



Binary Classification / Perceptron

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- Homework 1 available soon on eLearning and due in 2 weeks
 - Late homework **will not be accepted**
- Instructions for getting started with the course, e.g., joining Piazza, are on eLearning

- **Input:** $(x^{(1)}, y^{(1)}), \dots, (x^{(M)}, y^{(M)})$
 - $x^{(m)}$ is the m^{th} data item and $y^{(m)}$ is the m^{th} **label**
- **Goal:** find a function f such that $f(x^{(m)})$ is a “good approximation” to $y^{(m)}$
 - Can use it to predict y values for previously unseen x values

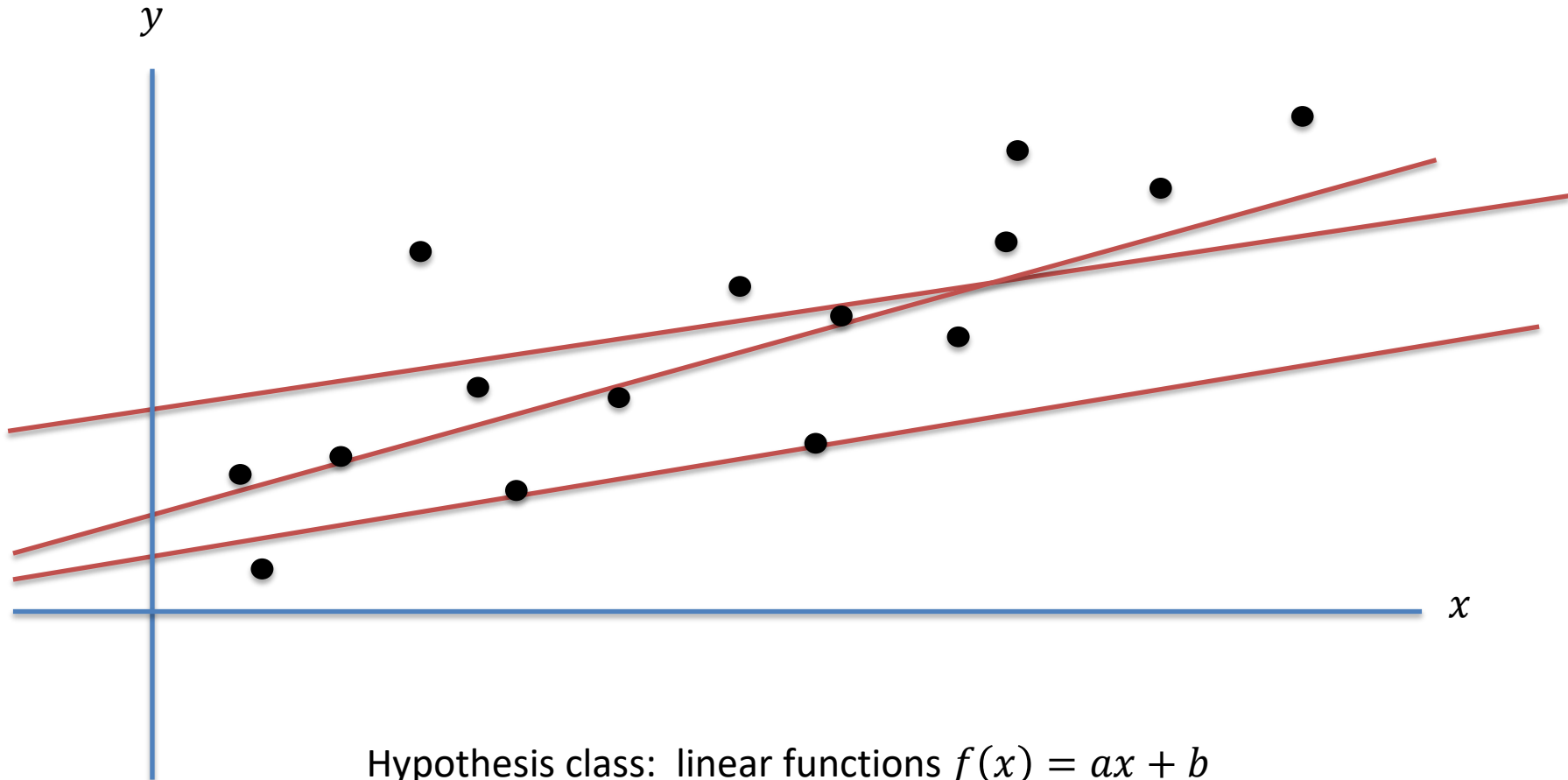
- **Hypothesis space**: set of allowable functions $f: X \rightarrow Y$
- Goal: find the “best” element of the hypothesis space
 - How do we measure the quality of f ?

Examples of Supervised Learning



- Spam email detection
- Handwritten digit recognition
- Stock market prediction
- More?

Regression



How do we measure the quality of the approximation?

Linear Regression



- In typical regression applications, measure the fit using a squared **loss function**

$$L(f) = \frac{1}{M} \sum_m (f(x^{(m)}) - y^{(m)})^2$$

- Want to minimize the average loss on the **training data**
- For 2-D linear regression, the learning problem is then

$$\min_{a,b} \frac{1}{M} \sum_m (ax^{(m)} + b - y^{(m)})^2$$

- For an unseen data point, x , the learning algorithm predicts $f(x)$

- **Select a hypothesis space** (elements of the space are represented by a collection of parameters)
- **Choose a loss function** (evaluates quality of the hypothesis as a function of its parameters)
- **Minimize loss function using gradient descent** (minimization over the parameters)
- **Evaluate quality of the learned model using test data** – that is, data on which the model was not trained

Binary Classification



- Regression operates over a continuous set of outcomes
- Suppose that we want to learn a function $f: \{0,1\}^3 \rightarrow \{0,1\}$
- As an example:

	x_1	x_2	x_3	y
1	0	0	1	0
2	0	1	0	1
3	1	1	0	1
4	1	1	1	0

How do we pick the hypothesis space?

How do we find the best f in this space?

Binary Classification



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- As an example:

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How many functions with three binary inputs and one binary output are there?

