Support Vector Machines

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Slides adapted from David Sontag and Vibhav Gogate

Announcements

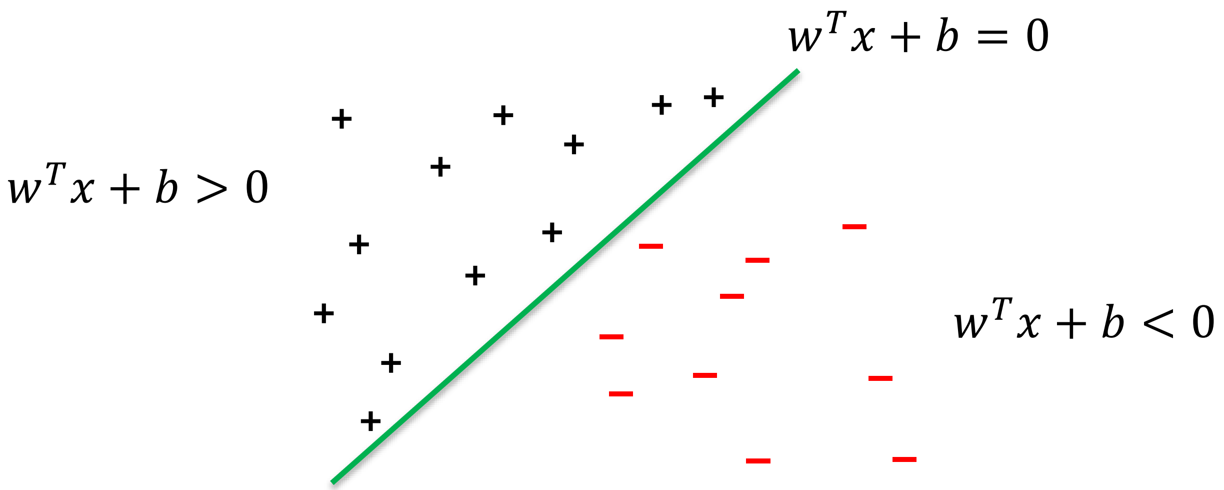
- Homework 1 is available soon

- Piazza discussion group?

- Reminder: my office hours are 10am-11am on Tuesdays

Binary Classification

- Input $(x^{(1)}, y^{(1)}), \dots, (x^{(n)}, y^{(n)})$ with $x^{(i)} \in \mathbb{R}^m$ and $y^{(i)} \in \{-1, +1\}$
- We can think of the observations as points in $\mathbb{R}^m$ with an associated sign (either +/- corresponding to 0/1)



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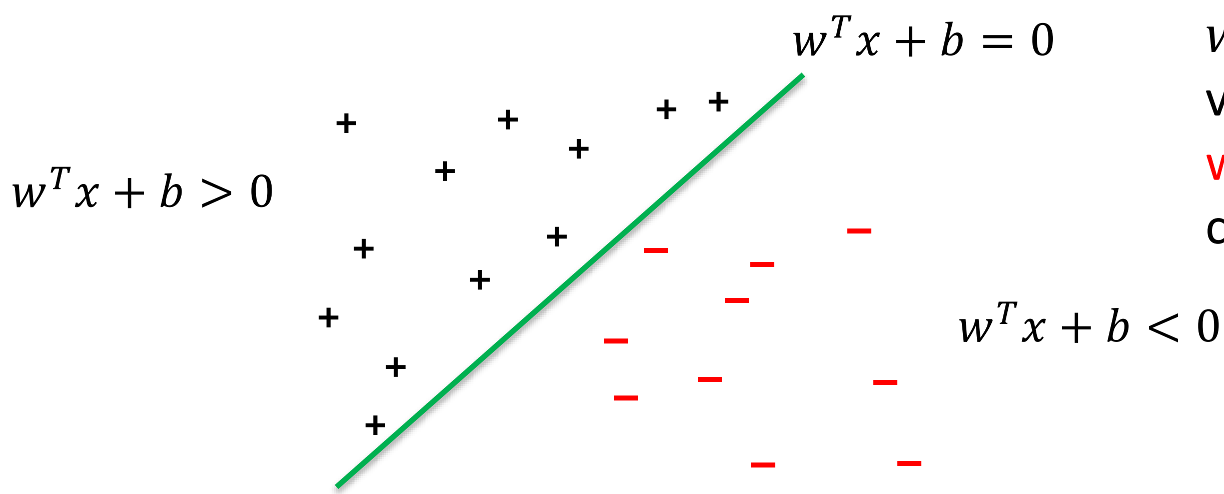

w is called the vector of weights and b is called the bias

What If the Data Isn‘t Separable?

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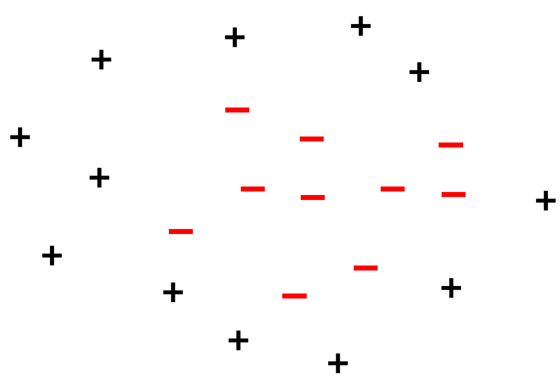

What is a good hypothesis space for this problem?

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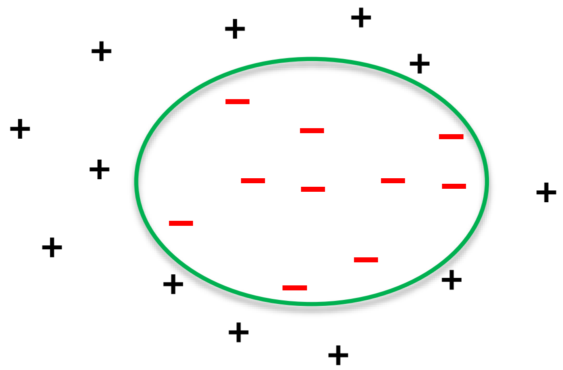


