CS 6347

Lecture 3

More Bayesian Networks

Recap

- Last time:
  - Complexity challenges
    - Representing distributions
    - Computing probabilities/doing inference
  - Introduction to Bayesian networks
- Today:
  - D-separation, I-maps, limits of Bayesian networks

Bayesian Networks

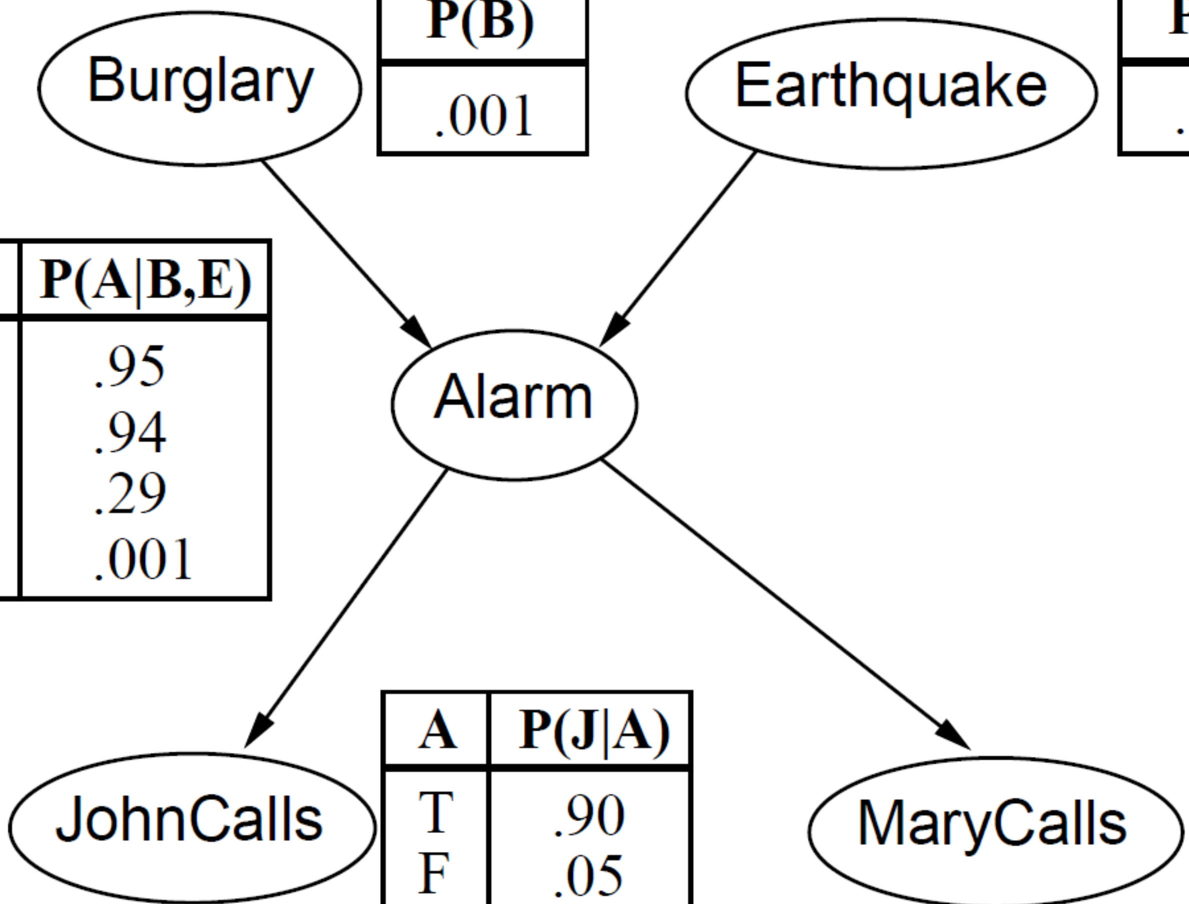
- A **Bayesian network** is a directed graphical model that represents a subset of the independence relationships of a given probability distribution
  - Directed acyclic graph (DAG), $G = (V, E)$
    - Edges are still pairs of vertices, but the edges $(1,2)$ and $(2,1)$ are now distinct in this model
  - One node for each random variable
  - One conditional probability distribution per node
  - Directed edge represents a direct statistical dependence