Binary Classification



	x_1	x_2	x_3	y
	0	0	0	?
1	0	0	1	0
2	0	1	0	1
	0	1	1	?
	1	0	0	?
	1	0	1	?
3	1	1	0	1
4	1	1	1	0

2^8 possible functions

2^4 are consistent with the observations

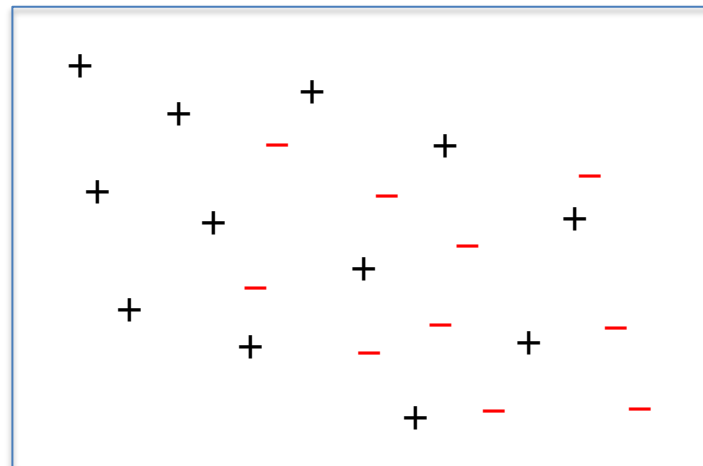
How do we choose the best one?

What if the observations are noisy?

Binary Classification



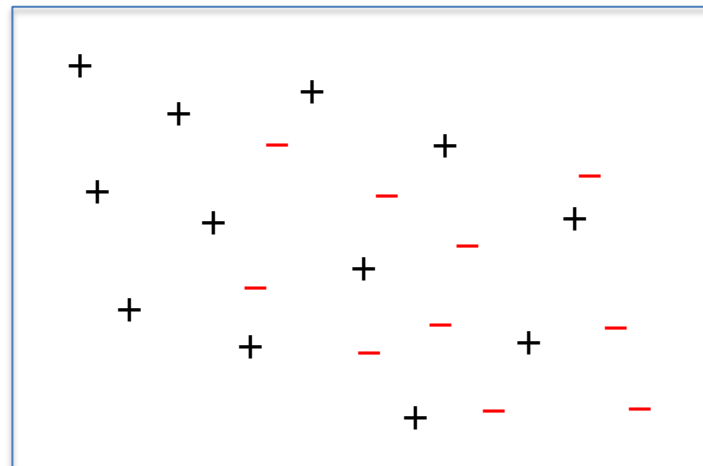
- Input $(x^{(1)}, y^{(1)}), \dots, (x^{(M)}, y^{(M)})$ with $x^{(m)} \in \mathbb{R}^n$ and $y^{(m)} \in \{-1, +1\}$
- We can think of the observations as points in \mathbb{R}^n with an associated sign (either +/- corresponding to 0/1)
- An example with $n = 2$



Binary Classification



- Input $(x^{(1)}, y^{(1)}), \dots, (x^{(M)}, y^{(M)})$ with $x^{(m)} \in \mathbb{R}^n$ and $y^{(m)} \in \{-1, +1\}$
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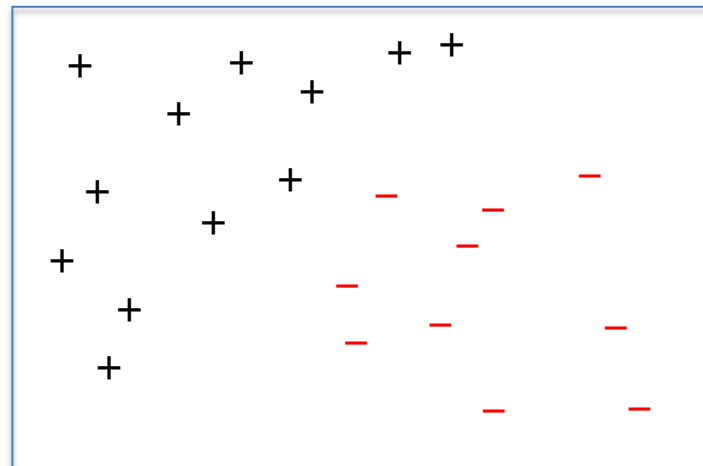


What is a good hypothesis space for this problem?

Binary Classification



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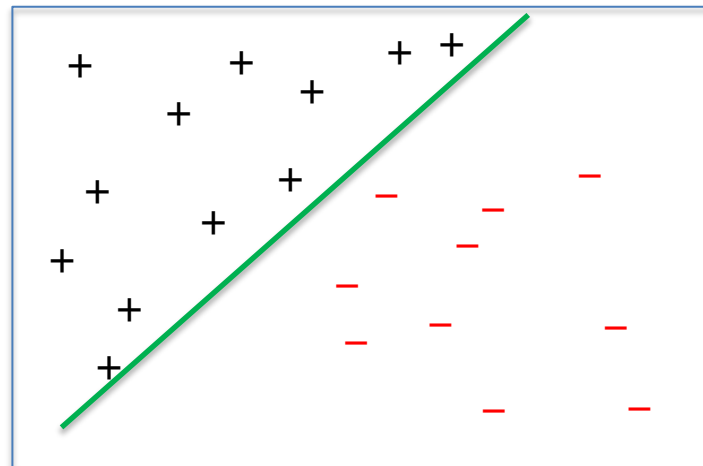


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Binary Classification



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In this case, we say that the observations are **linearly separable**

- In n dimensions, a hyperplane is a solution to the equation

$$w^T x + b = 0$$

with $w \in \mathbb{R}^n, b \in \mathbb{R}$

- Hyperplanes divide \mathbb{R}^n into two distinct sets of points (called open halfspaces)

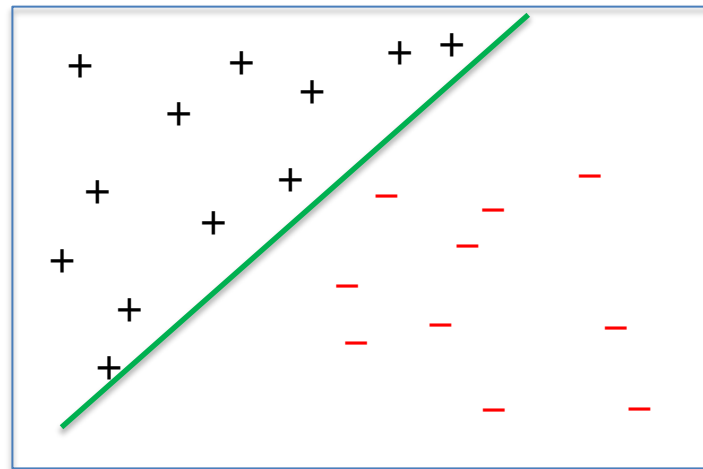
$$w^T x + b > 0$$

$$w^T x + b < 0$$

Binary Classification



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The Linearly Separable Case



- Input $(x^{(1)}, y^{(1)}), \dots, (x^{(M)}, y^{(M)})$ with $x^{(m)} \in \mathbb{R}^n$ and $y^{(m)} \in \{-1, +1\}$

- Hypothesis space: separating hyperplanes

$$f(x) = \text{sign}(w^T x + b)$$

- How should we choose the loss function?