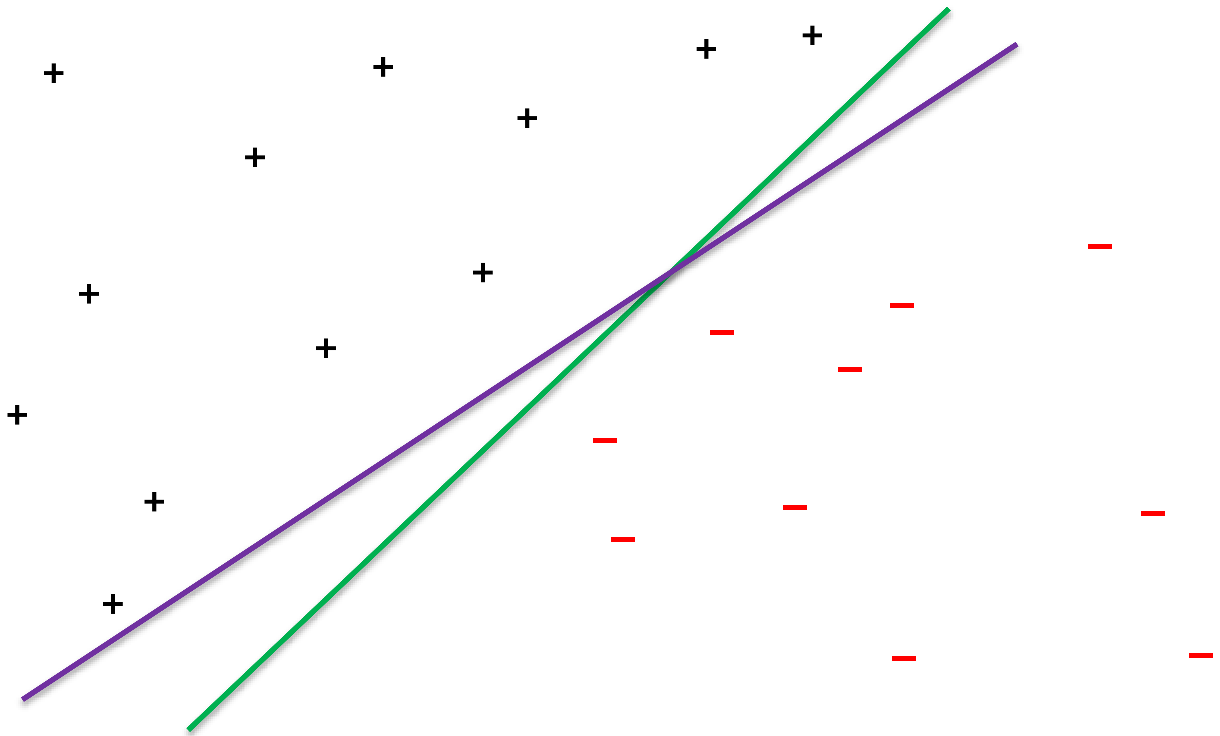
What is a good hypothesis space for this problem?

Adding Features

- The idea:
  - Given the observations $x^{(1)}, \dots, x^{(n)}$, construct a feature vectors $\phi\left(x^{(1)}\right), \dots, \phi(x^{(n)})$
  - Use $\phi\left(x^{(1)}\right), \dots, \phi\left(x^{(n)}\right)$ instead of $x^{(1)}, \dots, x^{(n)}$ in the learning algorithm
  - Goal is to choose ϕ so that $\phi\left(x^{(1)}\right), \dots, \phi\left(x^{(n)}\right)$ are linearly separable
  - Learn linear separators of the form $w^T \phi(x)$ (instead of $w^T x$)
- Warning: more expressive features can lead to overfitting!

Support Vector Machines

- How can we decide between perfect classifiers?



Support Vector Machines

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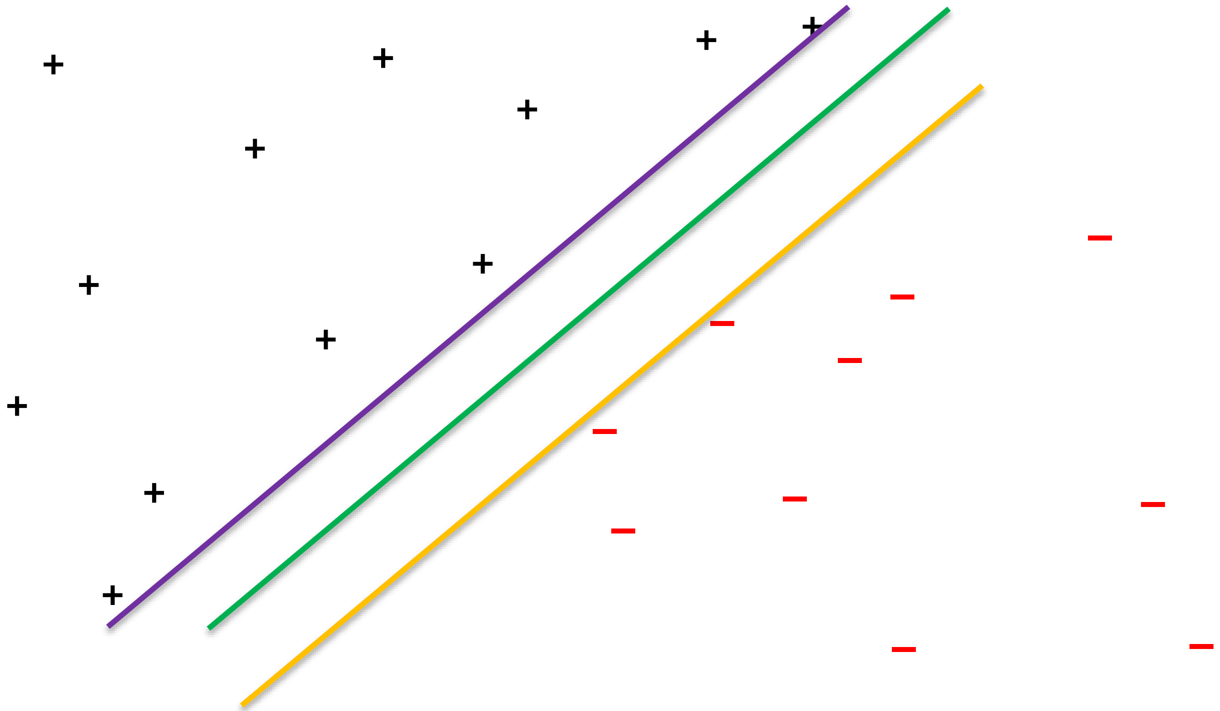
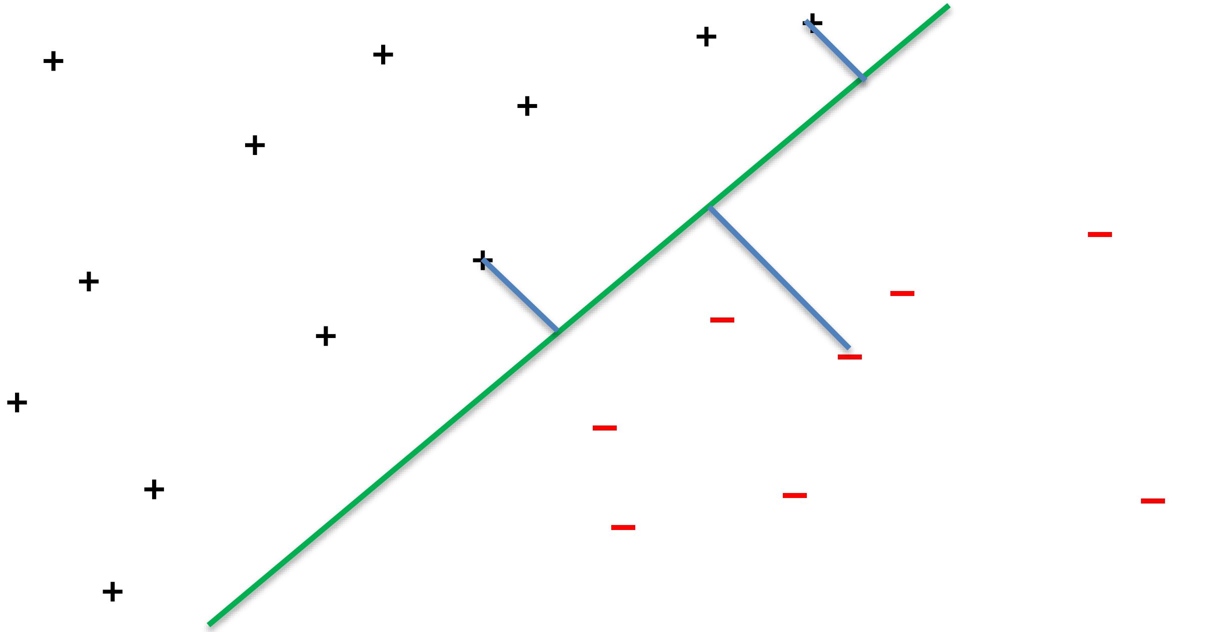


