

Statistical Methods in AI and ML

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The Course

One of the **most exciting** advances in AI/ML in the last decade

Judea Pearl won the Turing award for his work on Bayesian networks!

(among other achievements)

Prob. Graphical Models

Exploit **locality** and structural features of a given model in order to gain insight about **global properties**

The Course

- **What this course is:**
 - Probabilistic graphical models
 - Topics:
 - representing data
 - exact and approximate statistical inference
 - model learning
 - variational methods in ML

The Course

- **What you should be able to do at the end:**
 - **Design statistical models for applications in your domain of interest**
 - **Apply learning and inference algorithms to solve real problems (exactly or approximately)**
 - **Understand the complexity issues involved in the modeling decisions and algorithmic choices**

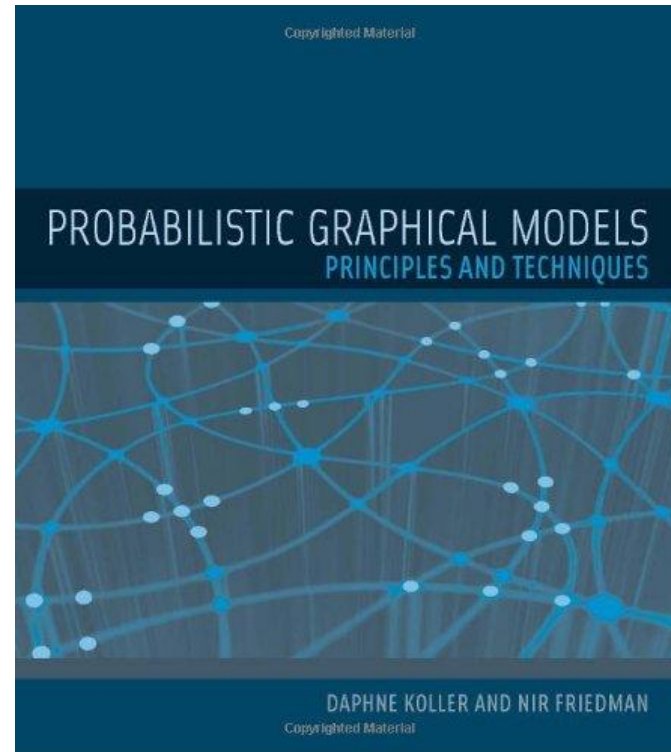
Prerequisites

- **CS 5343: Algorithm Analysis and Data Structures**
- **CS 3341: Probability and Statistics in Computer Science and Software Engineering**
- **Basically, comfort with probability and algorithms (machine learning is helpful, but not required)**

Suggested Textbook

Readings will be posted
online before each lecture

Check the course website
for additional resources and
papers



Textbook

- In addition, lecture notes, in book format, will (hopefully) be made available for each topic
- The idea is to build a set of notes that aligns well with the presentation of course material
- Comments, suggestions, corrections are welcome/encouraged

Grading

- **4-6 problem sets (70%)**
 - See collaboration policy on the web
- **Final project (25%)**
- **Class/Piazza participation & extra credit (5%)**

-subject to change-

Course Info.

- **Instructor: Nicholas Ruozzi**
 - Office: ECSS 3.409
 - Office hours: Tues. 10am – 11am and by appointment
- **TA: TBD**
 - Office hours and location TBD
- **Course website:**
<http://www.utdallas.edu/~nrr150130/cs6347/2017sp/>

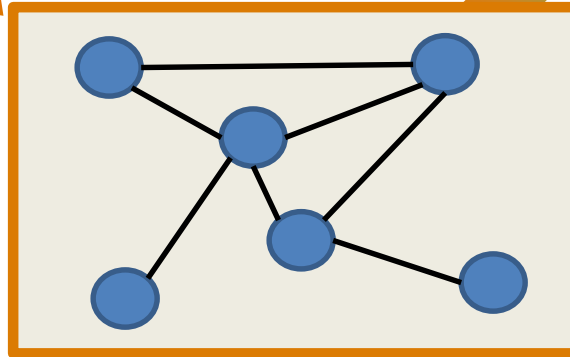
Main Ideas

- **Model the world (or at least the problem) as a collection of random variables related through some joint probability distribution**
 - **Compactly represent the distribution**
 - **Undirected graphical models**
 - **Directed graphical models**
- **Learn the distribution from observed data**
 - **Maximum likelihood, SVMs, etc.**
- **Make predictions (statistical inference)**

