Binary Classification / Perceptron

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Slides adapted from David Sontag and Vibhav Gogate
Supervised Learning

• **Input:** \((x^{(1)}, y^{(1)}), \ldots, (x^{(M)}, y^{(M)})\)
  
  • \(x^{(m)}\) is the \(m^{th}\) data item and \(y^{(m)}\) is the \(m^{th}\) label

• **Goal:** find a function \(f\) such that \(f(x^{(m)})\) is a “good approximation” to \(y^{(m)}\)
  
  • Can use it to predict \(y\) values for previously unseen \(x\) values
Supervised Learning

- **Hypothesis space**: set of allowable functions $f : X \rightarrow Y$

- **Goal**: find the “best” element of the hypothesis space
  
  - How do we measure the quality of $f$?
Examples of Supervised Learning

- Spam email detection
- Handwritten digit recognition
- Stock market prediction
- More?
Regression

Hypothesis class: linear functions $f(x) = ax + b$

How do we measure the quality of the approximation?
Linear Regression

- In typical regression applications, measure the fit using a squared loss function

\[ L(f) = \frac{1}{M} \sum_{m} (f(x^{(m)}) - y^{(m)})^2 \]

- Want to minimize the average loss on the training data

- For 2-D linear regression, the learning problem is then

\[ \min_{a,b} \frac{1}{M} \sum_{m} (ax^{(m)} + b - y^{(m)})^2 \]

- For an unseen data point, \( x \), the learning algorithm predicts \( f(x) \)
Supervised Learning

• **Select a hypothesis space** (elements of the space are represented by a collection of parameters)

• **Choose a loss function** (evaluates quality of the hypothesis as a function of its parameters)

• **Minimize loss function using gradient descent** (minimization over the parameters)

• **Evaluate quality of the learned model using test data** – that is, data on which the model was not trained
Binary Classification

- Input \((x^{(1)}, y^{(1)}), \ldots, (x^{(M)}, y^{(M)})\) with \(x^{(m)} \in \mathbb{R}^n\) and \(y^{(m)} \in \{-1, +1\}\)

- We can think of the observations as points in \(\mathbb{R}^n\) with an associated sign (either +/- corresponding to 0/1)

- An example with \(n = 2\)
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What is a good hypothesis space for this problem?
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In this case, we say that the observations are linearly separable
Linear Separators

- In $n$ dimensions, a hyperplane is a solution to the equation
  \[ w^T x + b = 0 \]
  with $w \in \mathbb{R}^n, b \in \mathbb{R}$

- Hyperplanes divide $\mathbb{R}^n$ into two distinct sets of points (called open halfspaces)
  \[ w^T x + b > 0 \]
  \[ w^T x + b < 0 \]
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The Linearly Separable Case

- Input \((x^{(1)}, y^{(1)}), \ldots, (x^{(M)}, y^{(M)})\) with \(x^{(m)} \in \mathbb{R}^n\) and \(y^{(m)} \in \{-1, +1\}\)

- Hypothesis space: separating hyperplanes
  \[
  f(x) = \text{sign} \ (w^T x + b)
  \]

- How should we choose the loss function?
The Linearly Separable Case

• Input \((x^{(1)}, y^{(1)}), \ldots, (x^{(M)}, y^{(M)})\) with \(x^{(m)} \in \mathbb{R}^n\) and \(y^{(m)} \in \{-1, +1\}\)

• Hypothesis space: separating hyperplanes
  \[ f(x) = \text{sign} (w^T x + b) \]

• How should we choose the loss function?
  • Count the number of misclassifications
    \[
    \text{loss} = \sum_{m} |y^{(m)} - \text{sign}(w^T x^{(m)} + b)|
    \]
  • Tough to optimize, gradient contains no information
The Linearly Separable Case

• Input \((x^{(1)}, y^{(1)}), \ldots, (x^{(M)}, y^{(M)})\) with \(x^{(m)} \in \mathbb{R}^n\) and \(y^{(m)} \in \{-1, +1\}\)

• Hypothesis space: separating hyperplanes

\[ f(x) = \text{sign} (w^T x + b) \]

• How should we choose the loss function?

• Penalize misclassification linearly by the size of the violation

\[ \text{perceptron loss} = \sum_m \max\{0, -y^{(m)}(w^T x^{(m)} + b)\} \]

• Modified hinge loss (this loss is convex, but not differentiable)
The Perceptron Algorithm

• Try to minimize the perceptron loss using gradient descent

  • The perceptron loss isn't differentiable, how can we apply gradient descent?

  • Need a generalization of what it means to be the gradient of a convex function
Gradients of Convex Functions

- For a differentiable convex function $g(x)$ its gradients are **linear underestimators**
Gradients of Convex Functions

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Gradients of Convex Functions

- For a differentiable convex function $g(x)$ its gradients are linear underestimators: zero gradient corresponds to a global optimum
Subgradients

- For a convex function $g(x)$, a subgradient at a point $x^0$ is given by any line, $l$, such that $l(x^0) = g(x^0)$ and $l(x) \leq g(x)$ for all $x$, i.e., it is a linear underestimator.
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If $\vec{0}$ is a subgradient at $x^0$, then $x^0$ is a global minimum.
Subgradients

• If a convex function is differentiable at a point $x$, then it has a unique subgradient at the point $x$ given by the gradient.

• If a convex function is not differentiable at a point $x$, it can have many subgradients.

  • E.g., the set of subgradients of the convex function $|x|$ at the point $x = 0$ is given by the set of slopes $[-1,1]$.