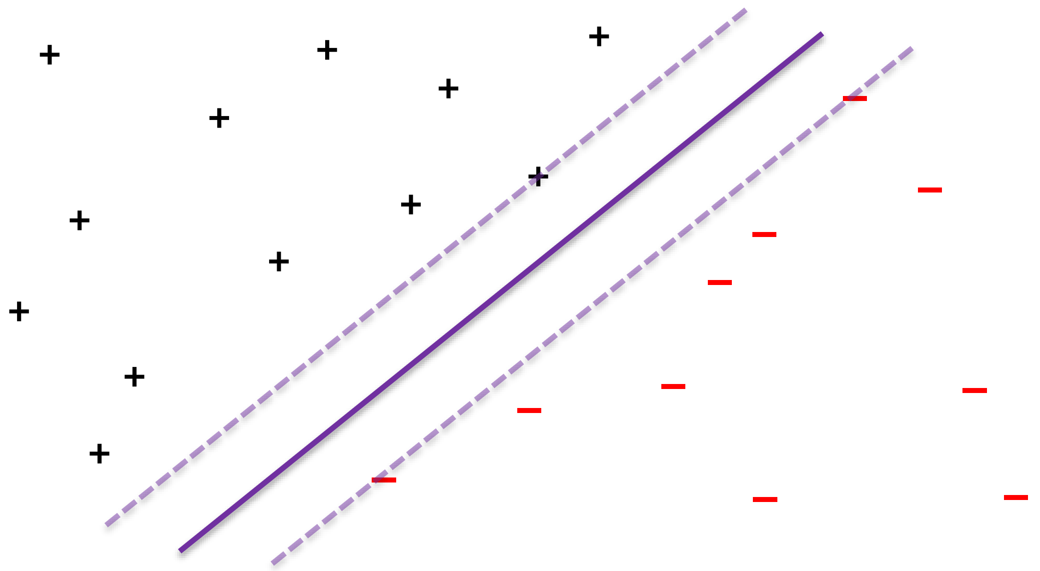
Support Vector Machines

- Define the margin to be the distance of the closest data point to the classifier



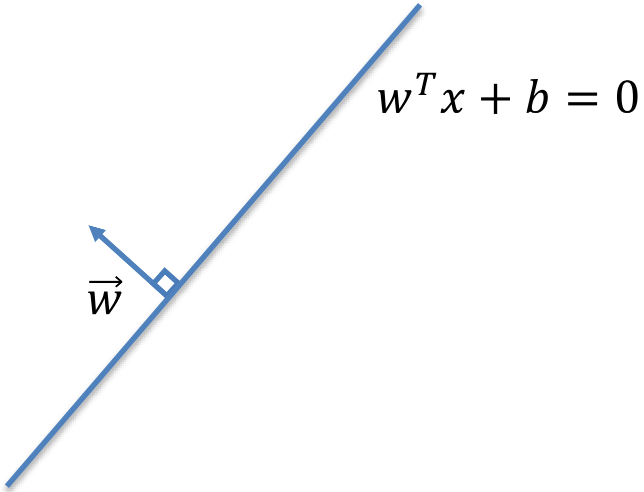
Support Vector Machines

- Support vector machines (SVMs)



- Choose the classifier with the largest margin
  - Has good practical and theoretical performance