The Linearly Separable Case



- Input $(x^{(1)}, y^{(1)}), \dots, (x^{(M)}, y^{(M)})$ with $x^{(m)} \in \mathbb{R}^n$ and $y^{(m)} \in \{-1, +1\}$

- Hypothesis space: separating hyperplanes

$$f(x) = \text{sign}(w^T x + b)$$

- How should we choose the loss function?

- Count the number of misclassifications

$$\text{loss} = \sum_m |y^{(m)} - \text{sign}(w^T x^{(m)} + b)|$$

- Tough to optimize, gradient contains no information

The Linearly Separable Case



- Input $(x^{(1)}, y^{(1)}), \dots, (x^{(M)}, y^{(M)})$ with $x^{(m)} \in \mathbb{R}^n$ and $y^{(m)} \in \{-1, +1\}$

- Hypothesis space: separating hyperplanes

$$f(x) = \text{sign}(w^T x + b)$$

- How should we choose the loss function?
 - Penalize misclassification linearly by the size of the violation

$$\text{perceptron loss} = \sum_m \max\{0, -y^{(m)}(w^T x^{(m)} + b)\}$$

- Modified hinge loss (this loss is **convex**, but not differentiable)

The Perceptron Algorithm

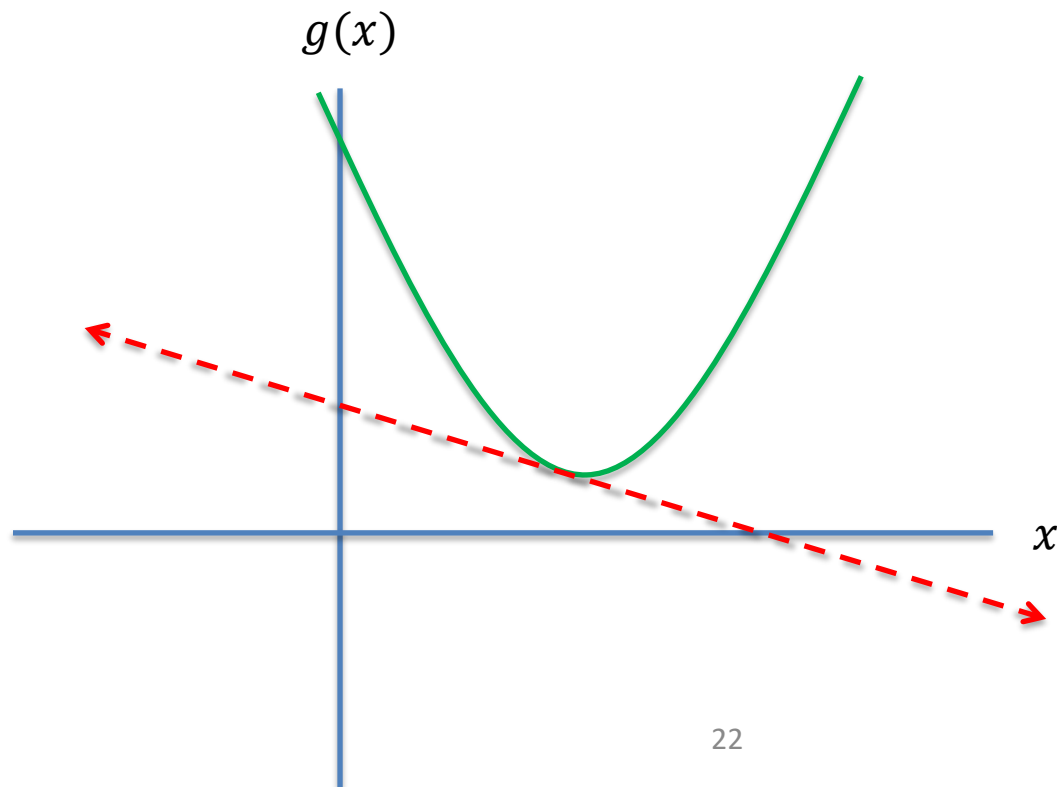


- Try to minimize the perceptron loss using gradient descent
 - The perceptron loss isn't differentiable, how can we apply gradient descent?
 - Need a generalization of what it means to be the gradient of a **convex** function

Gradients of Convex Functions



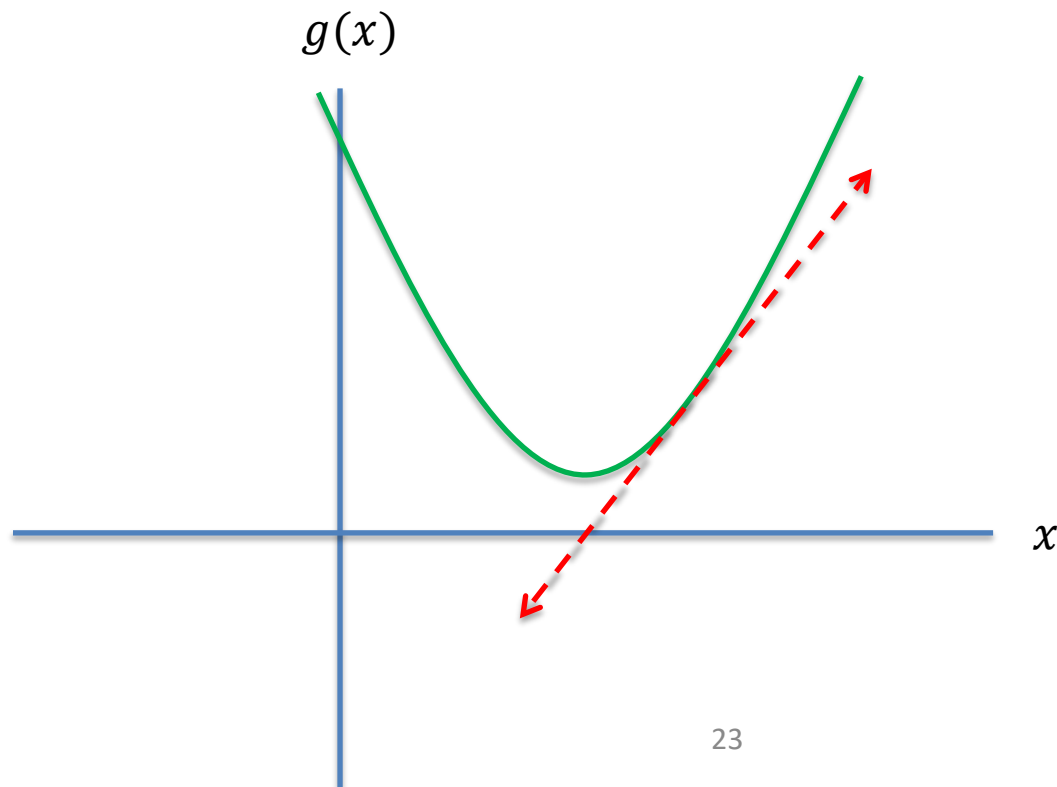
- For a differentiable convex function $g(x)$ its gradients yield **linear underestimators**



