UT DALLAS

CS 6347

Lecture 3

Bayesian Networks

Chain Rule

$$p(x_1, x_2) = p(x_1)p(x_2|x_1)$$

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$$p(x_1, \dots, x_n) = p(x_1)p(x_2|x_1) \dots p(x_n|x_1, \dots, x_{n-1})$$

Structured Distributions

- Consider a general joint distribution $p(X_1, \dots, X_n)$ over binary valued random variables
- If $X_1, \dots, X_n$ are all independent given a different random variable Y, then

$$p(x_1, \dots, x_n|y) = p(x_1|y) \dots p(x_n|y)$$

and

$$p(y, x_1, \dots, x_n) = p(y)p(x_1|y) \dots p(x_n|y)$$

- How much storage is needed to represent this model?

Structured Distributions

- Consider a different joint distribution $p(X_1, \dots, X_n)$ over binary valued random variables
- Suppose, for $i > 2$, X_i is independent of $X_1, \dots, X_{i-2}$ given X_{i-1}

$$p(x_1, \dots, x_n) = p(x_1)p(x_2|x_1) \dots p(x_n|x_1, \dots, x_{n-1})$$
$$= p(x_1)p(x_2|x_1)p(x_3|x_2) \dots p(x_n|x_{n-1})$$

- How much storage is needed to represent this model?
- This distribution is chain-like

Bayesian Network

- A **Bayesian network** is a directed graphical model that captures independence relationships of a given probability distribution
  - Directed acyclic graph (DAG), $G = (V, E)$
  - One node for each random variable
  - One conditional probability distribution per node
  - Directed edge represents a direct statistical dependence

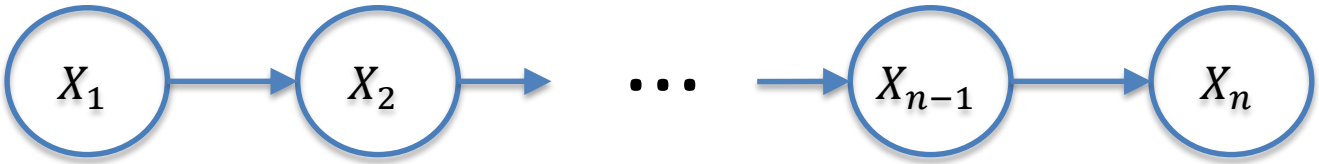
Bayesian Network

- A **Bayesian network** is a directed graphical model that captures independence relationships of a given probability distribution
  - Encodes **local Markov** independence assumptions that each node is independent of its non-descendants given its parents
  - Corresponds to a **factorization** of the joint distribution

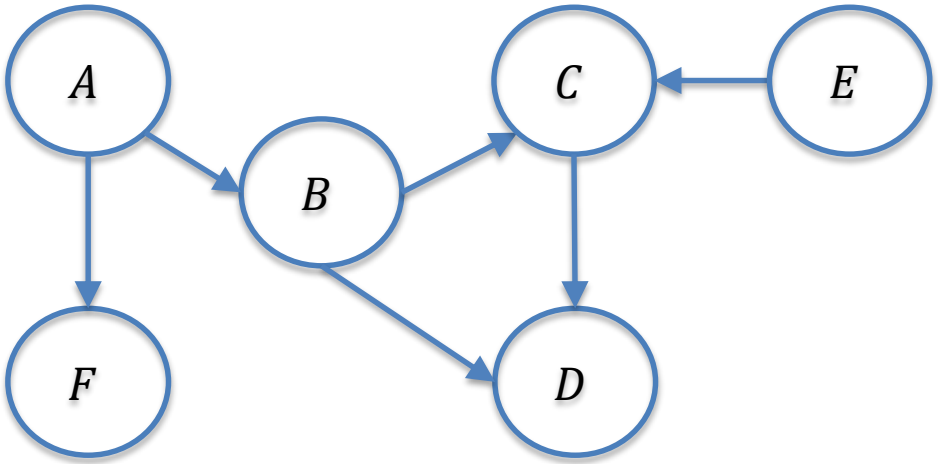
$$p(x_1, \ldots, x_n) = \prod_i p(x_i | x_{parents(i)})$$