Inference and Learning

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49 721 59 722 59 723 59 724 29 725 59 726 59 727 59 728 59 729 59 730 49 731
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Collect Data



**“Learn” a model
that represents the
observed data**

$$Z(\theta) = \sum_x p(x; \theta)$$

**Use the model to
do inference / make
predictions**

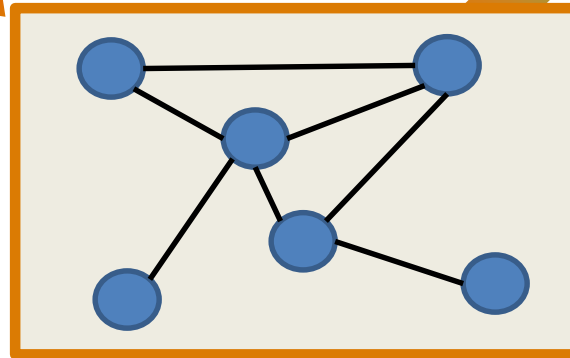
Inference and Learning

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```

**Data sets can
be large**

$$Z(\theta) = \sum_x p(x; \theta)$$

**Inference needs to
be fast**



**Data must be
compactly modeled**

Applications

- **Computer vision**
- **Natural language processing**
- **Robotics**
- **Computational biology**
- **Computational neuroscience**
- **Text translation**
- **Text-to-speech**
- **Many more...**

Graphical Models

- A graphical model is a graph together with "local interactions"
- The graph and interactions model a global optimization or learning problem
- The study of graphical models is concerned with how to exploit local structure to solve these problems either exactly or approximately

Probability Review

Discrete Probability

- **Sample space** specifies the set of possible outcomes
 - For example, $\Omega = \{H, T\}$ would be the set of possible outcomes of a coin flip
- Each element $\omega \in \Omega$ is associated with a number $p(\omega) \in [0,1]$ called a **probability**

$$\sum_{\omega \in \Omega} p(\omega) = 1$$

- For example, a biased coin might have $p(H) = .6$ and $p(T) = .4$

Discrete Probability

- An **event** is a subset of the sample space
 - Let $\Omega = \{1, 2, 3, 4, 5, 6\}$ be the 6 possible outcomes of a dice roll
 - $A = \{1, 5, 6\} \subseteq \Omega$ would be the event that the dice roll comes up as a one, five, or six
- The probability of an event is just the sum of all of the outcomes that it contains
 - $p(A) = p(1) + p(5) + p(6)$

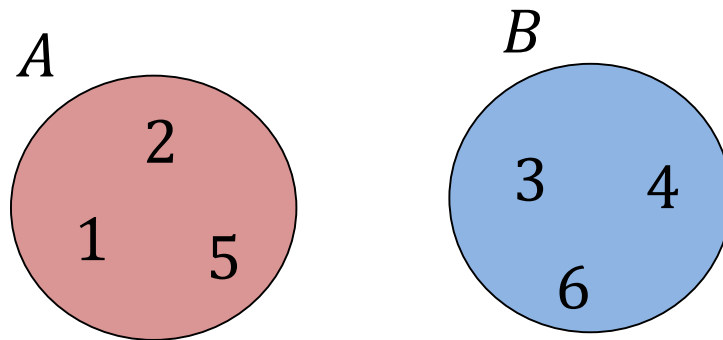
Independence

- Two events A and B are **independent** if

$$p(A \cap B) = p(A)p(B)$$

Let's suppose that we have a fair die: $p(1) = \dots = p(6) = 1/6$

If $A = \{1, 2, 5\}$ and $B = \{3, 4, 6\}$ are A and B independent?