Gradients of Convex Functions



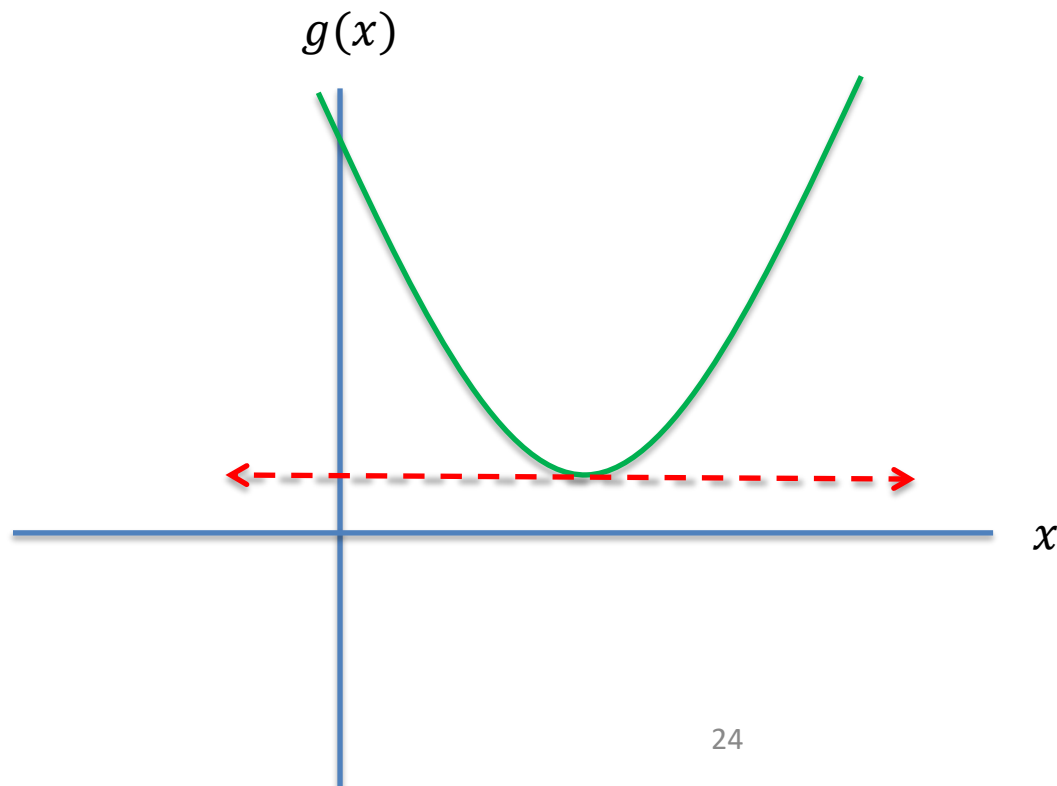
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Gradients of Convex Functions



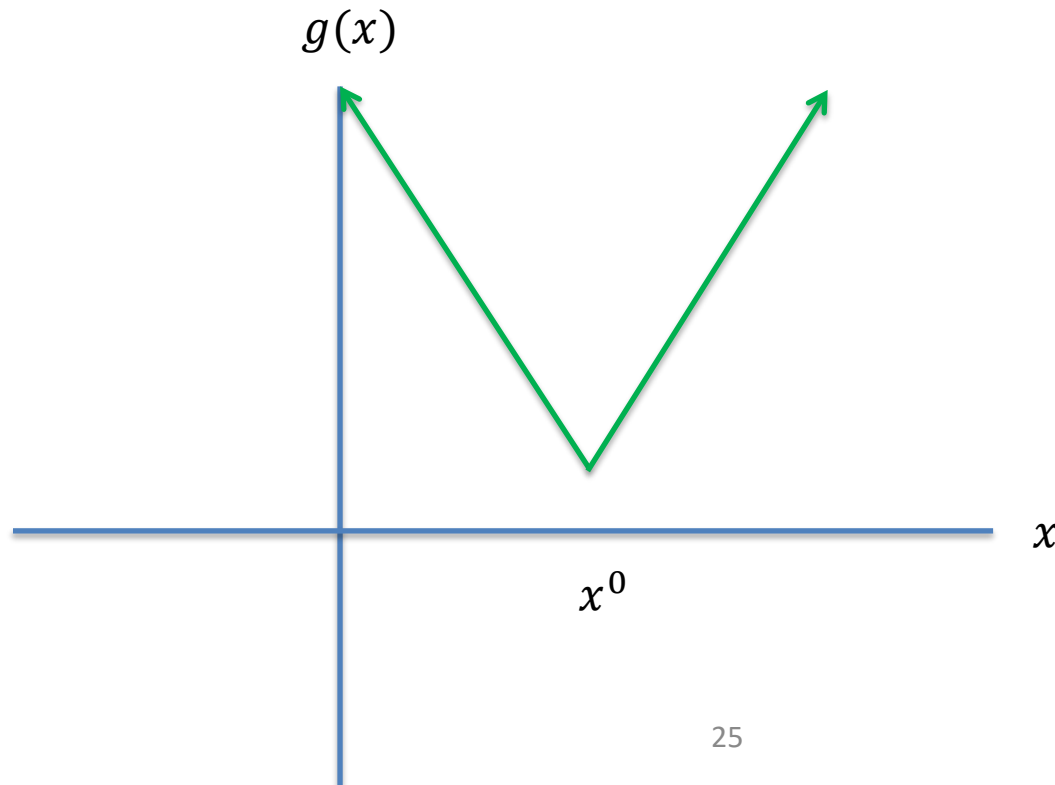
- For a differentiable convex function $g(x)$ its gradients yield **linear underestimators**: zero gradient corresponds to a global optimum