Directed Chain

$$p(x_1, \ldots, x_n) = p(x_1)p(x_2|x_1)p(x_3|x_2) \ldots p(x_n|x_{n-1})$$

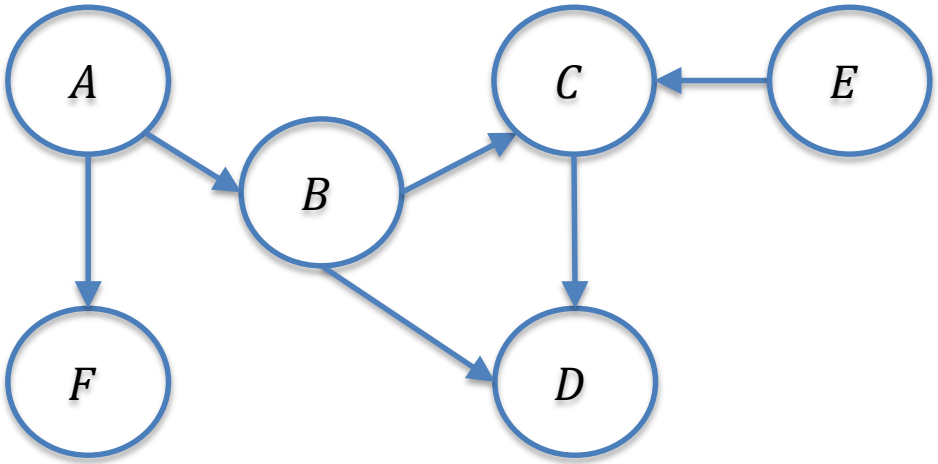


Example:



- Local Markov independence relations?
- Joint distribution?

Example:



- This list is not exhaustive:
  - How can we figure out which independence relationships the model represents?

D-separation

- Independence relationships can be figured out by looking at the graph structure!
  - Easier than looking at the tables and plugging into the definition
- We look at all of the paths from X to Y in the graph and determine whether or not they are **blocked**
  - $X \subset V$ is d-separated from $Y \subset V$ given $Z \subset V$ iff every path from X to Y in the graph is blocked by Z

D-separation

- Three types of situations can occur along any given path

(1) Sequential

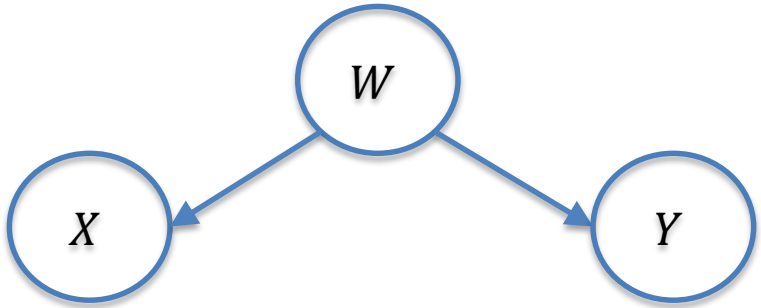
The path from X to Y is blocked if we condition on W

Intuitively, if we condition on W, then information about X does not affect Y and vice versa

D-separation

- Three types of situations can occur along any given path

(2) Divergent



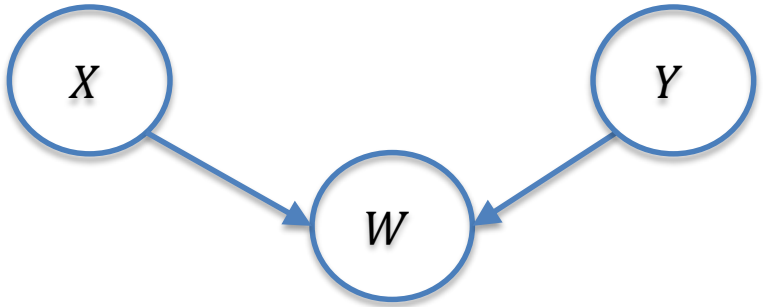
The path from X to Y is blocked if we condition on W

If we don't condition on W, then information about W could effect the probability of observing either X or Y

D-separation

- Three types of situations can occur along any given path

(3) Convergent



The path from X to Y is blocked if we **do not** condition on W or any of its descendants

Conditioning on W couples the variables X and Y: knowing whether or not X occurs impacts the probability that Y occurs