Bayesian Networks

- A **Bayesian network** is a directed graphical model that represents a subset of the independence relationships of a given probability distribution
  - Encodes **local Markov** independence assumptions that each node is independent of its non-descendants given its parents
  - Corresponds to a **factorization** of the joint distribution

$$p(x_1, \dots, x_n) = \prod_i p(x_i | x_{parents(i)})$$

An Example



P(B)
.001

P(E)
.002

B	E	P(A|B,E)
T	T	.95
T	F	.94
F	T	.29
F	F	.001

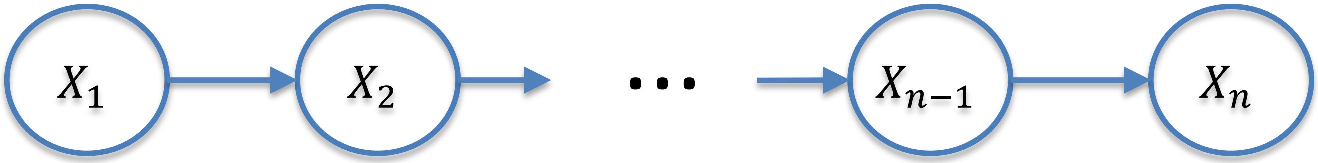
A	P(J|A)
T	.90
F	.05

A	P(M|A)
T	.70
F	.01

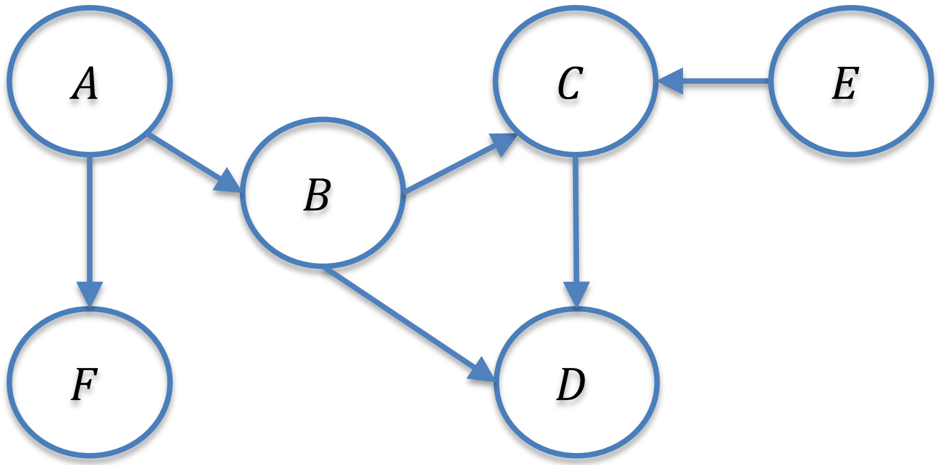
from Artificial Intelligence: A Modern Approach

Directed Chain

$$p(x_1, \ldots, x_n) = p(x_1)p(x_2|x_1)p(x_3|x_2) \ldots p(x_n|x_{n-1})$$



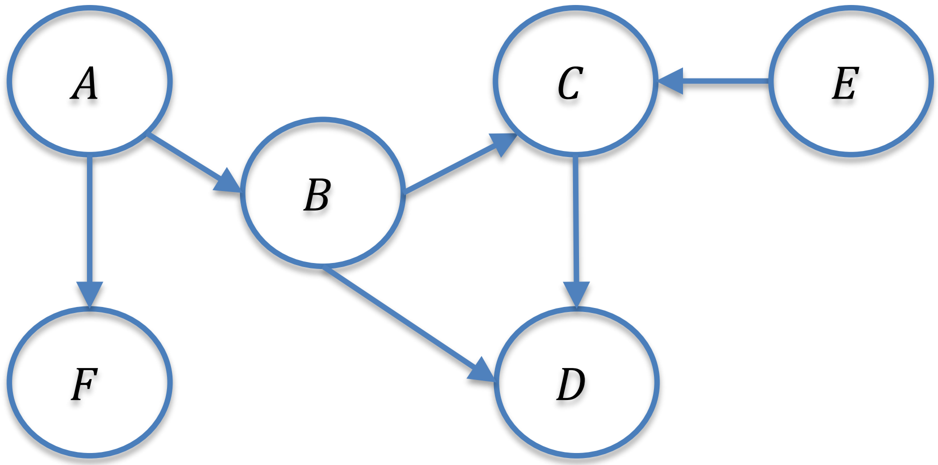
Example:



Suppose that a joint distribution factorizes over this graph…

- Local Markov independence relations?
- Joint distribution?

Example:



The local Markov independence relations are not exhaustive:

- How can we figure out which independence relationships the model represents?

D-separation

- Independence relationships can be figured out by looking at the graph structure!
  - Easier than looking at the tables and plugging into the definition
- We look at **all** of the paths from X to Y in the graph and determine whether or not they are **blocked**
  - $X \subset V$ is d-separated from $Y \subset V$ given $Z \subset V$ iff every path from X to Y in the graph is blocked by Z

D-separation

- Three types of situations can occur along any given path

(1) Sequential

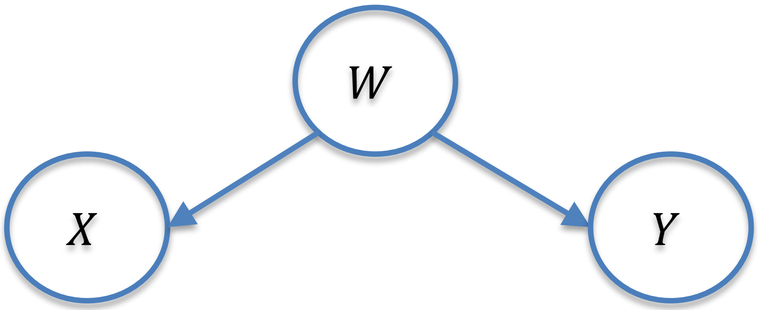
The path from X to Y is blocked if we condition on W

Intuitively, if we condition on W, then information about X does not affect Y and vice versa

D-separation

- Three types of situations can occur along any given path

(2) Divergent



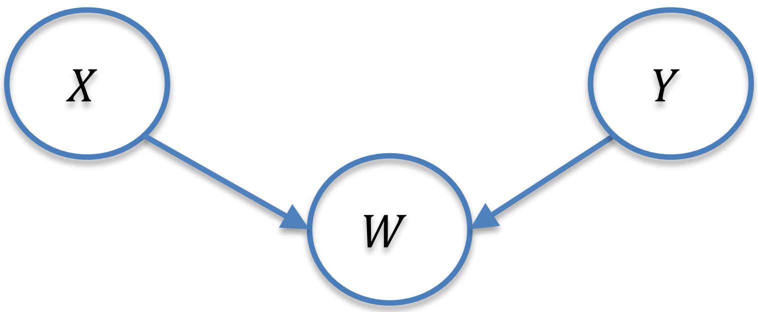
The path from X to Y is blocked if we condition on W

If we don't condition on W, then information about W could affect the probability of observing either X or Y