Some Geometry



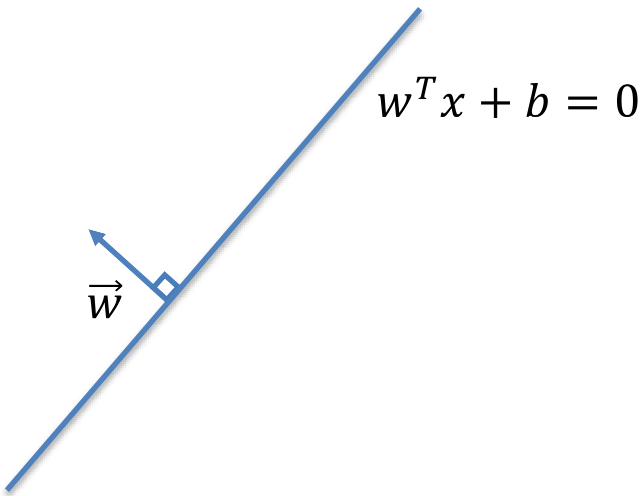
- In n dimensions, a hyperplane is a solution to the equation

$$w^T x + b = 0$$

with $w \in \mathbb{R}^n, b \in \mathbb{R}$

- The vector w is sometimes called the normal vector of the hyperplane

Some Geometry



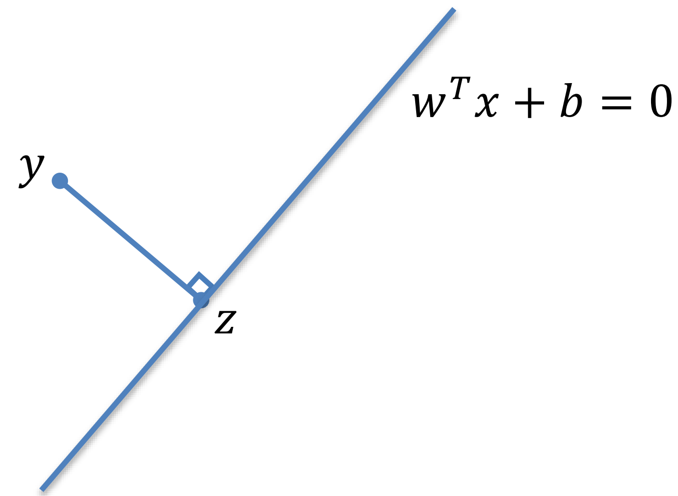
- In n dimensions, a hyperplane is a solution to the equation

$$w^T x + b = 0$$

- Note that this equation is scale invariant for any scalar c

$$c \cdot (w^T x + b) = 0$$

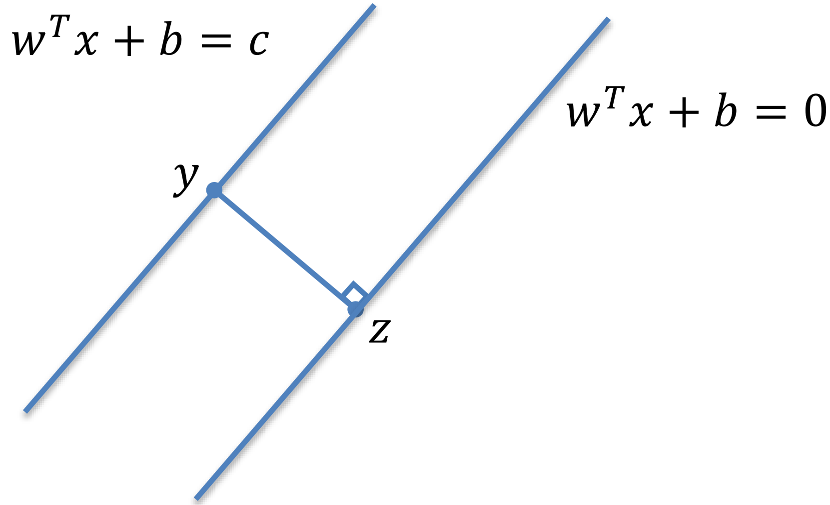
Some Geometry



- The distance between a point y and a hyperplane $w^T + b = 0$ is the length of the segment perpendicular to the line to the point y

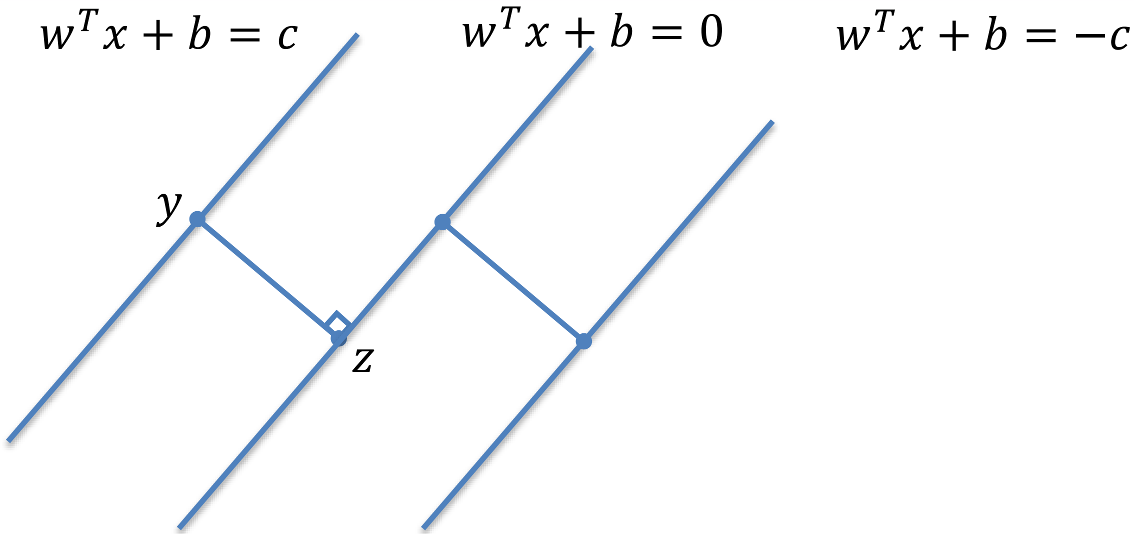
$$y - z = ||y - z|| \frac{w}{||w||}$$

Scale Invariance



- By scale invariance, we can assume that $c = 1$
- The maximum margin is always attained by choosing $w^T x + b = 0$ so that it is equidistant from the closest data point classified as +1 and the closest data point classified as -1