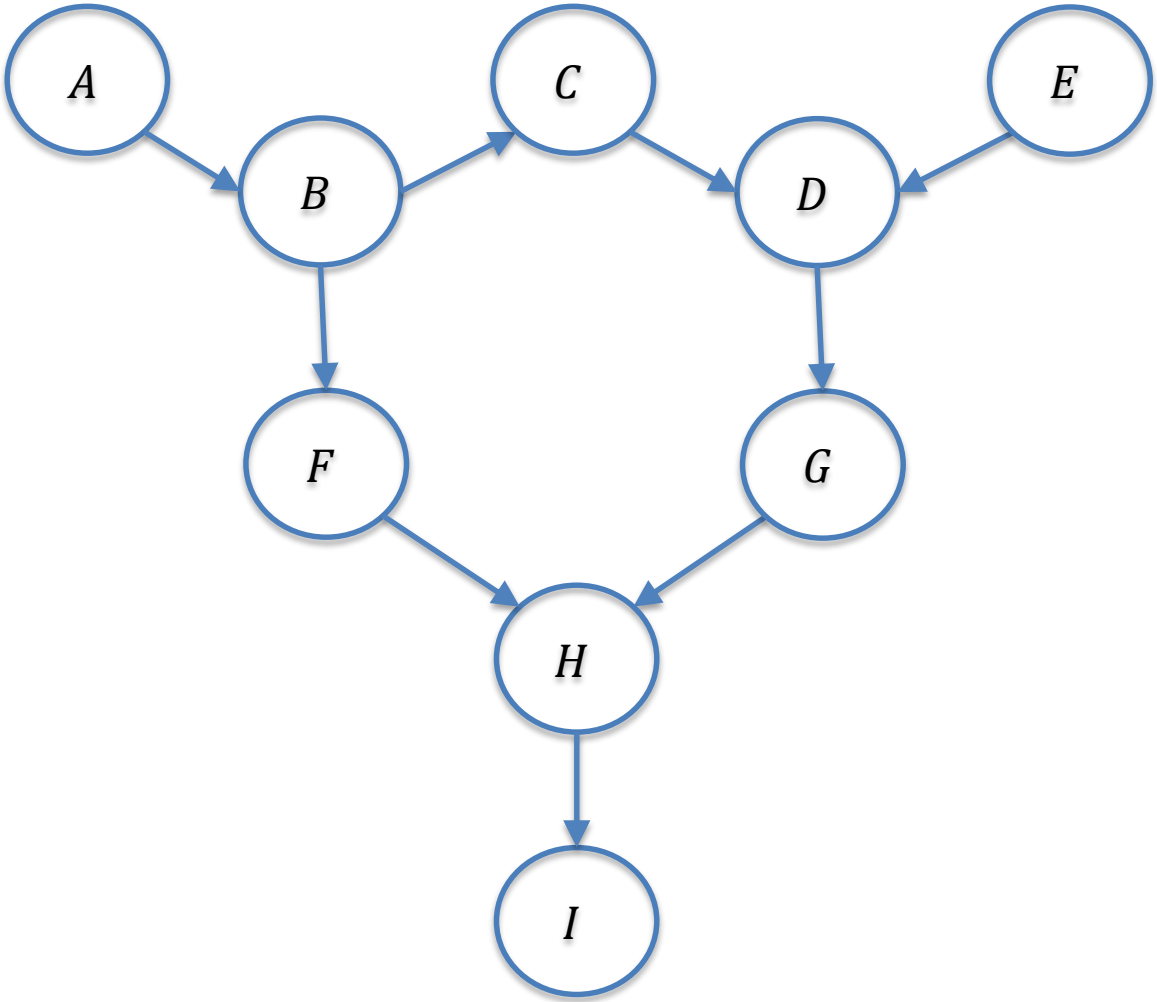
D-separation

- If the joint probability distribution factorizes with respect to the DAG $G = (V, E)$, then X is d-separated from Y given Z implies $X \perp Y \mid Z$
  - We can use this to quickly check independence assertions by using the graph
  - In general, these are only a subset of all independence relationships that are actually present in the joint distribution
  - If X and Y are not d-separated in G given Z, then there is some distribution that factorizes over G in which X and Y dependent

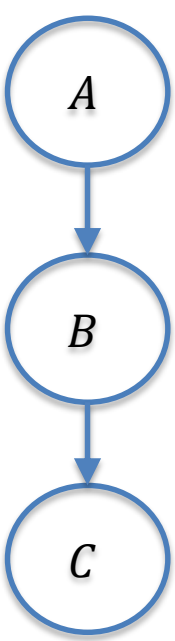
D-separation

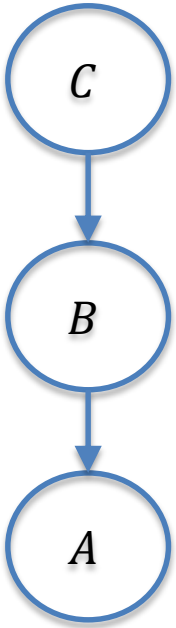
- Let $I(p)$ be the set of all independence relationships in the joint distribution p and $I(G)$ be the set of all independence relationships in the graph G
- We say that G is an I-map for $I(p)$ if $I(G) \subseteq I(p)$
- Theorem: the joint probability distribution, p, factorizes with respect to the DAG $G = (V, E)$ iff G is an I-map for $I(p)$
- An I-map is perfect if $I(G) = I(p)$
  - Not always possible to perfectly represent all of the independence relations with a graph