D-separation

- Three types of situations can occur along any given path

(3) Convergent



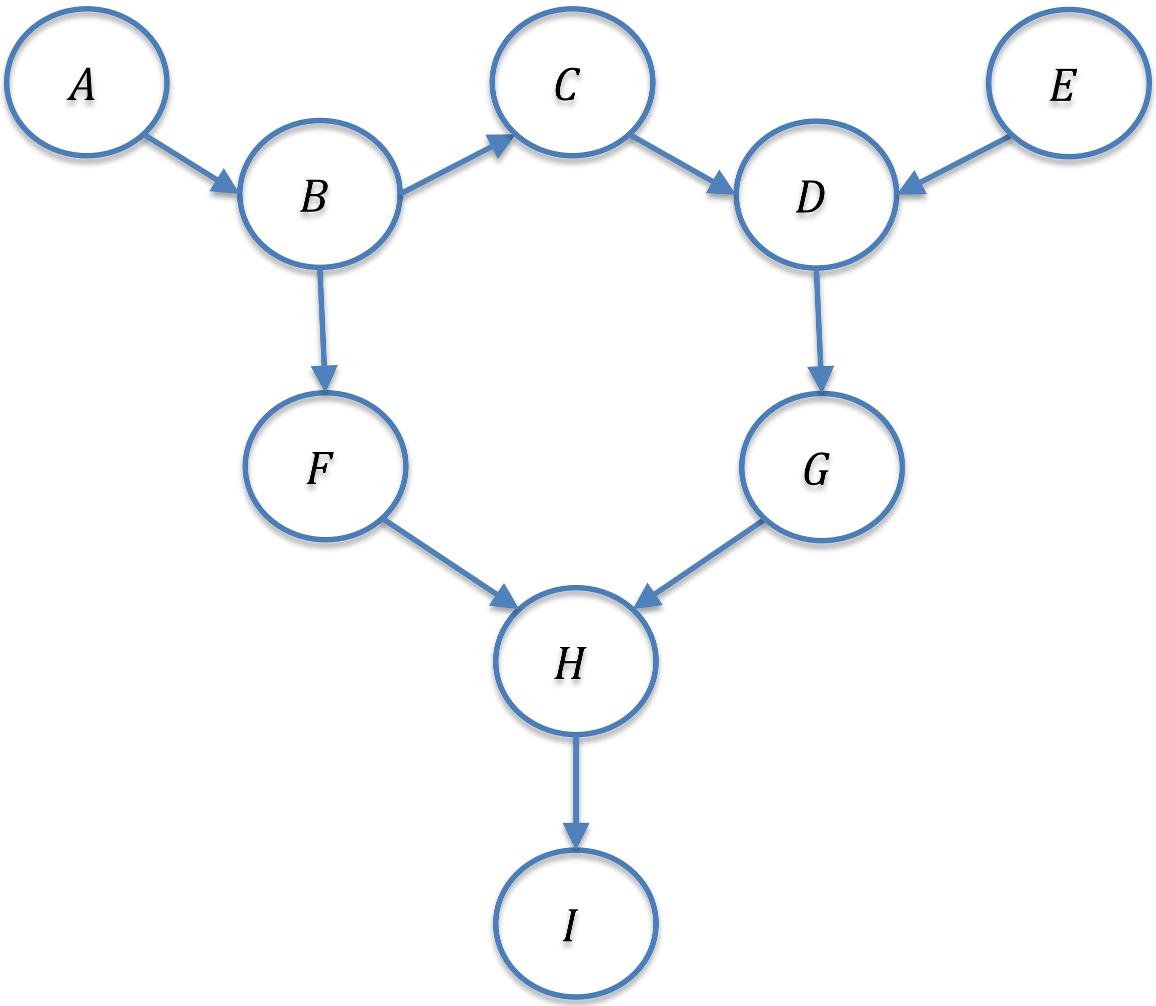
The path from X to Y is blocked if we **do not** condition on W or any of its descendants

Conditioning on W couples the variables X and Y: knowing whether or not X occurs impacts the probability that Y occurs