Scale Invariance

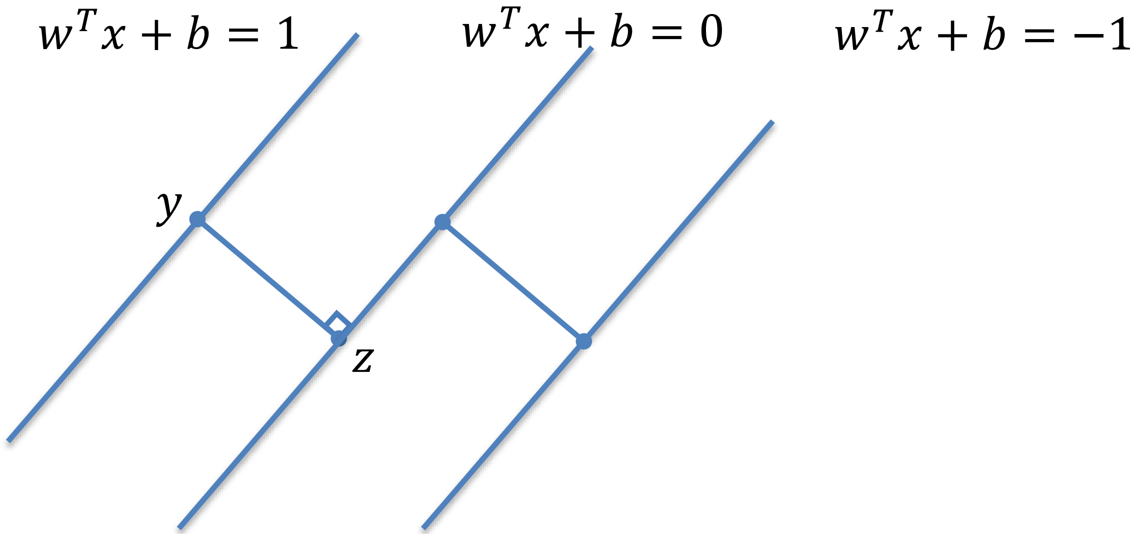


- We want to maximize the margin subject to the constraints that

$$y^{(i)}\left(w^T x^{(i)} + b\right) \geq 1$$

- But how do we compute the size of the margin?

Some Geometry



Putting it all together

$$y - z = \|y - z\| \frac{w}{\|w\|}$$

and

$$w^Ty + b = 1$$
$$w^Tz + b = 0$$

$$w^T(y - z) = 1$$

and

$$w^T(y - z) = \|y - z\| \|w\|$$

which gives

$$\|y - z\| = 1/\|w\|$$

SVMs

- This analysis yields the following optimization problem

$$\max_{w} \frac{1}{||w||}$$

such that

$$y^{(i)}\left(w^T x^{(i)} + b\right) \geq 1, \text{for all } i$$

- Or, equivalently,

$$\min_{w} ||w||^2$$

such that

$$y^{(i)}\left(w^T x^{(i)} + b\right) \geq 1, \text{for all } i$$

SVMs

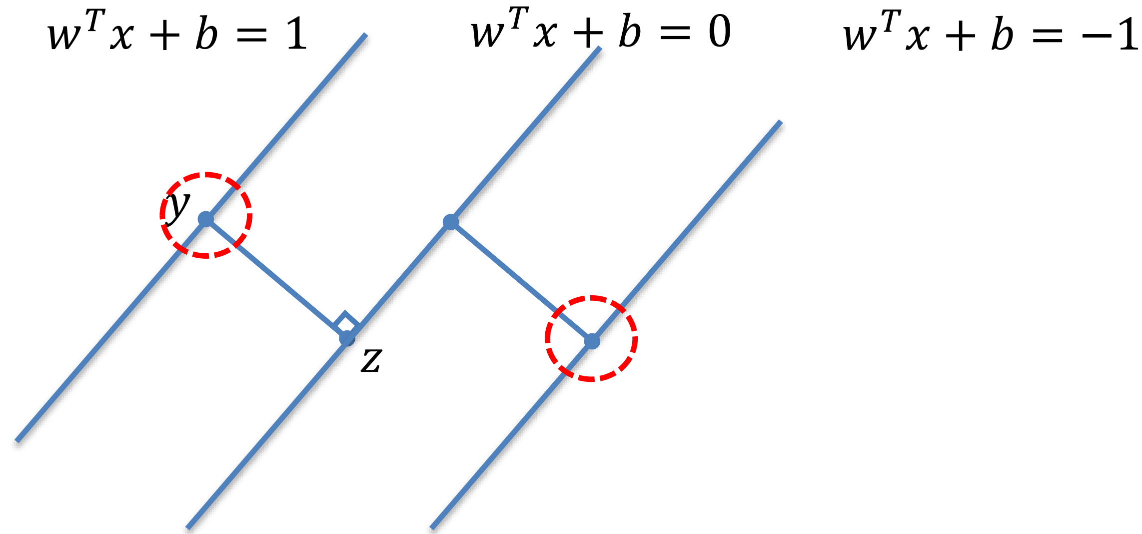
$$\min_{w} ||w||^2$$

such that

$$y^{(i)}\left(w^T x^{(i)} + b\right) \geq 1, \text{for all } i$$

- This is a standard quadratic programming problem
  - Falls into the class of convex optimization problems
  - Can be solved with many specialized optimization tools (e.g., quadprog() in MATLAB)