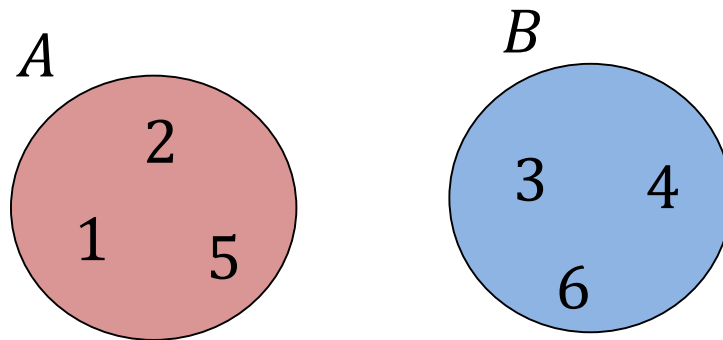
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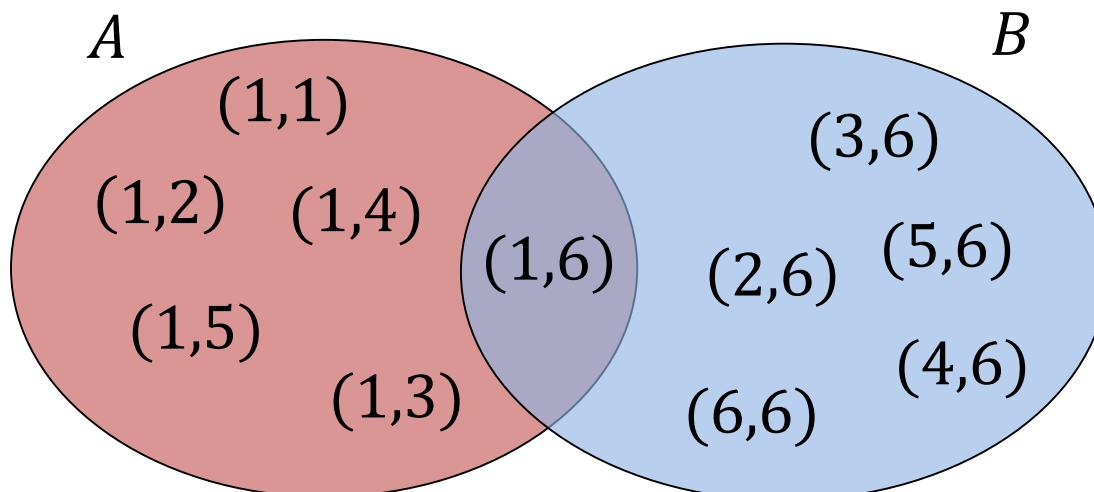


No!