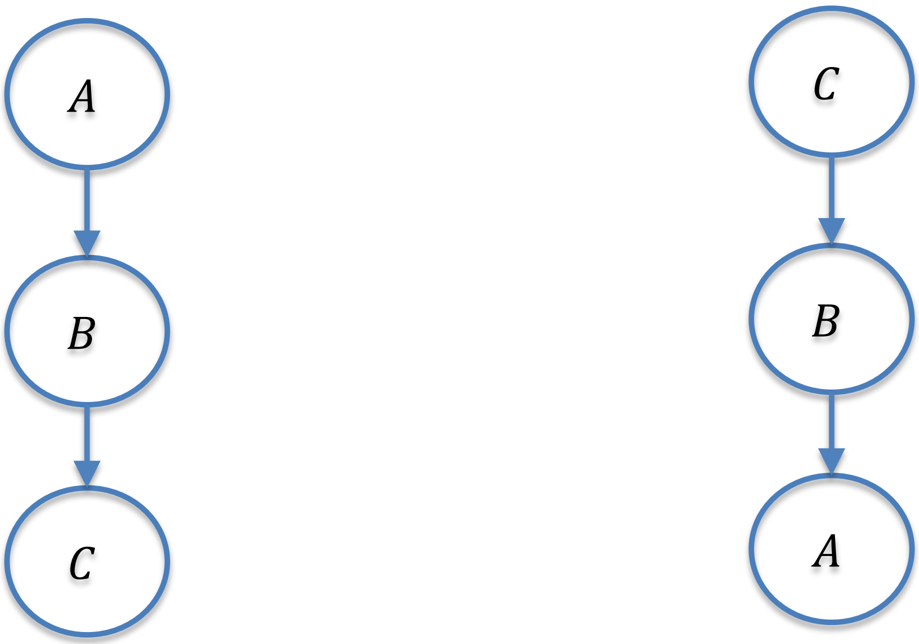
D-separation

- If the joint probability distribution factorizes with respect to the DAG $G = (V, E)$, then X is d-separated from Y given Z implies $X \perp Y \mid Z$
  - We can use this to quickly check independence assertions by using the graph
  - In general, these are only a subset of all independence relationships that are actually present in the joint distribution
  - If X and Y are not d-separated in G given Z, then there is *some distribution* that factorizes over G in which X and Y are dependent

D-separation Example

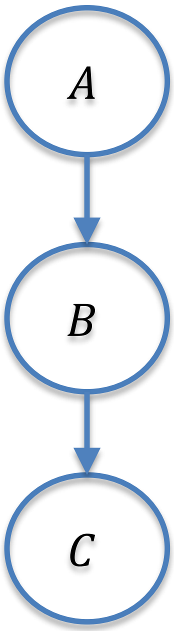


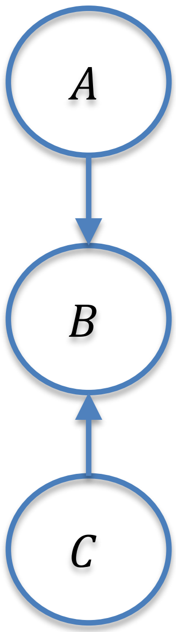
Equivalent Models?



Do these models represent the same independence relations?

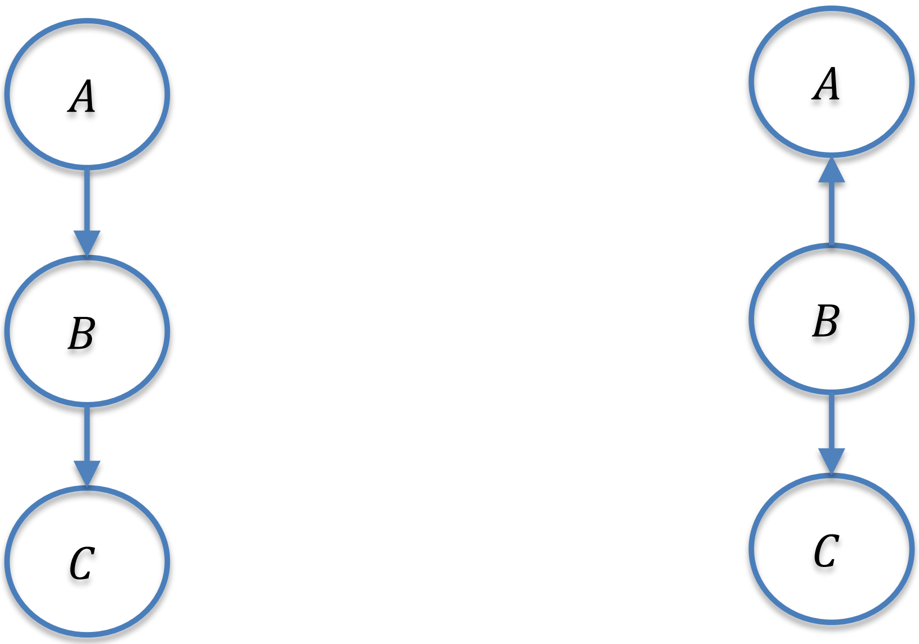
Equivalent Models?