SVMs

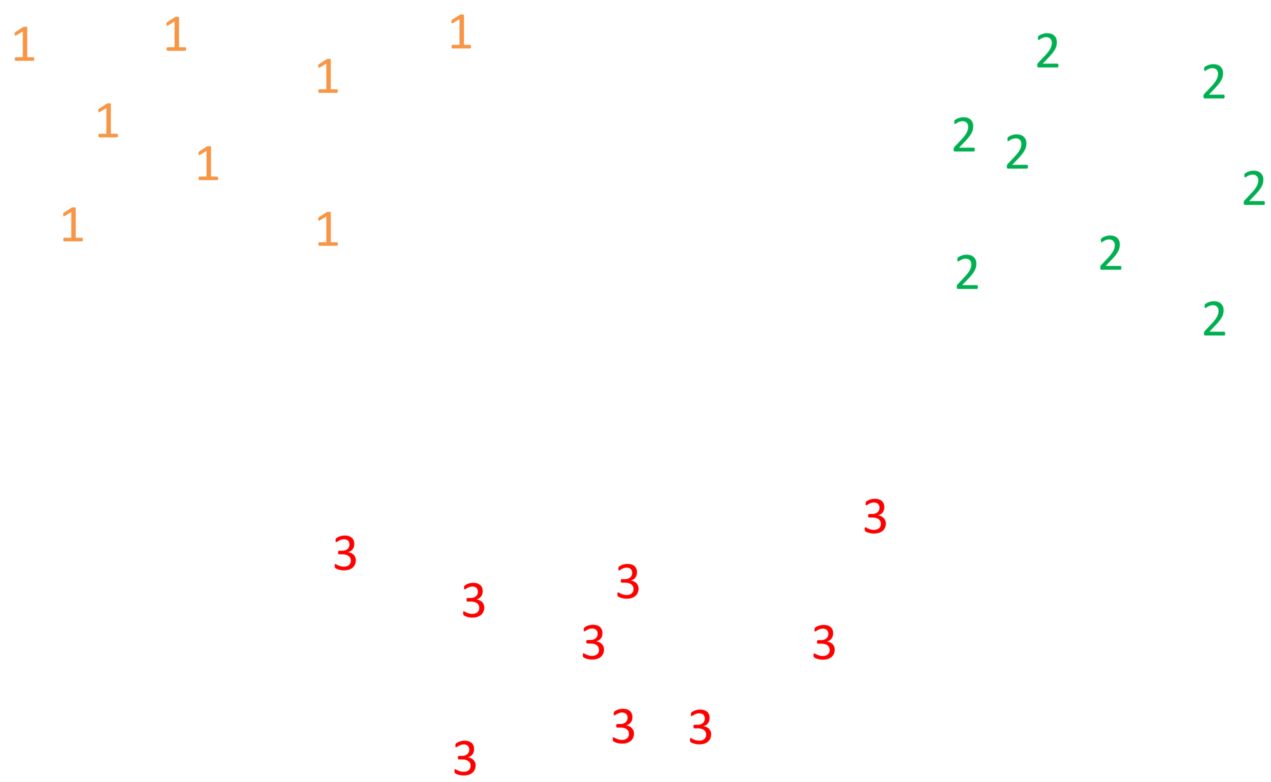


- Where does the name come from?
  - The set of all data points such that $y^{(i)}(w^T x^{(i)} + b) = 1$ are called support vectors

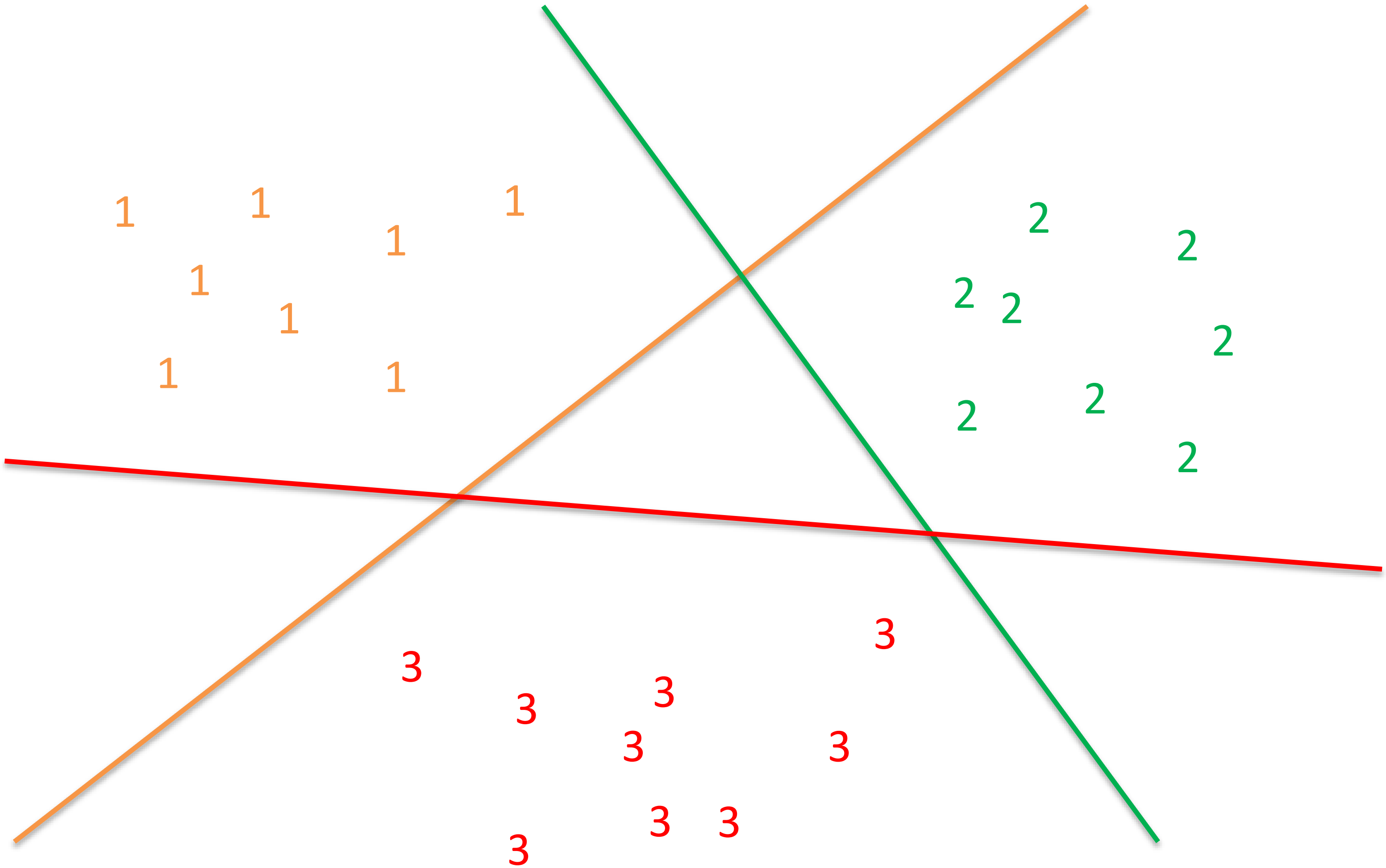
SVMs

- What if the data isn't linearly separable?
  - Use feature vectors
  - Relax the constraints (coming soon)
- What if we want to do more than just binary classification (i.e., if $y \in \{1,2,3\}$)?

