# Statistical Methods in AI and ML

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University of Texas at Dallas

# The Course

A powerful and flexible set of tools for modeling problems in AI/ML

Judea Pearl won the Turing award for his work on Bayesian networks!

(among other achievements)

# Prob. Graphical Models

Exploit **locality** and structural features of a given model in order to gain insight about **global properties**

# The Course

- What this course is:
  - Probabilistic graphical models
  - Topics:
    - representing data
    - exact and approximate statistical inference
    - model learning
    - variational methods in ML

# The Course

- What you should be able to do at the end:
  - Design statistical models for applications in your domain of interest
  - Apply learning and inference algorithms to solve real problems (exactly or approximately)
  - Understand the complexity issues involved in the modeling decisions and algorithmic choices

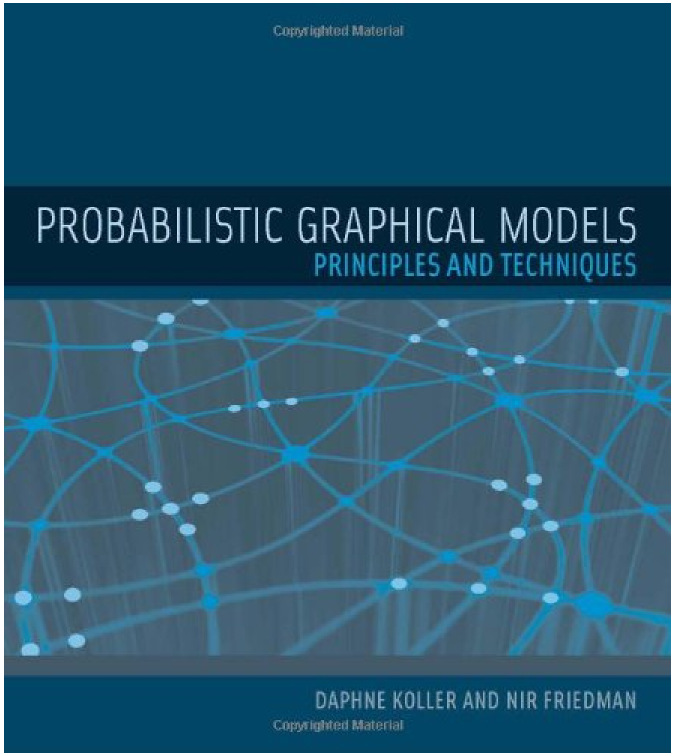
# Prerequisites

- CS 5343: Algorithm Analysis and Data Structures
- CS 3341: Probability and Statistics in Computer Science and Software Engineering
- Basically, comfort with probability and algorithms (machine learning is helpful, but not required)

# Suggested Textbook

Readings will be posted online before each lecture

Check the course website for additional resources and papers



# Textbook

- In addition, some lecture notes, in book format, will be made available for the main topics
- The idea is to build a set of notes that aligns well with the presentation of course material
- Comments, suggestions, corrections are welcome/encouraged

# Grading

- 4-6 problem sets (70%)
  - See collaboration policy on the web
- Final project (25%)
- Class/Piazza participation & extra credit (5%)

*-subject to change-*

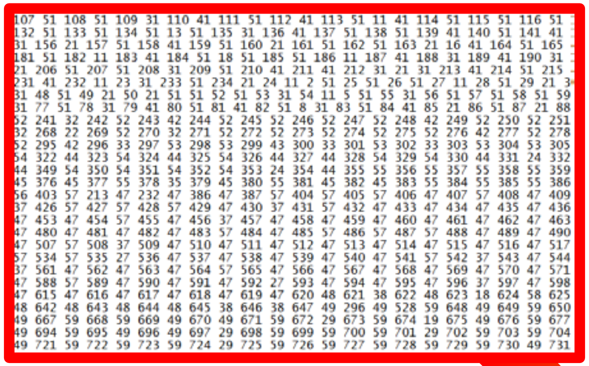
# Course Info.

