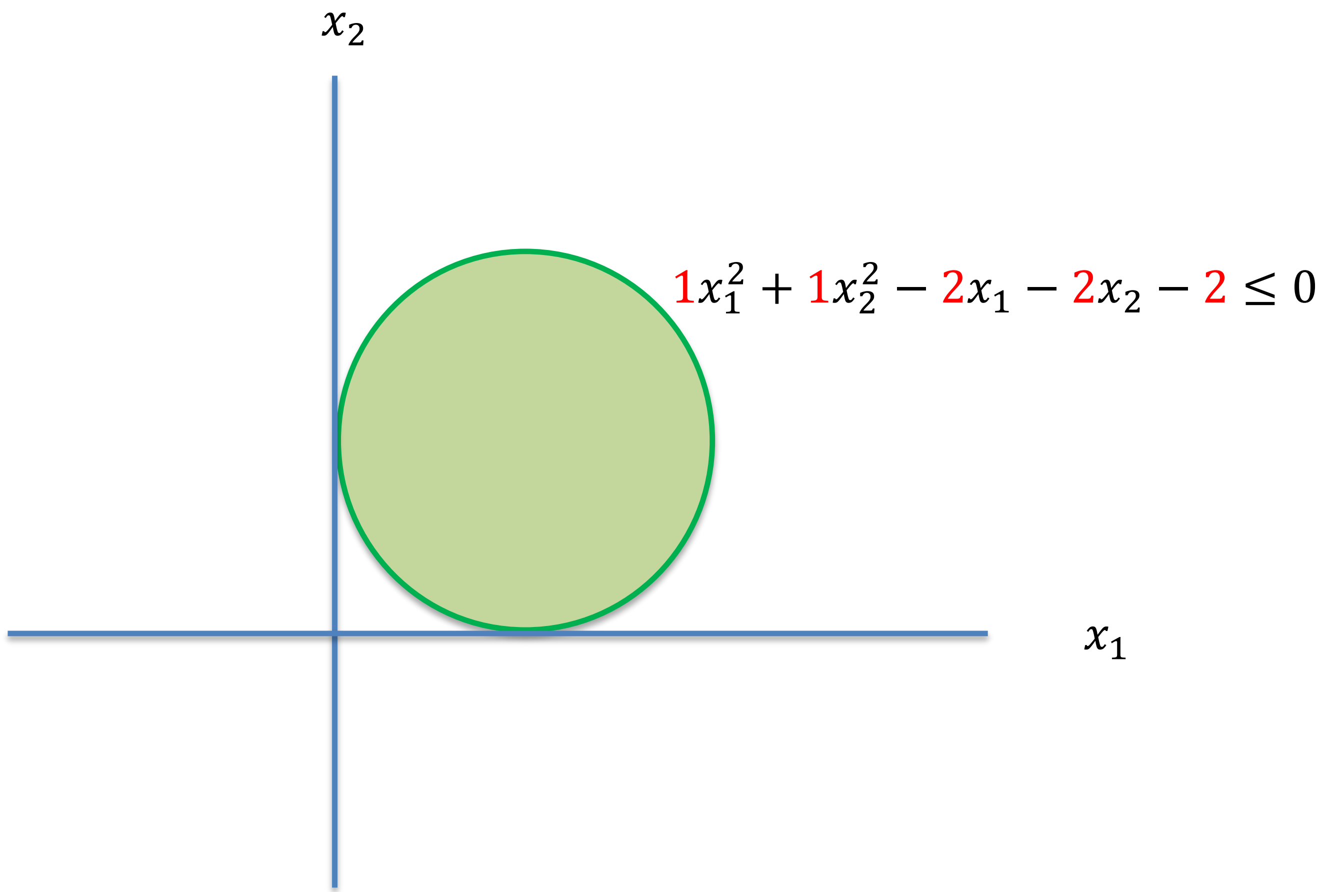
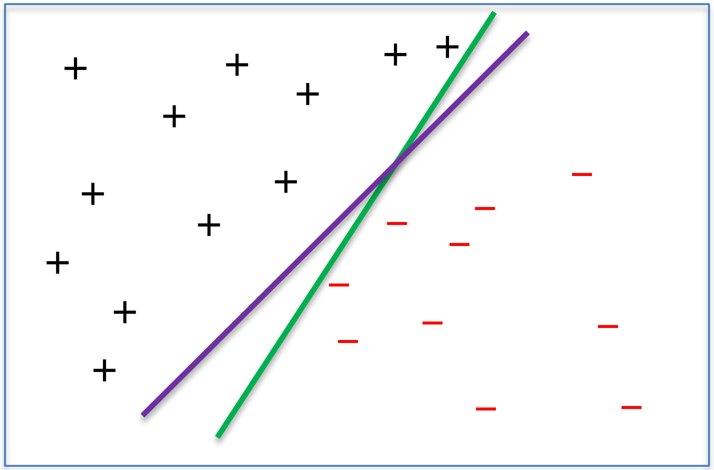


Support Vector Machines

- How can we decide between two perfect classifiers?



- What is the practical difference between these two solutions?