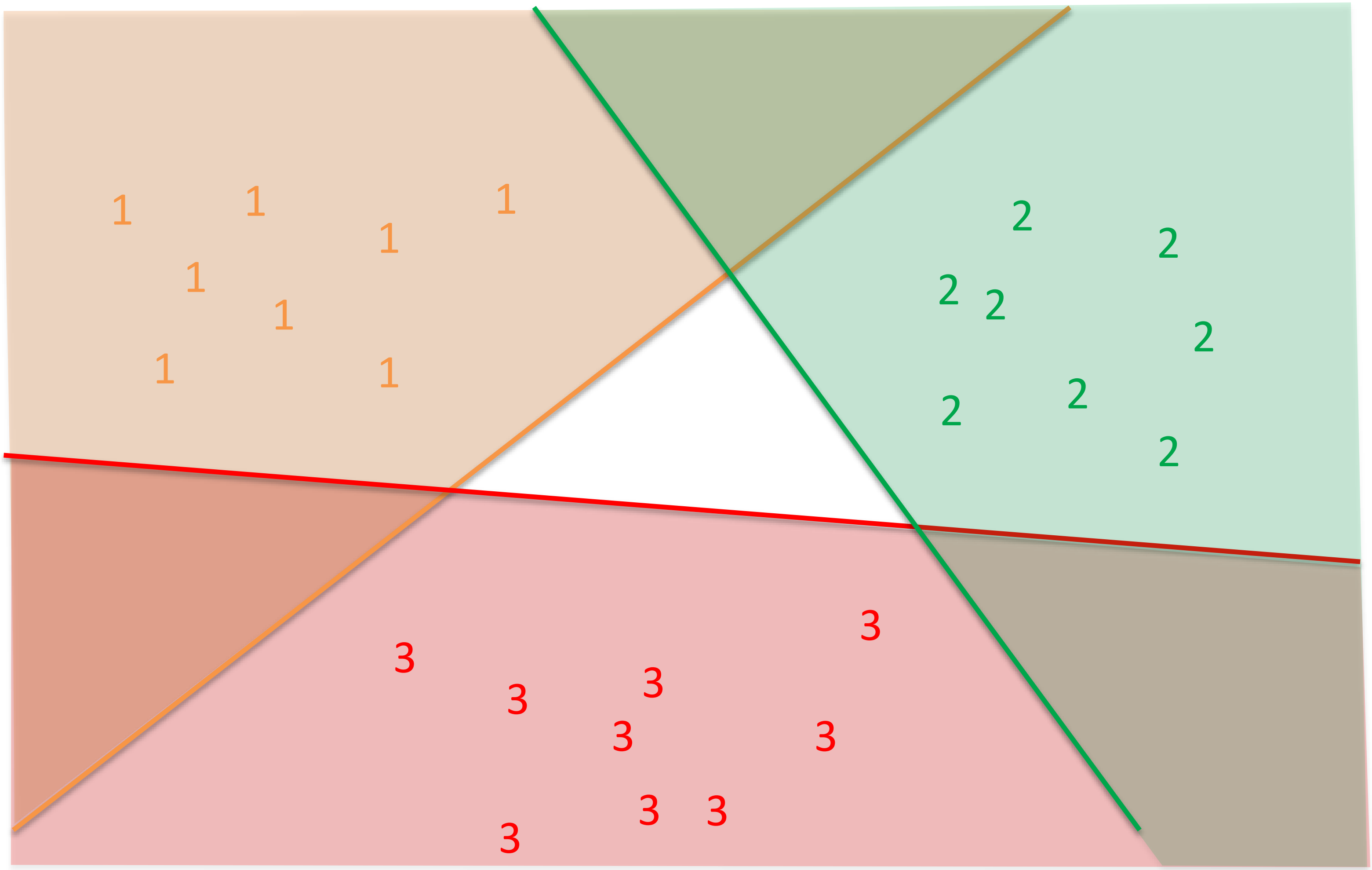
Multiclass Classification



One-Versus-All SVMs



One-Versus-All SVMs



Regions correctly classified by exactly one classifier

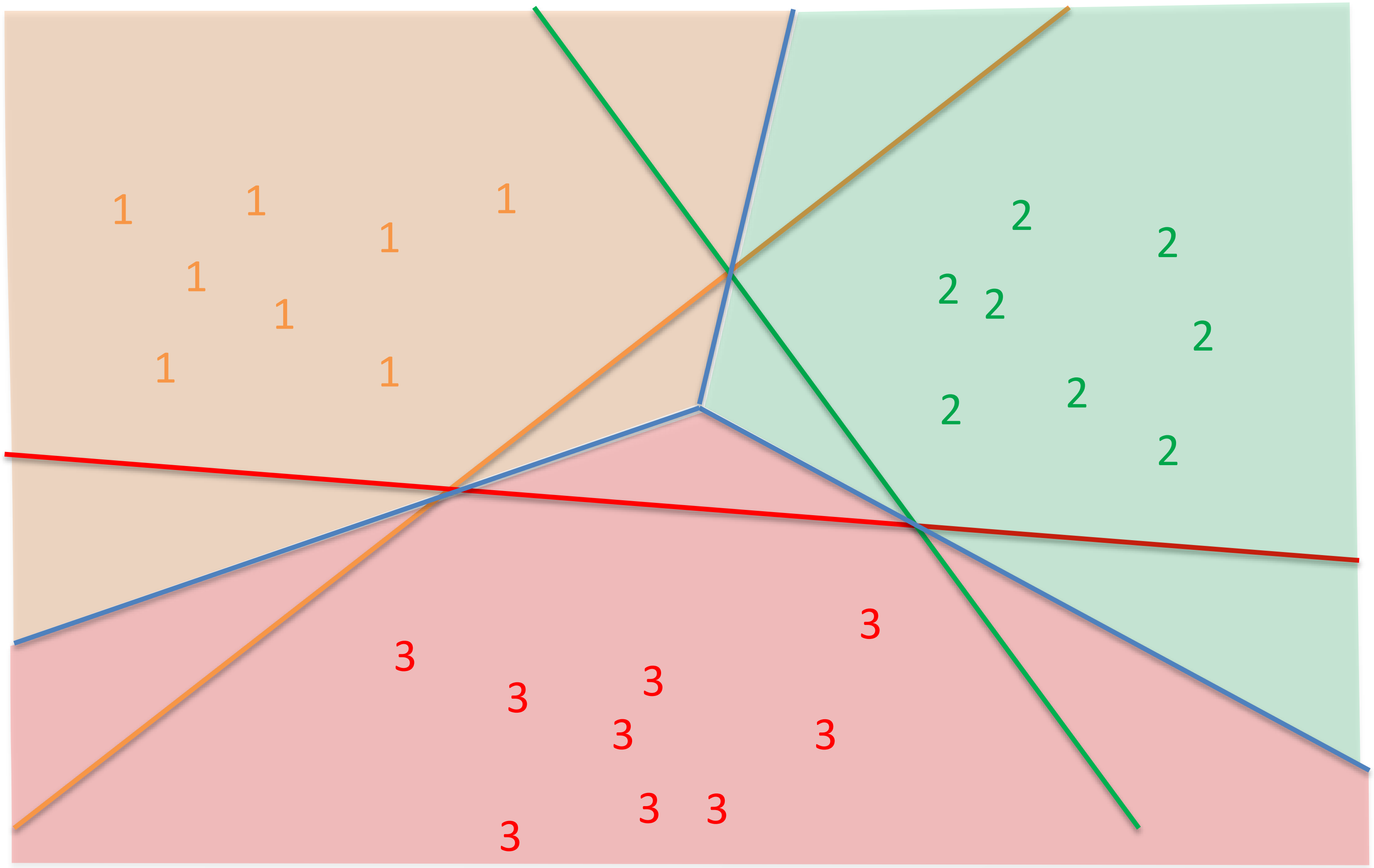
One-Versus-All SVMs

- Compute a classifier for each label versus the remaining labels (i.e., and SVM with the selected label as plus and the remaining labels changed to minuses)
- Let $f^k(x) = w^{(k)^T} x + b^{(k)}$ be the classifier for the k^{th} label
- For a new datapoint x, classify it as

$$k' \in \operatorname{argmax}_k f^k(x)$$

- Drawbacks:
  - If there are L possible labels, requires learning L classifiers over the entire data set
  - Doesn't make sense if the classifiers are not comparable