D-separation Example



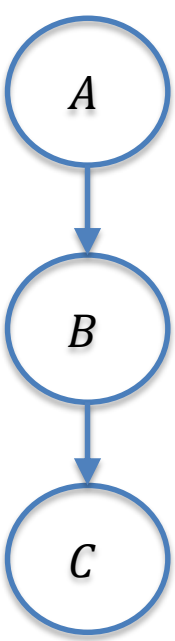
Equivalent Models?

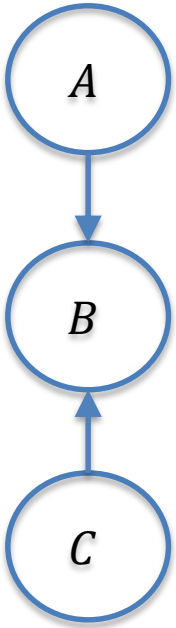




Do these models represent the same independence relations?

Equivalent Models?





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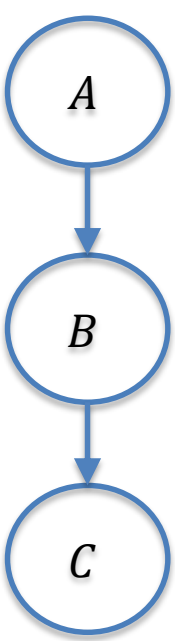
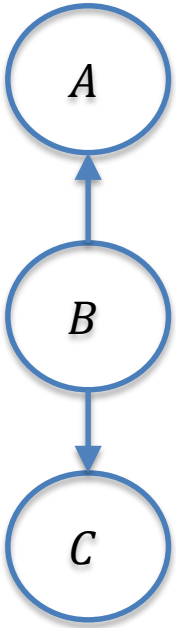
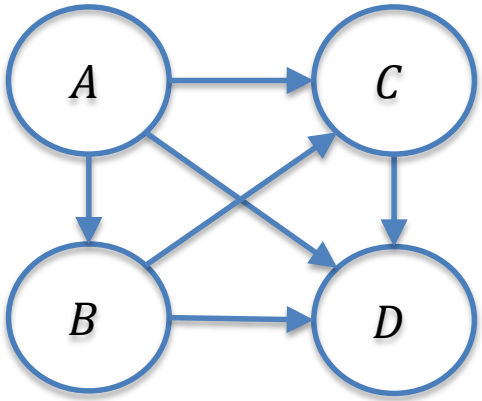
Equivalent Models?





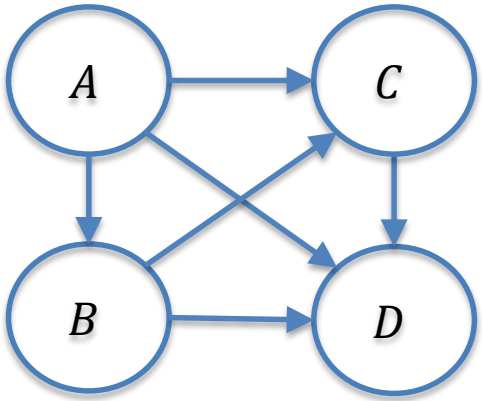
Do these models represent the same independence relations?

I-Maps



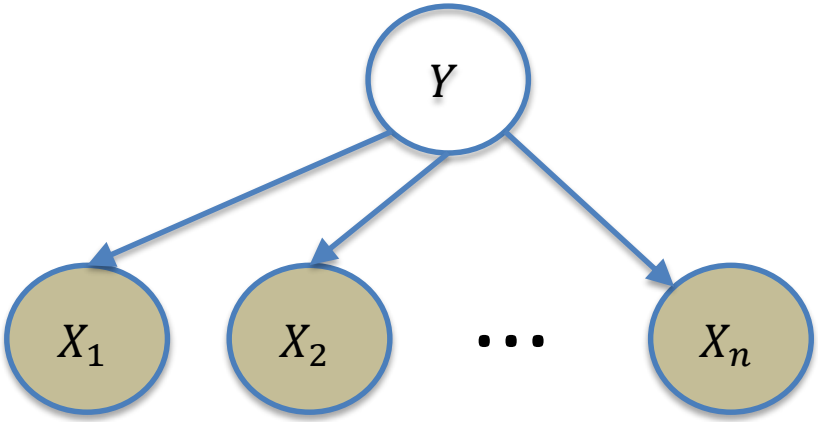
What independence relations does this model imply?