$$p(A \cap B) = 0 \neq \frac{1}{4}$$

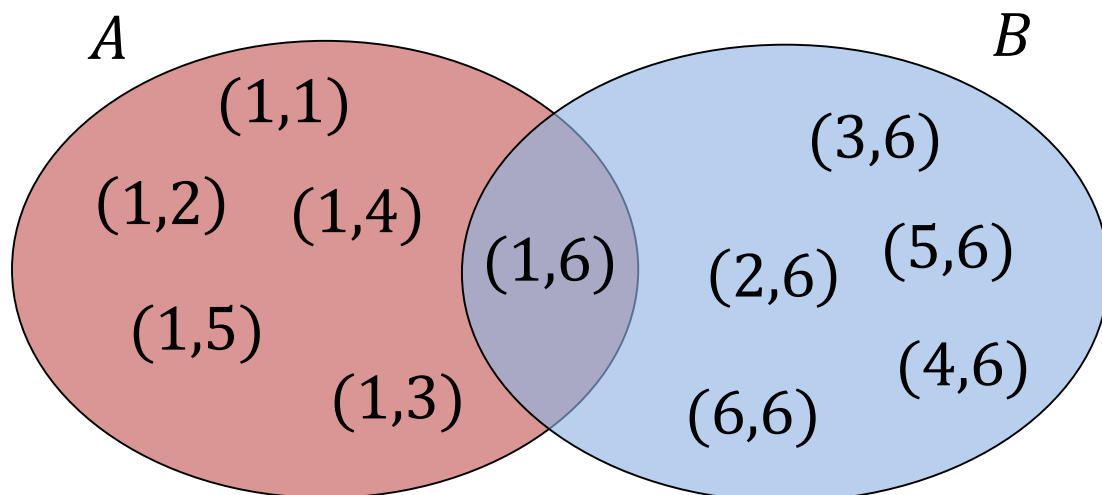
Independence

- Now, suppose that $\Omega = \{(1,1), (1,2), \dots, (6,6)\}$ is the set of all possible rolls of two **unbiased** dice
- Let $A = \{(1,1), (1,2), (1,3), \dots, (1,6)\}$ be the event that the first die is a one and let $B = \{(1,6), (2,6), \dots, (6,6)\}$ be the event that the second die is a six
- Are A and B independent?



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- Are A and B independent?



Yes!

$$p(A \cap B) = \frac{1}{36} = \frac{1}{6} * \frac{1}{6}$$

Conditional Probability

- The **conditional probability** of an event A given an event B with $p(B) > 0$ is defined to be

$$p(A|B) = \frac{p(A \cap B)}{P(B)}$$

- This is the probability of the event $A \cap B$ over the sample space $\Omega' = B$
- Some properties:
 - $\sum_{\omega \in B} p(\omega|B) = 1$
 - If A and B are independent, then $p(A|B) = p(A)$

Simple Example

Cheated	Grade	Probability
Yes	A	.15
Yes	F	.05
No	A	.5
No	F	.3

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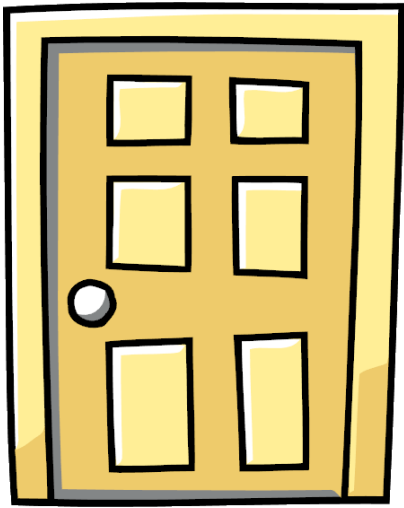
$$p(\text{Cheated} = \text{Yes} \mid \text{Grade} = \text{F}) = ?$$

Simple Example

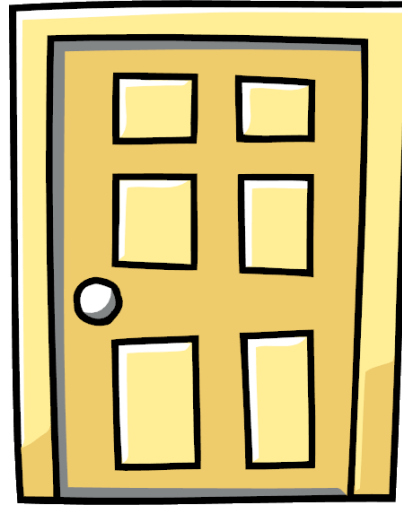
Cheated	Grade	Probability
Yes	A	.15
Yes	F	.05
No	A	.5
No	F	.3

$$p(\text{Cheated} = \text{Yes} | \text{Grade} = \text{F}) = \frac{.05}{.35} \approx .14$$

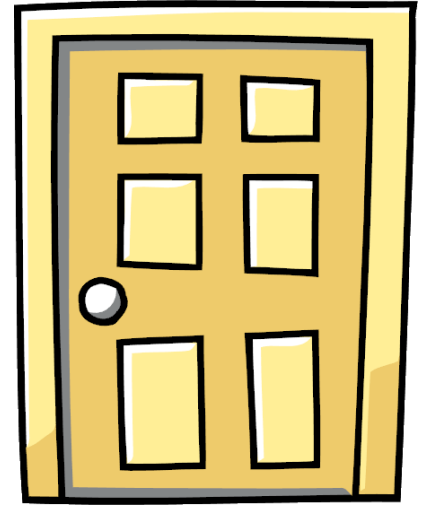
The Monty Hall Problem



1



2



3

Chain Rule

$$p(A \cap B) = p(A)p(B|A)$$

$$\begin{aligned} p(A \cap B \cap C) &= p(A \cap B)p(C|A \cap B) \\ &= p(A)p(B|A)p(C|A \cap B) \end{aligned}$$