Subgradients



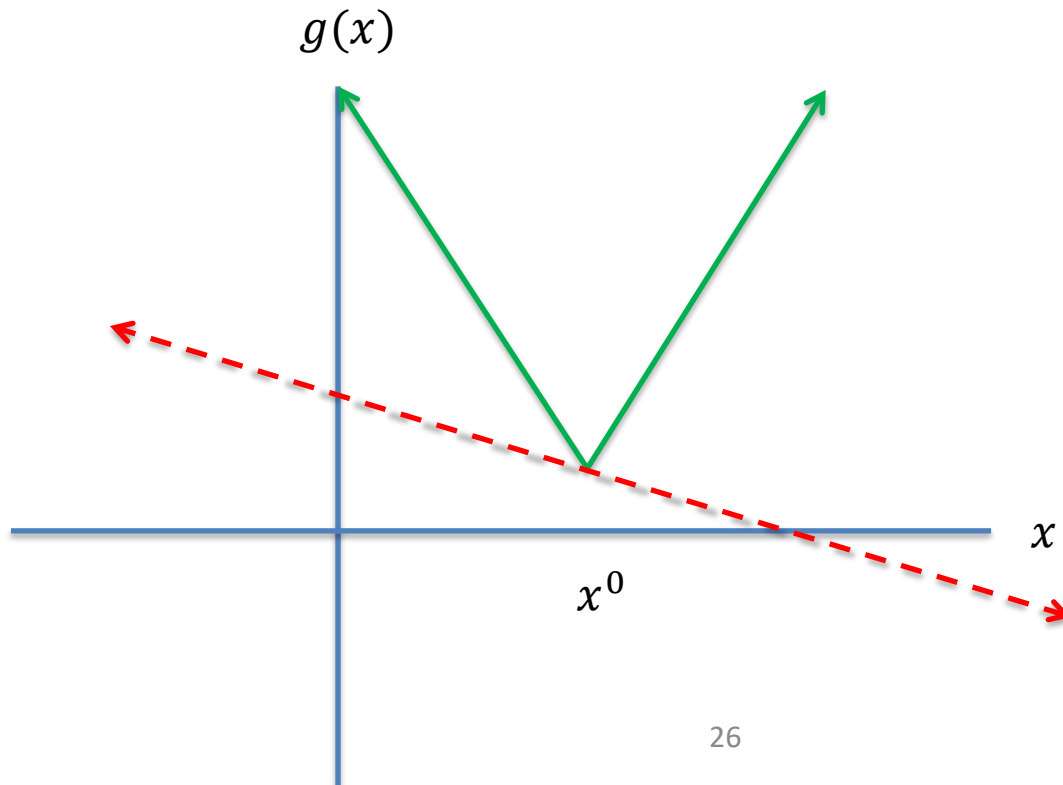
- For a convex function $g(x)$, a **subgradient** at a point x^0 is given by any line, l , such that $l(x^0) = g(x^0)$ and $l(x) \leq g(x)$ for all x , i.e., it is a linear underestimator



Subgradients



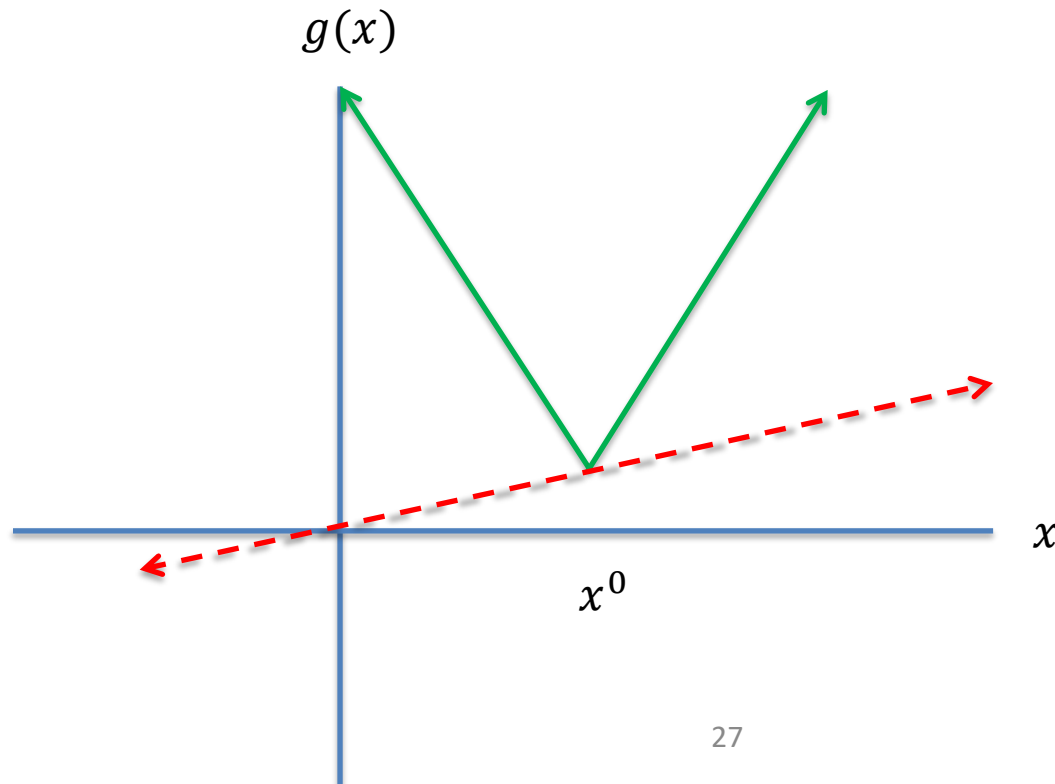
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Subgradients



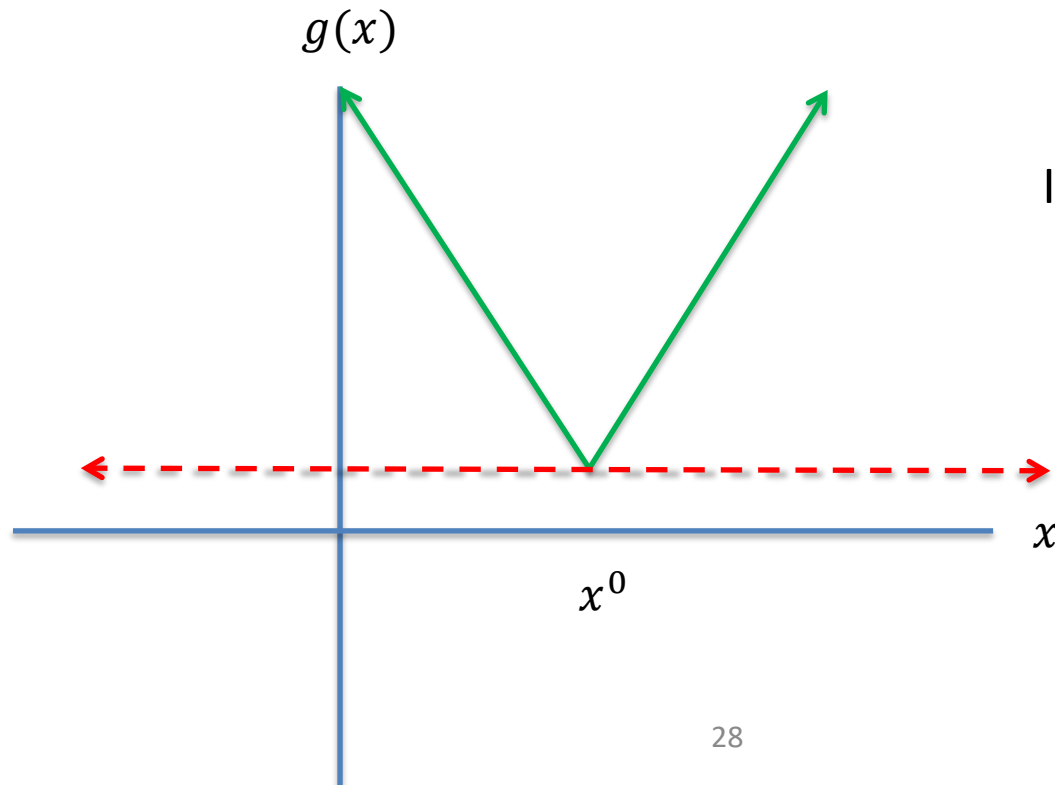
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Subgradients



