Latent/Missing Variables & Hidden Markov Models

Nicholas Ruozzi

University of Texas at Dallas

Unobserved Variables

- Latent or hidden variables in the model are never observed
    - We may or may not be interested in their values, but their existence is crucial to the model
- Some observations in a particular sample may be missing
    - Missing information on surveys or medical records (quite common)
    - We may need to model how the variables are missing

Missing Data

- Data can be missing from the model in many different ways
  - Missing completely at random: the probability that a data item is missing is independent of the observed data and the other missing data
  - Missing at random: the probability that a data item is missing can depend on the observed data
  - Missing not at random: the probability that a data item is missing can depend on the observed data and the other missing data

Modelling Missing Data

- Add additional binary variable m_i to the model for each possible observed variable x_i that indicates whether or not that variable is observed

$$p(x_{obs}, x_{mis}, m) = p(m|x_{obs}, x_{mis})p(x_{obs}, x_{mis})$$

Modelling Missing Data

- Add additional binary variable m_i to the model for each possible observed variable x_i that indicates whether or not that variable is observed

$$p(x_{obs}, x_{mis}, m) = \underbrace{p(m|x_{obs}, x_{mis})}_{\substack{\text{Explicit model of the missing} \\ \text{data} \\ \text{(missing not at random)}}} p(x_{obs}, x_{mis})$$

Modelling Missing Data

- Add additional binary variable m_i to the model for each possible observed variable x_i that indicates whether or not that variable is observed

$$p(x_{obs}, x_{mis}, m) = \underbrace{p(m|x_{obs})}_{\text{Missing at random}} p(x_{obs}, x_{mis})$$

Modelling Missing Data

- Add additional binary variable m_i to the model for each possible observed variable x_i that indicates whether or not that variable is observed

$$p(x_{obs}, x_{mis}, m) = \underbrace{p(m)}_{\text{Missing completely at random}} p(x_{obs}, x_{mis})$$

Modelling Missing Data

- Add additional binary variable m_i to the model for each possible observed variable x_i that indicates whether or not that variable is observed

$$p(x_{obs}, x_{mis}, m) = p(m)p(x_{obs}, x_{mis})$$

How can you model latent variables in this framework?

Learning with Missing Data

- In order to design learning algorithms for models with missing data, we will make two assumptions

  1. The data is missing at random

  2. The model parameters corresponding to the missing data (δ) are separate from the model parameters of the observed data (θ)

- That is

$$p(x_{obs}, m|\theta, \delta) = p(m|x_{obs}, \delta)p(x_{obs}|\theta)$$

- Derivation of the algorithm in this case then follows similarly to the previous discussion

Learning with Latent Variables

- Log-likelihood with latent variables:

$$\log l(\theta) = \sum_{i=1}^{N} \log p(x^{(i)}|\theta)$$
$$= \sum_{i=1}^{N} \log \sum_{y} p(x^{(i)}, y|\theta)$$

  - Again, this is typically not a concave function of θ
    - We will apply the same trick that we did with GMMs last lecture

Expectation Maximization

$$\log l(\theta) = \sum_{i=1}^{N} \log p(x^{(i)}|\theta)$$

$$= \sum_{i=1}^{N} \log \sum_{y} p(x^{(i)}, y|\theta)$$

$$= \sum_{i=1}^{N} \log \sum_{y} q_i(y) \cdot \frac{p\left(x^{(i)}, y\middle|\theta\right)}{q_{i(y)}}$$

$$\geq \sum_{i=1}^{N} \sum_{y} q_i(y) \log \frac{p\left(x^{(i)}, y\middle|\theta\right)}{q_{i(y)}}$$

Expectation Maximization

$$F(q,\theta) \equiv \sum_{i=1}^{N} \sum_{y} q_i(y) \log \frac{p\left(x^{(i)}, y \middle| \theta\right)}{q_i(y)}$$

- Maximizing F is equivalent to the maximizing the log-likelihood
- Maximize it using coordinate ascent

$$q^{t+1} = \arg \max_{q_1,\ldots,q_K} F(q, \theta^t)$$

$$\theta^{t+1} = \operatorname*{argmax}_{\theta} F(q^{t+1}, \theta)$$

Expectation Maximization

$$\sum_{i=1}^{N} \sum_{y} q_i(y) \log \frac{p\left(x^{(i)}, y \middle| \theta^t\right)}{q_i(y)}$$

- Maximized when $q_i(y) = p\left(y \middle| x^{(i)}, \theta^t\right)$

- Can reformulate the EM algorithm as

$$\theta^{t+1} = \operatorname*{argmax}_{\theta} \sum_{i=1}^{N} \sum_{y} p(y|x^{(i)}, \theta^t) \log p\left(x^{(i)}, y \middle| \theta\right)$$

Latent Variable Models

- Many real-world models contain latent variables
- Because we will need to marginalize out over the latent variables in MLE, the presence of latent variables in the model can make performing MLE much harder
  - As before, we will make simplifying assumptions about the probability distribution of the latent variables

Markov Chains

- A Markov chain is a sequence of random variables $X_1, \dots, X_T \in S$ such that