• Subgradients only guaranteed to exist for convex functions.
The Perceptron Algorithm

- Try to minimize the perceptron loss using (sub)gradient descent
The Perceptron Algorithm

• Try to minimize the perceptron loss using (sub)gradient descent

\[
\nabla_w (\text{perceptron loss}) = - \sum_{m=1}^{M} \left( y^{(m)} x^{(m)} \cdot 1_{-y^{(m)} f_{w,b}(x^{(m)}) \geq 0} \right)
\]

\[
\nabla_b (\text{perceptron loss}) = - \sum_{m=1}^{M} \left( y^{(m)} \cdot 1_{-y^{(m)} f_{w,b}(x^{(m)}) \geq 0} \right)
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The Perceptron Algorithm

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\nabla_w (\text{perceptron loss}) = - \sum_{m=1}^{M} (y^{(m)} x^{(m)} \cdot 1_{y^{(m)} f_{w,b}(x^{(m)}) \geq 0})
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\nabla_b (\text{perceptron loss}) = - \sum_{m=1}^{M} (y^{(m)} \cdot 1_{y^{(m)} f_{w,b}(x^{(m)}) \geq 0})
\]

Is equal to zero if the \(m^{th}\) data point is correctly classified and one otherwise
The Perceptron Algorithm

• Try to minimize the perceptron loss using (sub)gradient descent

\[
\begin{align*}
    w^{(t+1)} &= w^{(t)} + \gamma_t \sum_{m=1}^{M} \left( y^{(m)} x^{(m)} \cdot 1_{-y^{(m)} f_{w,b}(x^{(m)}) \ge 0} \right) \\
    b^{(t+1)} &= b^{(t)} + \gamma_t \sum_{m=1}^{M} \left( y^{(m)} \cdot 1_{-y^{(m)} f_{w,b}(x^{(m)}) \ge 0} \right)
\end{align*}
\]

• With step size \( \gamma_t \) (also called the learning rate)

• Note that, for convergence of subgradient methods, a diminishing step size, e.g., \( \gamma_t = \frac{1}{1+t} \) is required
Stochastic Gradient Descent

- To make the training more practical, stochastic (sub)gradient descent is often used instead of standard gradient descent.

- Approximate the gradient of a sum by sampling a few indices (as few as one) uniformly at random and averaging

\[
\nabla_x \left[ \sum_{m=1}^{M} g_m(x) \right] \approx \frac{1}{K} \sum_{k=1}^{K} \nabla_x g_{m_k}(x)
\]

here, each \( m_k \) is sampled uniformly at random from \( \{1, \ldots, M\} \)

- Stochastic gradient descent converges to the global optimum under certain assumptions on the step size.
• Setting $K = 1$, we pick a random observation $m$ and perform the following update

if the $m^{th}$ data point is misclassified:

$$w^{(t+1)} = w^{(t)} + \gamma_t y^{(m)} x^{(m)}$$

$$b^{(t+1)} = b^{(t)} + \gamma_t y^{(m)}$$

if the $m^{th}$ data point is correctly classified:

$$w^{(t+1)} = w^{(t)}$$

$$b^{(t+1)} = b^{(t)}$$

• Sometimes, you will see the perceptron algorithm specified with $\gamma_t = 1$ for all $t$
Applications of Perceptron

- Spam email classification
  - Represent emails as vectors of counts of certain words (e.g., sir, madam, Nigerian, prince, money, etc.)
  - Apply the perceptron algorithm to the resulting vectors
  - To predict the label of an unseen email
    - Construct its vector representation, $x'$
    - Check whether or not $w^T x' + b$ is positive or negative
Perceptron Learning Drawbacks

- No convergence guarantees if the observations are not linearly separable
- Can overfit
  - There can be a number of perfect classifiers, but the perceptron algorithm doesn’t have any mechanism for choosing between them
What If the Data Isn’t Separable?

• Input \((x^{(1)}, y^{(1)}), ..., (x^{(M)}, y^{(M)})\) with \(x^{(m)} \in \mathbb{R}^n\) and \(y^{(m)} \in \{-1, +1\}\)

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• An example with \(n = 2\)

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What is a good hypothesis space for this problem?
• Perceptron algorithm only works for linearly separable data

Can add **features** to make the data linearly separable in a higher dimensional space!

Essentially the same as higher order polynomials for linear regression!
Adding Features

• The idea, choose a feature map $\phi: \mathbb{R}^n \to \mathbb{R}^k$

• Given the observations $x^{(1)}, \ldots, x^{(M)}$, construct feature vectors $\phi(x^{(1)}), \ldots, \phi(x^{(M)})$

• Use $\phi(x^{(1)}), \ldots, \phi(x^{(M)})$ instead of $x^{(1)}, \ldots, x^{(M)}$ in the learning algorithm

• Goal is to choose $\phi$ so that $\phi(x^{(1)}), \ldots, \phi(x^{(M)})$ are linearly separable in $\mathbb{R}^k$

• Learn linear separators of the form $w^T \phi(x)$ (instead of $w^T x$)

• **Warning**: more expressive features can lead to overfitting!
Adding Features: Examples

- \( \phi \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \)

  - This is just the input data, without modification

- \( \phi \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ x_1 \\ x_2 \\ x_1^2 \\ x_2^2 \end{bmatrix} \)

  - This corresponds to a second degree polynomial separator, or equivalently, elliptical separators in the original space
Adding Features

\[(x_1 - 1)^2 + (x_2 - 1)^2 - 1 \leq 0\]
Adding Features

\[ \begin{align*}
1x_1^2 + 1x_2^2 - 2x_1 - 2x_2 - 2 & \leq 0
\end{align*} \]
Support Vector Machines

- How can we decide between two perfect classifiers?

- What is the practical difference between these two solutions?