- Instructor:  Nicholas Ruozzi
  - Office:  ECSS 3.409
  - Office hours:  Tues. 10am – 11am and by appointment
- TA:  TBD
  - Office hours and location TBD
- Course website: http://www.utdallas.edu/~nrr150130/cs6347/2018sp/

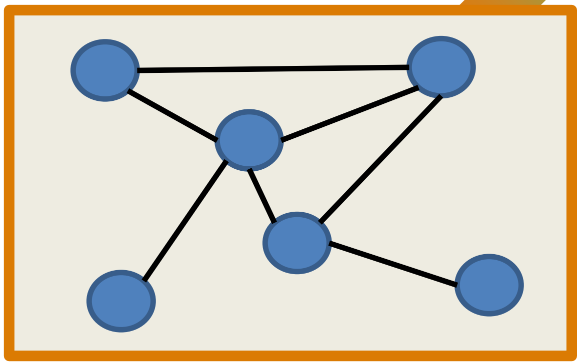
# Main Ideas

- Model the world (or at least the problem) as a collection of random variables related through some joint probability distribution
  - Compactly represent the distribution
  - Undirected graphical models
  - Directed graphical models
- Learn the distribution from observed data
  - Maximum likelihood, SVMs, etc.
- Make predictions (statistical inference)

# Inference and Learning



Collect Data

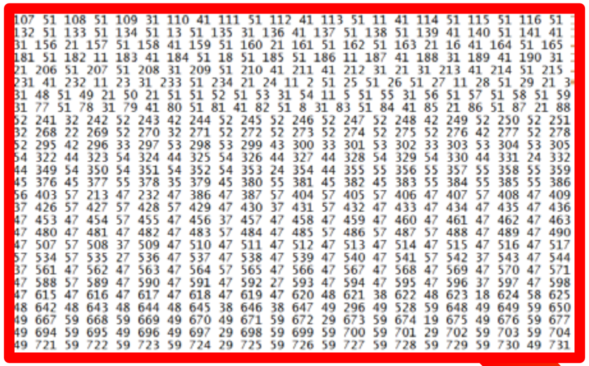


"Learn" a model that represents the observed data

$$Z(\theta) = \sum_{x} p(x;\theta)$$

Use the model to do inference / make predictions

# Inference and Learning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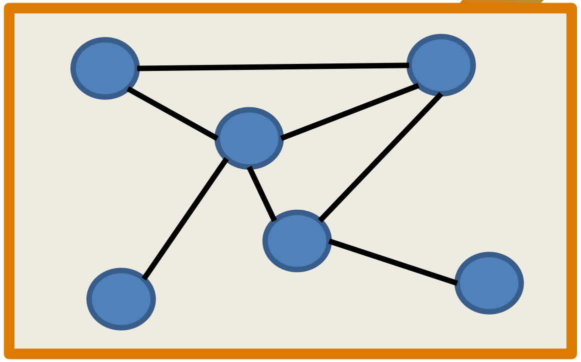
Data sets can be large



Data must be compactly modeled

$$Z(\theta) = \sum_{x} p(x;\theta)$$

Inference needs to be fast

# Applications

- Computer vision
- Natural language processing
- Robotics
- Computational biology
- Computational neuroscience
- Text translation
- Text-to-speech
- Many more…

# Graphical Models

- A graphical model is a graph together with "local interactions"
- The graph and interactions model a global optimization or learning problem
- The study of graphical models is concerned with how to exploit local structure to solve these problems either exactly or approximately

# Probability Review

# Discrete Probability

- **Sample space** specifies the set of possible outcomes
  - For example, $\Omega = \{\mathrm{H}, \mathrm{T}\}$ would be the set of possible outcomes of a coin flip
- Each element $\omega \in \Omega$ is associated with a number $\mathrm{p}(\omega) \in [0,1]$ called a **probability**

$$\sum_{\omega \in \Omega} p(\omega) = 1$$

  - For example, a biased coin might have $p(H) = .6$ and $p(T) = .4$

# Discrete Probability

- An **event** is a subset of the sample space
  - Let $\Omega = \{1, 2, 3, 4, 5, 6\}$ be the 6 possible outcomes of a dice role
  - $A = \{1, 5, 6\} \subseteq \Omega$ would be the event that the dice roll comes up as a one, five, or six
- The probability of an event is just the sum of all of the outcomes that it contains
  - $p(A) = p(1) + p(5) + p(6)$

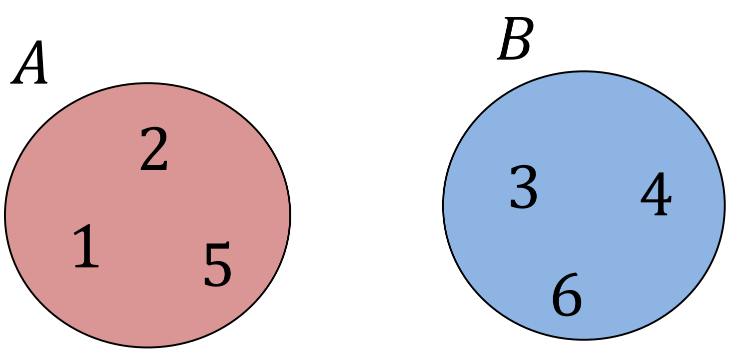
# Independence

- Two events A and B are **independent** if

$$p(A \cap B) = p(A)P(B)$$

Let's suppose that we have a fair die: $p(1) = \ldots = p(6) = 1/6$

If $A = \{1, 2, 5\}$ and $B = \{3, 4, 6\}$ are $A$ and $B$ indpendent?