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$$p\left(\bigcap_{i=1}^n A_i\right) = p(A_1)p(A_2|A_1) \dots p(A_n|A_1 \cap \dots \cap A_{n-1})$$

Conditional Independence

- Two events A and B are independent if learning something about B tells you nothing about A (and vice versa)
- Two events A and B are **conditionally independent** given C if

$$p(A \cap B|C) = p(A|C)p(B|C)$$

- This is equivalent to

$$p(A|B \cap C) = p(A|C)$$

- That is, given C , information about B tells you nothing about A (and vice versa)

Conditional Independence

- Let $\Omega = \{(H, H), (H, T), (T, H), (T, T)\}$ be the outcomes resulting from tossing two different fair coins
- Let A be the event that the first coin is heads
- Let B be the event that the second coin is heads
- Let C be the event that both coins are heads or both are tails
- A and B are independent, but A and B are not independent given C

Discrete Random Variables

- A discrete **random variable**, X , is a function from the state space Ω into a discrete space D

- For each $x \in D$,

$$p(X = x) \equiv p(\{\omega \in \Omega : X(\omega) = x\})$$

is the probability that X takes the **value** x

- $p(X)$ defines a probability distribution

- $\sum_{x \in D} p(X = x) = 1$

- Random variables partition the state space into disjoint events

Example: Pair of Dice

- Let Ω be the set of all possible outcomes of rolling a pair of dice
- Let p be the uniform probability distribution over all possible outcomes in Ω
- Let $X(\omega)$ be equal to the sum of the value showing on the pair of dice in the outcome ω
 - $p(X = 2) = ?$
 - $p(X = 8) = ?$

Example: Pair of Dice

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$$- p(X = 2) = \frac{1}{36}$$

$$- p(X = 8) = ?$$

Example: Pair of Dice

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- Let $X(\omega)$ be equal to the sum of the value showing on the pair of dice in the outcome ω

$$- p(X = 2) = \frac{1}{36}$$

$$- p(X = 8) = \frac{5}{36}$$

Discrete Random Variables

- We can have vectors of random variables as well

$$X(\omega) = [X_1(\omega), \dots, X_n(\omega)]$$

- The **joint distribution** is $p(X_1 = x_1, \dots, X_n = x_n)$ is

$$p(X_1 = x_1 \cap \dots \cap X_n = x_n)$$

typically written as

$$p(x_1, \dots, x_n)$$

- Because $X_i = x_i$ is an event, all of the same rules - independence, conditioning, chain rule, etc. - still apply

Discrete Random Variables

- Two random variables X_1 and X_2 are independent if

$$p(X_1 = x_1, X_2 = x_2) = p(X_1 = x_1)p(X_2 = x_2)$$

for all values of x_1 and x_2

- Similar definition for conditional independence
- The conditional distribution of X_1 given $X_2 = x_2$ is

$$p(X_1 | X_2 = x_2) = \frac{p(X_1, X_2 = x_2)}{p(X_2 = x_2)}$$

this means that this relationship holds for all choices of x_1

Expected Value

- The **expected value** of a real-valued random variable is the weighted sum of its outcomes

$$E[X] = \sum_{x \in D} p(X = d) \cdot d$$

- Expected value is linear

$$E[X + Y] = E[X] + E[Y]$$

Expected Value: Lotteries

- Powerball Lottery currently has a grand prize of \$106 million
- Odds of winning the grand prize are $1/292,201,338$
- Tickets cost \$2 each
- Expected value of the game

$$= \frac{-2 \cdot 292,201,337}{292,201,338} + \frac{106,000,000 - 2}{292,201,338} \approx \$ - 1.6$$

Variance

- The **variance** of a random variable measures its squared deviation from its mean

$$\text{var}(X) = E[(X - E[X])^2]$$

- Estimates the square of the expected amount by which a random variable deviates from its expected value