- For a convex function $g(x)$, a **subgradient** at a point x^0 is given by any line, l , such that $l(x^0) = g(x^0)$ and $l(x) \leq g(x)$ for all x , i.e., it is a linear underestimator



If $\vec{0}$ is a subgradient at x^0 , then x^0 is a global minimum

- If a convex function is differentiable at a point x , then it has a unique subgradient at the point x given by the gradient
- If a convex function is not differentiable at a point x , it can have many subgradients
 - E.g., the set of subgradients of the convex function $|x|$ at the point $x = 0$ is given by the set of slopes $[-1,1]$
- Subgradients only guaranteed to exist for convex functions

The Perceptron Algorithm



- Try to minimize the perceptron loss using (sub)gradient descent

The Perceptron Algorithm



- Try to minimize the perceptron loss using (sub)gradient descent

$$\nabla_w(\text{perceptron loss}) = - \sum_{m=1}^M \left(y^{(m)} x^{(m)} \cdot \mathbf{1}_{-y^{(m)} f_{w,b}(x^{(m)}) \geq 0} \right)$$

$$\nabla_b(\text{perceptron loss}) = - \sum_{m=1}^M \left(y^{(m)} \cdot \mathbf{1}_{-y^{(m)} f_{w,b}(x^{(m)}) \geq 0} \right)$$

The Perceptron Algorithm



- Try to minimize the perceptron loss using (sub)gradient descent

$$\nabla_w(\text{perceptron loss}) = - \sum_{m=1}^M \left(y^{(m)} x^{(m)} \cdot \mathbf{1}_{-y^{(m)}(w^T x^{(m)} + b) \geq 0} \right)$$

$$\nabla_b(\text{perceptron loss}) = - \sum_{m=1}^M \left(y^{(m)} \cdot \mathbf{1}_{-y^{(m)}(w^T x^{(m)} + b) \geq 0} \right)$$

Is equal to zero if the m^{th} data point is correctly classified and one otherwise

The Perceptron Algorithm



- Try to minimize the perceptron loss using (sub)gradient descent

$$w^{(t+1)} = w^{(t)} + \gamma_t \sum_{m=1}^M \left(y^{(m)} x^{(m)} \cdot \mathbf{1}_{-y^{(m)}(w^T x^{(m)} + b) \geq 0} \right)$$

$$b^{(t+1)} = b^{(t)} + \gamma_t \sum_{m=1}^M \left(y^{(m)} \cdot \mathbf{1}_{-y^{(m)}(w^T x^{(m)} + b) \geq 0} \right)$$

- With step size γ_t (also called the learning rate)
- Note that, for convergence of subgradient methods, a diminishing step size, e.g., $\gamma_t = \frac{1}{1+t}$ is required

- To make the training more practical, **stochastic (sub)gradient descent** is often used instead of standard gradient descent
- Approximate the gradient of a sum by sampling a few indices (as few as one) uniformly at random and averaging

$$\nabla_x \left[\sum_{m=1}^M g_m(x) \right] \approx \frac{1}{K} \sum_{k=1}^K \nabla_x g_{m_k}(x)$$

here, each m_k is sampled uniformly at random from $\{1, \dots, M\}$

- Stochastic gradient descent converges to the global optimum under certain assumptions on the step size

- Setting $K = 1$, we pick a random observation m and perform the following update

if the m^{th} data point is misclassified:

$$w^{(t+1)} = w^{(t)} + \gamma_t y^{(m)} x^{(m)}$$

$$b^{(t+1)} = b^{(t)} + \gamma_t y^{(m)}$$

if the m^{th} data point is correctly classified:

$$w^{(t+1)} = w^{(t)}$$

$$b^{(t+1)} = b^{(t)}$$

- Sometimes, you will see the perceptron algorithm specified with $\gamma_t = 1$ for all t

Applications of Perceptron

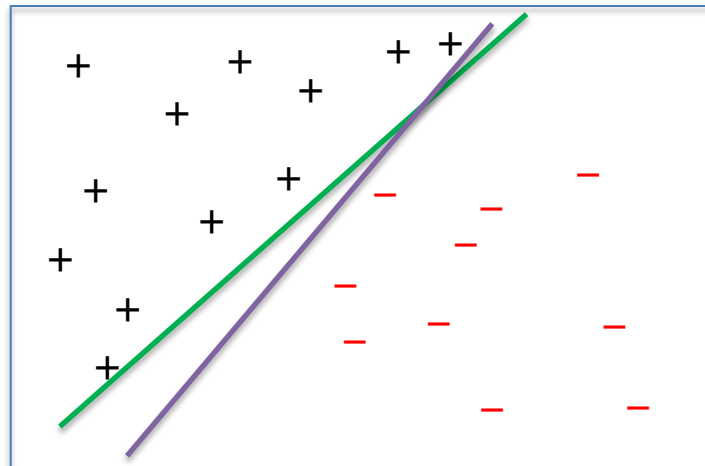


- Spam email classification
 - Represent emails as vectors of counts of certain words (e.g., sir, madam, Nigerian, prince, money, etc.)
 - Apply the perceptron algorithm to the resulting vectors
 - To predict the label of an unseen email
 - Construct its vector representation, x'
 - Check whether or not $w^T x' + b$ is positive or negative

Perceptron Learning Drawbacks



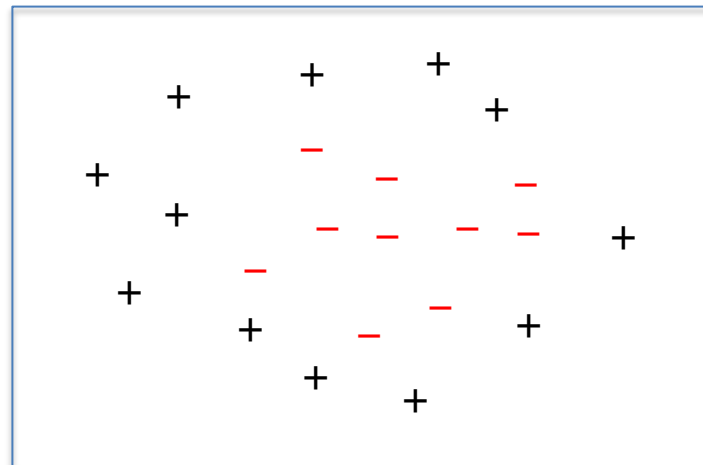
- No convergence guarantees if the observations are not linearly separable
- Can overfit
 - There can be a number of perfect classifiers, but the perceptron algorithm doesn't have any mechanism for choosing between them



What If the Data Isn't Separable?