Do these models represent the same independence relations?

Equivalent Models?



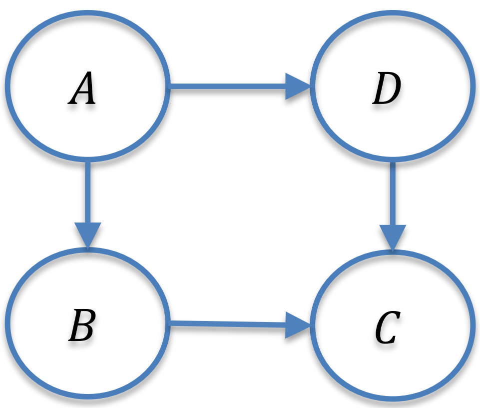
Do these models represent the same independence relations?

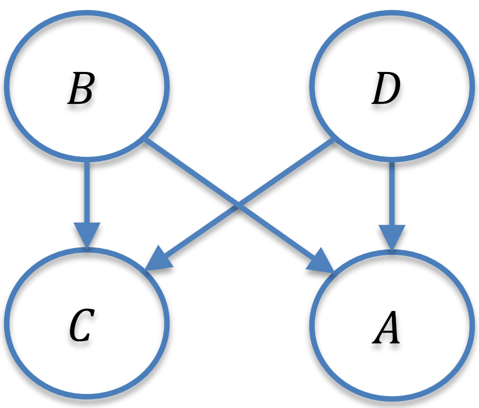
D-separation

- Let $I(p)$ be the set of all independence relationships in the joint distribution p and $I(G)$ be the set of all independence relationships implied by the graph G
- We say that G is an I-map for $I(p)$ if $I(G) \subseteq I(p)$
- Theorem: the joint probability distribution, p, factorizes with respect to the DAG $G = (V, E)$ iff G is an I-map for $I(p)$
- An I-map is perfect if $I(G) = I(p)$
    - Not always possible to perfectly represent all of the independence relations with a graph

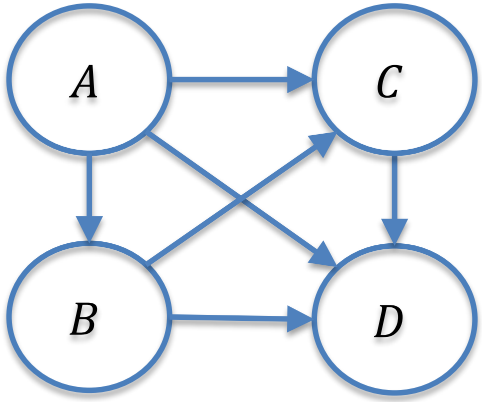
Limits of Bayesian Networks

- Not all sets of independence relations can be captured by a Bayesian network, e.g., suppose these are the only independence relationships allowed
  - $A \perp C \mid B, D$
  - $B \perp D \mid A, C$
- Possible DAGs that represent only these independence relationships?



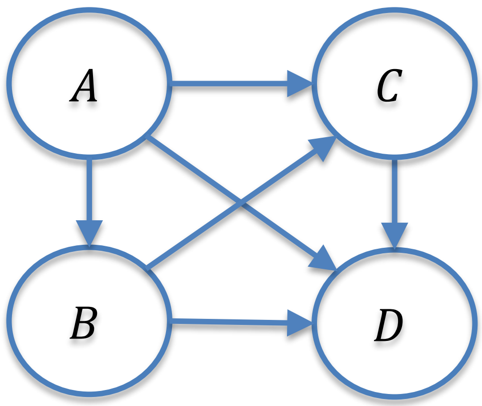


Limits of Bayesian Networks



What independence relations does this model imply?

Limits of Bayesian Networks



$I(G) = \emptyset$, this is an I-map for any joint distribution on four variables!