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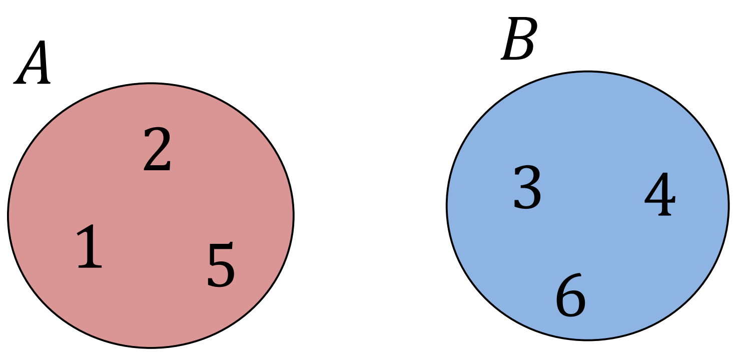



*No!*

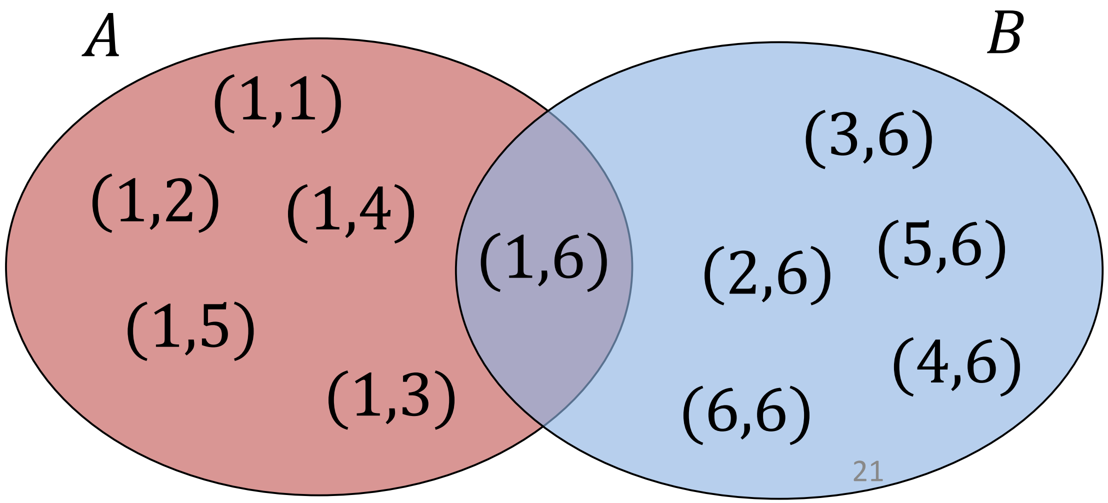
$$p(A \cap B) = 0 \neq \frac{1}{4}$$

# Independence

- Now, suppose that $\Omega = \{(1,1), (1,2), \dots, (6,6)\}$ is the set of all possible rolls of two **unbiased** dice

- Let $A = \{(1,1), (1,2), (1,3), \dots, (1,6)\}$ be the event that the first die is a one and let $B = \{(1,6), (2,6), \dots, (6,6)\}$ be the event that the second die is a six

- Are $A$ and $B$ independent?