- Input $(x^{(1)}, y^{(1)}), \dots, (x^{(M)}, y^{(M)})$ with $x^{(m)} \in \mathbb{R}^n$ and $y^{(m)} \in \{-1, +1\}$
- We can think of the observations as points in \mathbb{R}^n with an associated sign (either +/- corresponding to 0/1)
- An example with $n = 2$

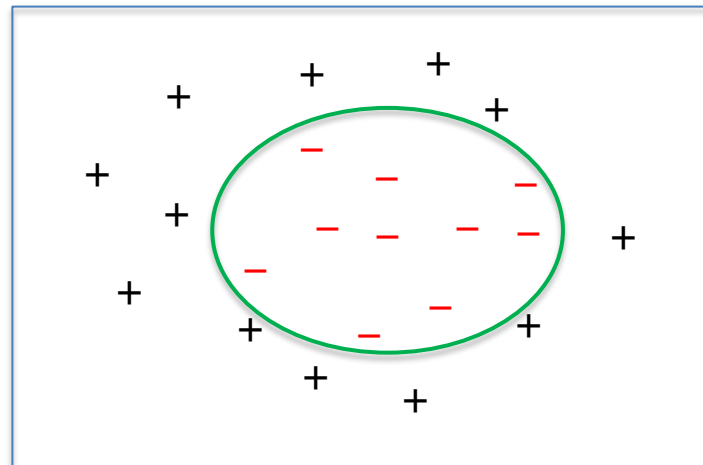


What is a good hypothesis space for this problem?

What If the Data Isn't Separable?



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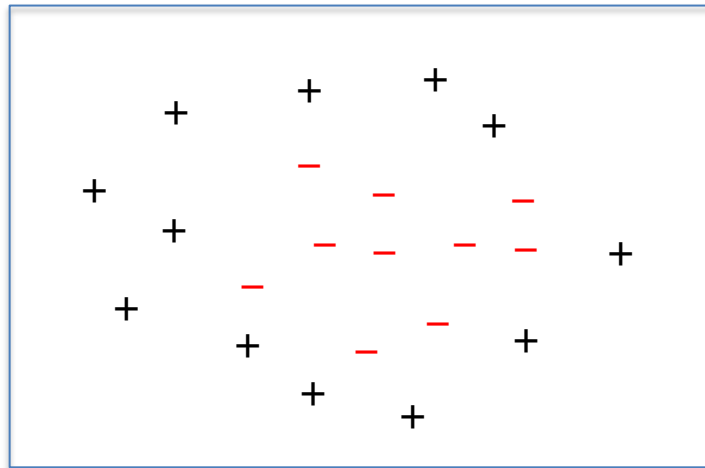


What is a good hypothesis space for this problem?

Adding Features



- Perceptron algorithm only works for linearly separable data



Can add **features** to make the data linearly separable in a higher dimensional space!

Essentially the same as higher order polynomials for linear regression!

- The idea, choose a feature map $\phi: \mathbb{R}^n \rightarrow \mathbb{R}^k$
 - Given the observations $x^{(1)}, \dots, x^{(M)}$, construct feature vectors $\phi(x^{(1)}), \dots, \phi(x^{(M)})$
 - Use $\phi(x^{(1)}), \dots, \phi(x^{(M)})$ instead of $x^{(1)}, \dots, x^{(M)}$ in the learning algorithm
 - Goal is to choose ϕ so that $\phi(x^{(1)}), \dots, \phi(x^{(M)})$ are linearly separable in \mathbb{R}^k
 - Learn linear separators of the form $w^T \phi(x)$ (instead of $w^T x$)
- **Warning:** more expressive features can lead to overfitting!

Adding Features: Examples



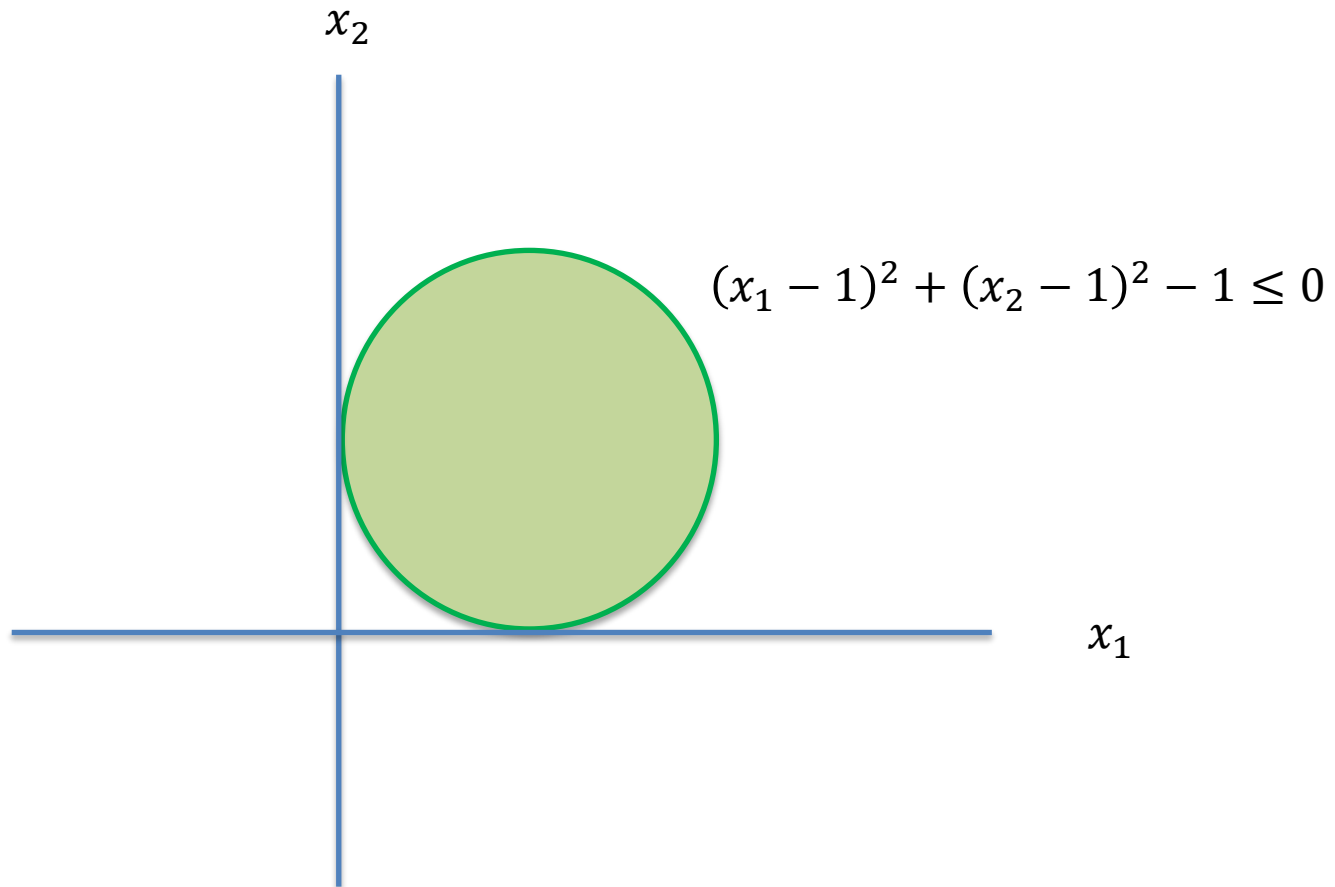
- $\phi \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