One-Versus-All SVMs

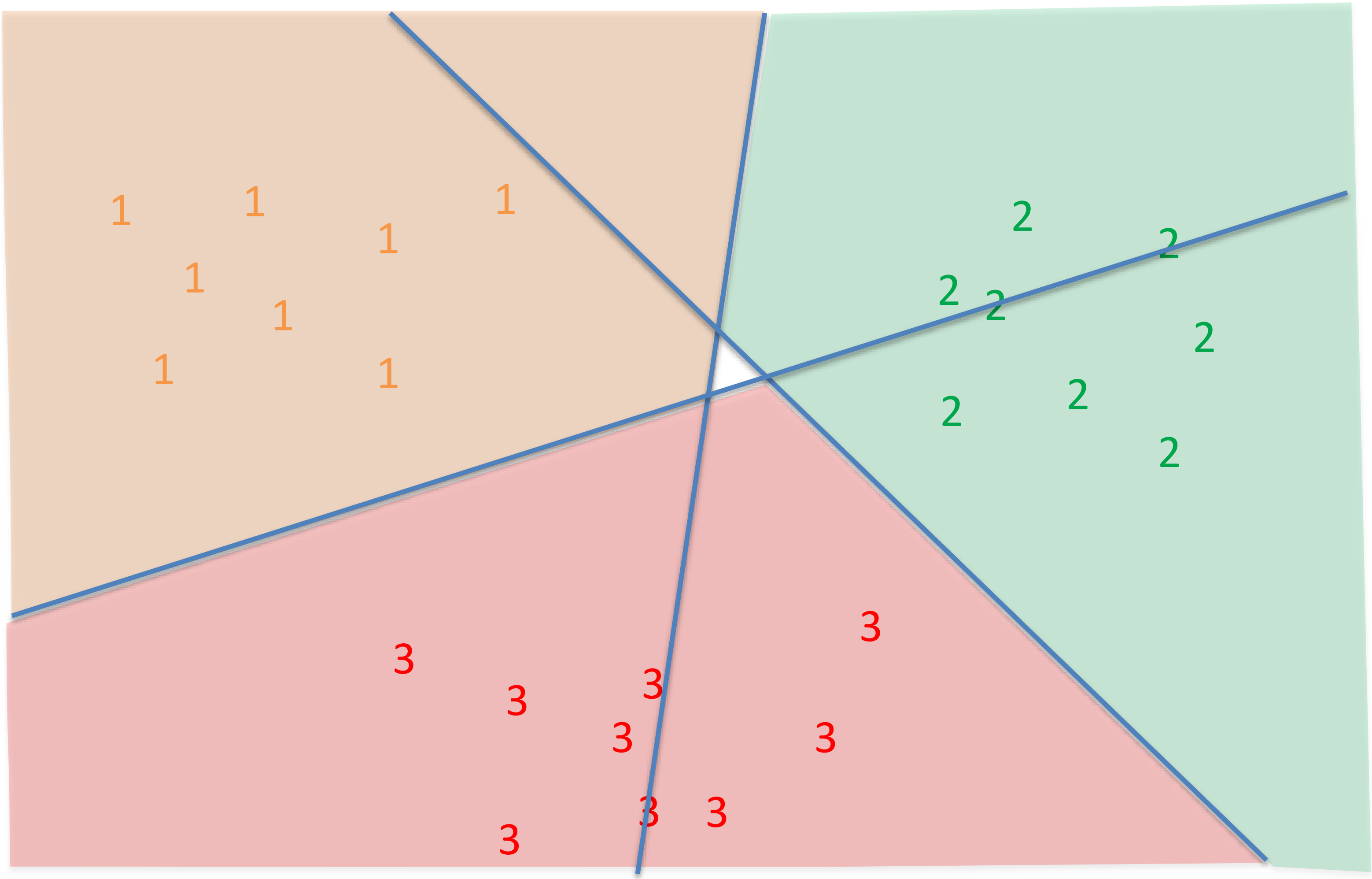


Regions in which points are classified by highest value of $w^T x + b$

One-Versus-One SVMs

- Alternative strategy is to construct a classifier for all possible pairs of labels
- Given a new data point, can classify it by majority vote (i.e., find the most common label among all of the possible classifiers)
- If there are L labels, requires computing $\binom{L}{2}$ different classifiers each of which uses only a fraction of the data
- Drawbacks: Can overfit if some pairs of labels do not have a significant amount of data

One-Versus-One SVMs



Regions determined by majority vote over the classifiers