# Independence

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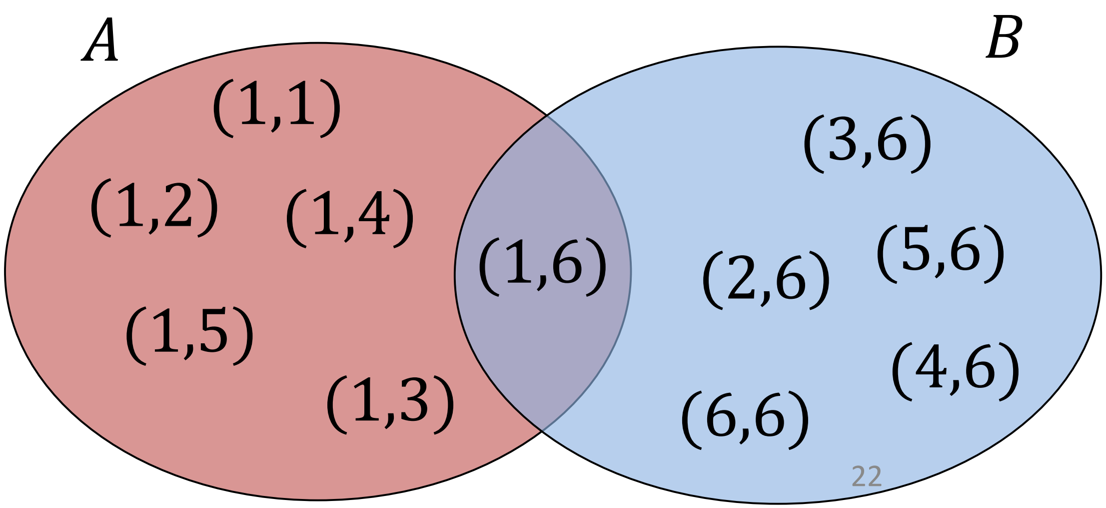
- Are $A$ and $B$ independent?




*Yes!*

$$p(A \cap B) = \frac{1}{36} = \frac{1}{6} * \frac{1}{6}$$

# Conditional Probability

- The **conditional probability** of an event $A$ given an event $B$ with $p(B) > 0$ is defined to be

$$p(A|B) = \frac{p(A \cap B)}{P(B)}$$

- This is the probability of the event $A \cap B$ over the sample space $\Omega' = B$

- Some properties:
  - $\sum_{\omega \in B} p(\omega|B) = 1$
  - If $A$ and $B$ are independent, then $p(A|B) = p(A)$

# Simple Example

| Cheated | Grade | Probability |
|---|---|---|
| Yes | A | .15 |
| Yes | F | .05 |
| No | A | .5 |
| No | F | .3 |

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| Cheated | Grade | Probability |
|---|---|---|
| Yes | A | .15 |
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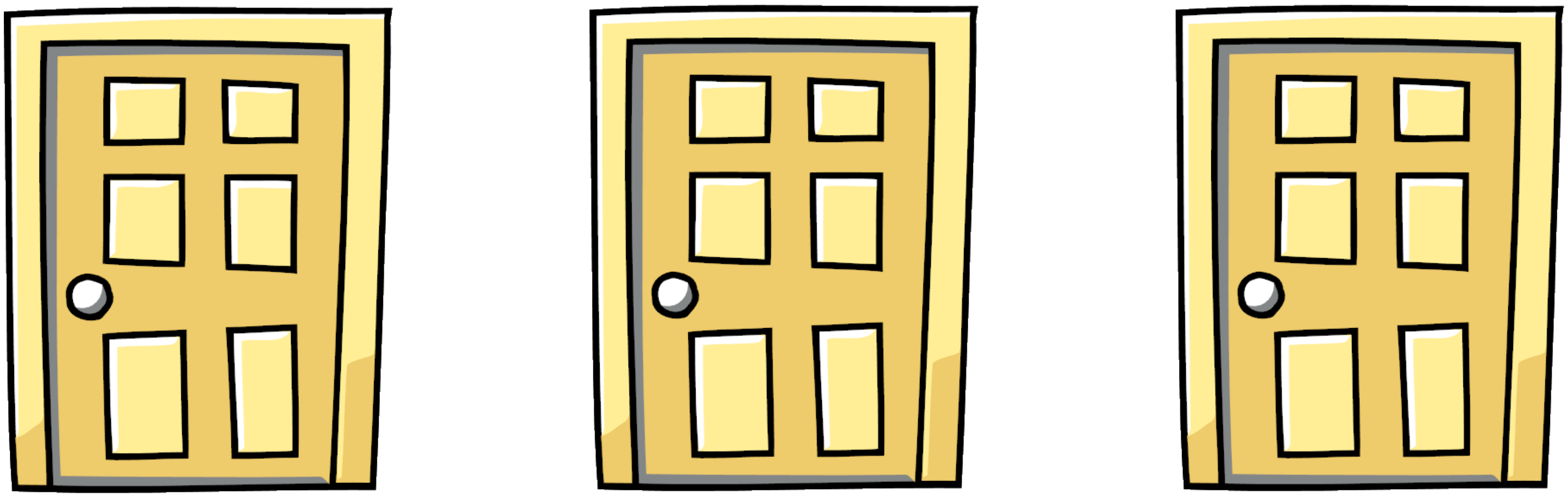
$$p(Cheated = Yes \mid Grade = F) = ?$$