- This is just the input data, without modification

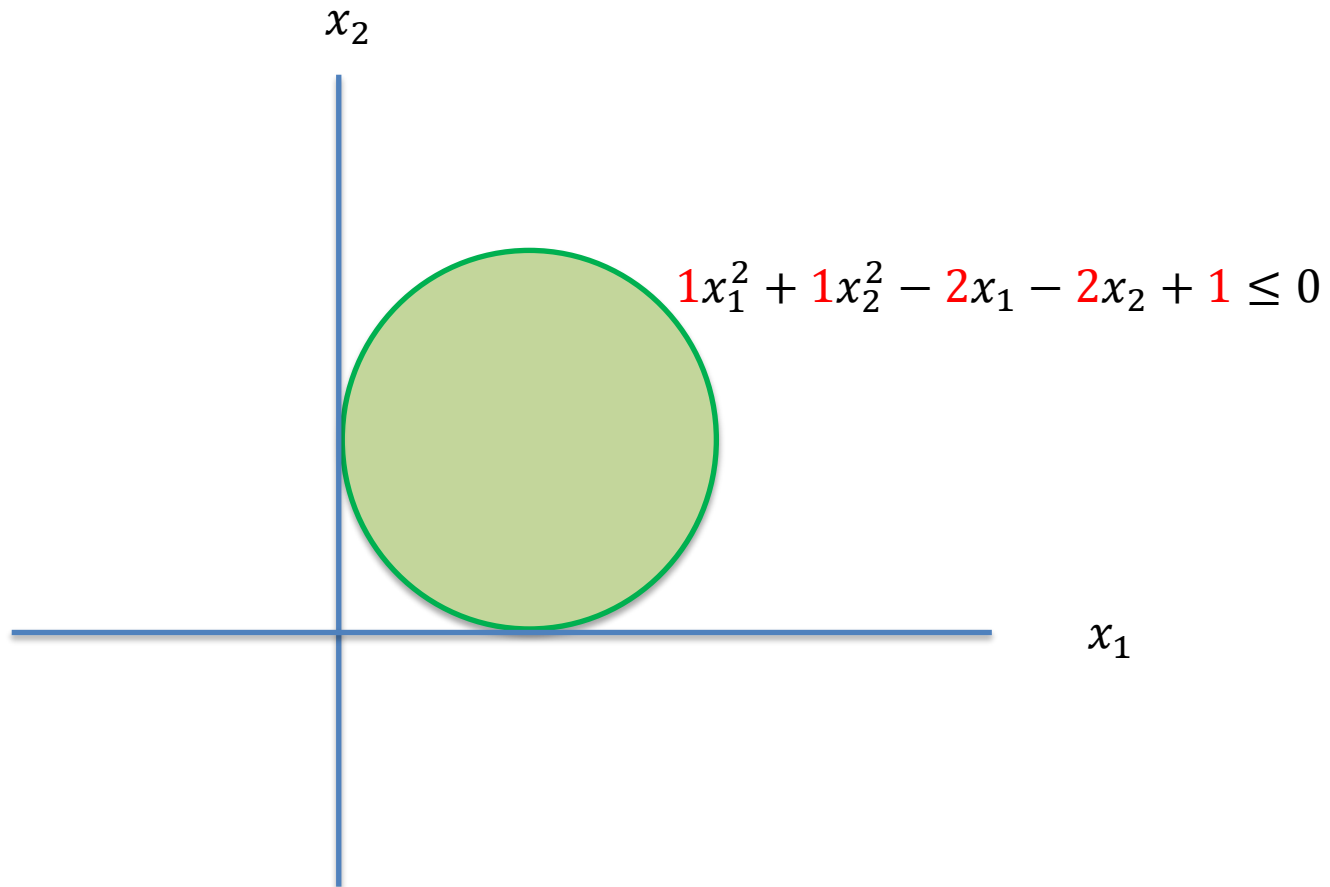
- $\phi \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ x_1 \\ x_2 \\ x_1^2 \\ x_2^2 \end{bmatrix}$

- This corresponds to a second degree polynomial separator, or equivalently, elliptical separators in the original space

Adding Features



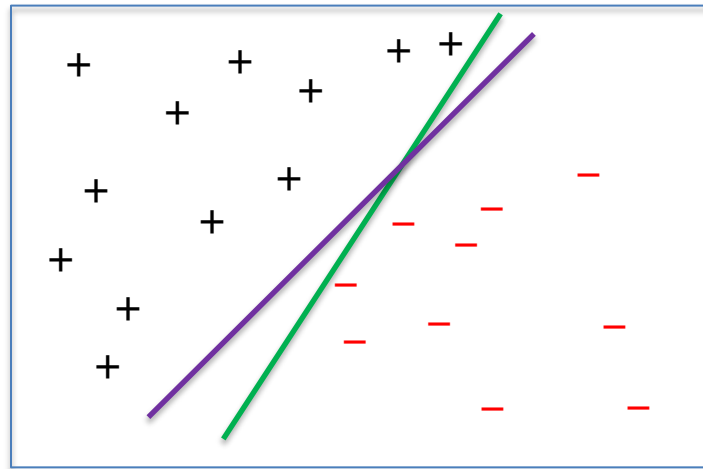
Adding Features



Support Vector Machines



- How can we decide between two perfect classifiers?



- What is the practical difference between these two solutions?