I-Maps



$I(G) = \emptyset$, this is an I-map for any joint distribution on four variables!

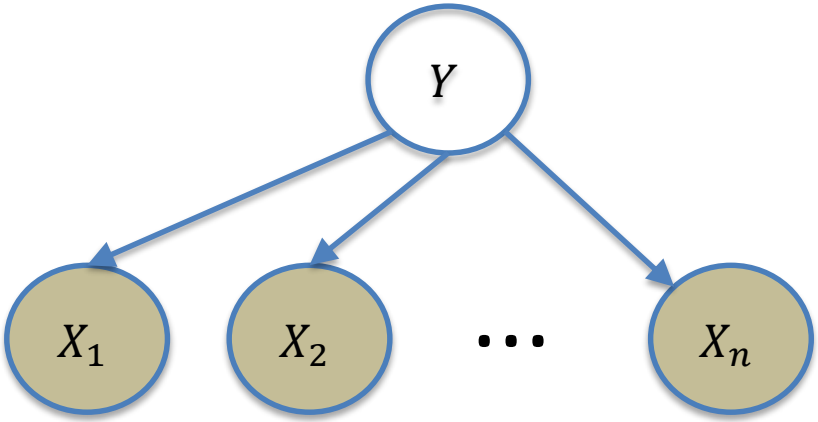
Naïve Bayes



$$p(y, x_1, \ldots, x_n) = p(y)p(x_1|y) \ldots p(x_n|y)$$

- In practice, we often have variables that we observe directly and those that can only be observed indirectly

Naïve Bayes



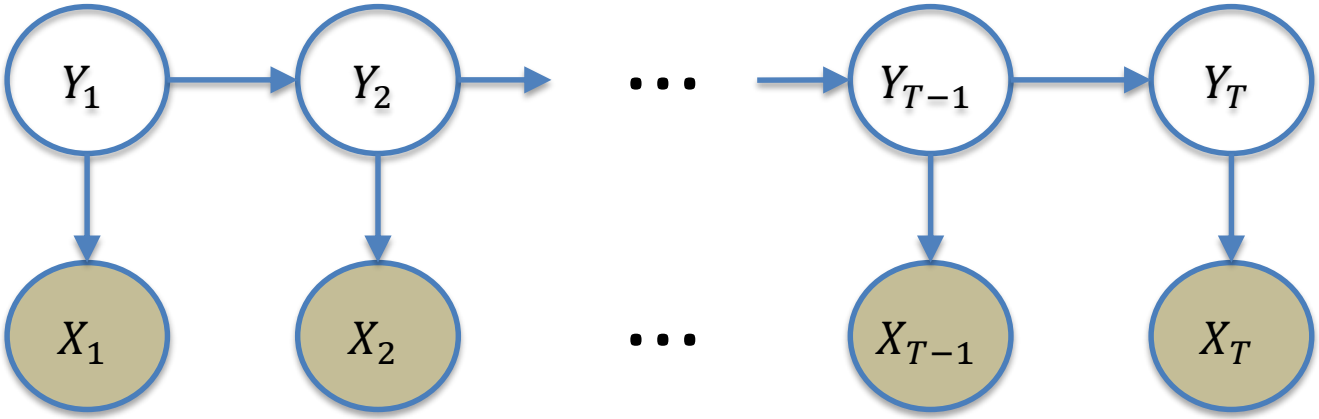
$$p(y, x_1, \ldots, x_n) = p(y)p(x_1|y) \ldots p(x_n|y)$$

- This model assumes that $X_1, \ldots, X_n$ are independent given Y, sometimes called naïve Bayes

Example: Naïve Bayes

- Let Y be a binary random variable indicating whether or not an email is a piece of spam
- For each word in the dictionary, create a binary random variable X_i indicating whether or not word i appears in the email
- For simplicity, we will assume that $X_1, \ldots, X_n$ are independent given Y
- How do we compute the probability that an email is spam?

Hidden Markov Models



$$p(x_1, \ldots, x_T, y_1, \ldots, y_T) = p(y_1)p(x_1|y_1)\prod_t p(y_t|y_{t-1})p(x_t|y_t)$$

- Used in coding, speech recognition, etc.
- Independence assertions?