# Simple Example

| Cheated | Grade | Probability |
|---|---|---|
| Yes | A | .15 |
| Yes | F | .05 |
| No | A | .5 |
| No | F | .3 |

$$p(Cheated = Yes\,|Grade = F) = \frac{.05}{.35} \approx .14$$

# The Monty Hall Problem



1 2 3

# Chain Rule

$$p(A \cap B) = p(A)p(B|A)$$

$$\begin{aligned} p(A \cap B \cap C) &= p(A \cap B)p(C|A \cap B) \\ &= p(A)p(B|A)p(C|A \cap B) \end{aligned}$$

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$$p\left(\bigcap_{i=1}^{n} A_i\right) = p(A_1)p(A_2|A_1)\dots p(A_n|A_1 \cap \dots \cap A_{n-1})$$

# Conditional Independence

- Two events $A$ and $B$ are independent if learning something about $B$ tells you nothing about $A$ (and vice versa)

- Two events $A$ and $B$ are **conditionally independent** given $C$ if

$$p(A \cap B|C) = p(A|C)p(B|C)$$

- This is equivalent to

$$p(A|B \cap C) = p(A|C)$$

  - That is, given $C$, information about $B$ tells you nothing about $A$ (and vice versa)

# Conditional Independence

- Let $\Omega = \{(H,H),(H,T),(T,H),(T,T)\}$ be the outcomes resulting from tossing two different fair coins

- Let $A$ be the event that the first coin is heads
- Let $B$ be the event that the second coin is heads
- Let $C$ be the even that both coins are heads or both are tails

- $A$ and $B$ are independent, but $A$ and $B$ are not independent given $C$

# Discrete Random Variables

- A discrete **random variable**, $X$, is a function from the state space $\Omega$ into a discrete space $D$
  - For each $x \in D$,

$$p(X = x) \equiv p(\{\omega \in \Omega : X(\omega) = x\})$$

    is the probability that $X$ takes the **value** $x$

  - $p(X)$ defines a probability distribution
    - $\sum_{x \in D} p(X = x) = 1$
- Random variables partition the state space into disjoint events

# Example: Pair of Dice

- Let $\Omega$ be the set of all possible outcomes of rolling a pair of dice
- Let $p$ be the uniform probability distribution over all possible outcomes in $\Omega$
- Let $X(\omega)$ be equal to the sum of the value showing on the pair of dice in the outcome $\omega$
  - $p(X = 2) = ?$
  - $p(X = 8) = ?$

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  - $p(X = 8) = ?$

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- Let $X(\omega)$ be equal to the sum of the value showing on the pair of dice in the outcome $\omega$
  - $p(X = 2) = \mathbf{\frac{1}{36}}$
  - $p(X = 8) = \mathbf{\frac{5}{36}}$

# Discrete Random Variables

- We can have vectors of random variables as well

$$X(\omega) = [X_1(\omega), \dots, X_n(\omega)]$$

- The **joint distribution** is $p(X_1 = x_1, \dots, X_n = x_n)$ is

$$p(X_1 = x_1 \cap \cdots \cap X_n = x_n)$$

typically written as

$$p(x_1, \dots, x_n)$$

- Because $X_i = x_i$ is an event, all of the same rules - independence, conditioning, chain rule, etc. - still apply

# Discrete Random Variables

- Two random variables $X_1$ and $X_2$ are independent if

$$p(X_1 = x_1, X_2 = x_2) = p(X_1 = x_1)p(X_2 = x_2)$$

for all values of $x_1$ and $x_2$

- Similar definition for conditional independence

- The conditional distribution of $X_1$ given $X_2 = x_2$ is

$$p(X_1 | X_2 = x_2) = \frac{p(X_1, X_2 = x_2)}{p(X_2 = x_2)}$$

this means that this relationship holds for all choices of $x_1$

# Expected Value

- The expected value of a real-valued random variable is the weighted sum of its outcomes

$$E[X] = \sum_{x \in \mathrm{D}} p(X = d) \cdot d$$

- Expected value is linear

$$E[X + Y] = E[X] + E[Y]$$

# Expected Value: Lotteries

- Powerball Lottery currently has a grand prize of $40 million
- Odds of winning the grand prize are 1/292,201,338
- Tickets cost $2 each
- Expected value of the game

$$= \frac{-2 \cdot 292{,}201{,}337}{292{,}201{,}338} + \frac{40{,}000{,}000 \ - 2}{292{,}201{,}338} \approx \$ - 1.86$$

# Variance

- The variance of a random variable measures its squared deviation from its mean

$$var(X) = E[(X - E[X])^2]$$

  - Estimates the square of the expected amount by which a random variable deviates from its expected value