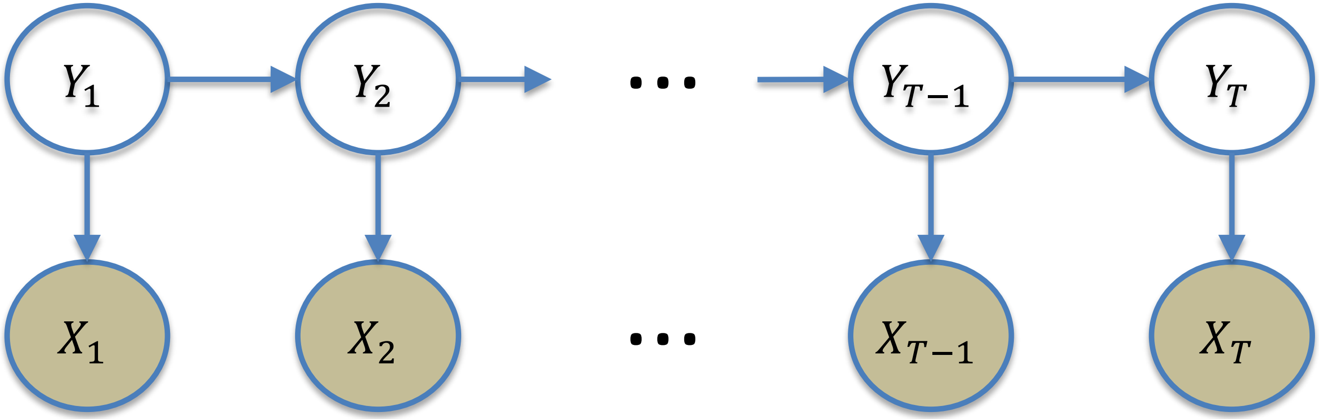
$$p(x_{t+1}|x_1, \dots, x_T) = p(x_{t+1}|x_t\ )$$

- The set S is called the state space, and $p(X_{t+1} = j|X_t = i)$ is the probability of transitioning from state i to state j at step t

Markov Chains

- When the probability of transitioning between two states does not depend on time, we call it a time homogeneous Markov chain
  - Represent it by a $|S| \times |S|$ transition matrix A
    - $A_{ij} = p(X_{t+1} = j | X_t = i)$
    - A is a **stochastic** matrix (all rows sum to one)

Hidden Markov Models



$$p(x_1, \dots, x_T, y_1, \dots, y_T) = p(y_1)p(x_1|y_1)\prod_t p(y_t|y_{t-1})p(x_t|y_t)$$

- X’s are observed variables, Y’s are latent/hidden
- Time homogenous: $p(y_t = j|y_{t-1} = i) = p(y_{t'} = j|y_{t'-1} = i)$
- For learning, we are given sequences of observations

Hidden Markov Models

- Well suited to problems/models that evolve over time
- Examples:
  - Observations correspond sizes of tree growth rings for one year, the latent variables correspond to average temperature
  - Observations correspond to noisy missile location, latent variables correspond to true missile locations

Learning HMMs

- A bit of notation:
  - $\pi_i = p(Y_1 = i)$
  - $A_{ij} = p(Y_t = j | Y_{t-1} = i)$
  - $b_j(x_t) = p(X_t = x_t | Y_t = j)$
- These parameters describe an HMM, $\theta = \{\pi, A, b\}$
  - We'll derive the updates in the case that the observations X_t are discrete random variables

Learning HMMs

$$\sum_y p(y|x,\theta^s)\log p(x,y|\theta) =$$

$$= \sum_y p(y|x,\theta^s)\log\left(p(y_1)p(x_1|y_1)\prod_{t=2}^{T} p(y_t|y_{t-1})p(x_t|y_t)\right)$$

$$= \sum_y p(y|x,\theta^s)\log\left(\pi_{y_1} b_{y_1}(x_1)\prod_{t=2}^{T} A_{y_t,y_{t-1}} b_{y_t}(x_t)\right)$$

$$= \sum_y p(y|x,\theta^s)\log\pi_{y_1} + \sum_y p(y|x,\theta^s)\left(\sum_{t=1}^{T}\log b_{y_t}(x_t)\right) + \sum_y p(y|x,\theta^s)\left(\sum_{t=2}^{T}\log A_{y_t,y_{t-1}}\right)$$

$$= \sum_i p(Y_1 = i|x,\theta^s)\log\pi_i + \sum_{t=1}^{T}\sum_i p(Y_t = i|x,\theta^s)\log b_i(x_t) + \sum_{t=2}^{T}\sum_i\sum_j p(Y_t = i, Y_{t-1} = j|x,\theta^s)\log A_{i,j}$$

Learning HMMs

$$p(y|x,\theta^s) = \pi_{y_1}^{s-1} b_{y_1}^{s-1}(x_1) \prod_{t=2}^{T} A_{y_t,y_{t-1}}^{s-1} b_{y_t}^{s-1}(x_t)$$

$$\pi_i^s = p(Y_1 = i|x,\theta^s)$$

$$b_i^s(k) = \frac{\sum_{t=1}^{T} p(Y_t = i|x,\theta^s)\delta(x_t = k)}{\sum_{t=1}^{T} p(Y_t = i|x,\theta^s)}$$

$$A_{ij}^s = \frac{\sum_{t=2}^{T} p(Y_t = i, Y_{t-1} = j|x,\theta^s)}{\sum_{t=2}^{T} p(Y_{t-1} = j|x,\theta^s)}$$

Prediction in HMMs

- Once we learn the model, given a new sequence of observations, $x_1, \dots, x_T$, we want to predict y_T
  - In the tree application, this corresponds to finding the temperature at a specific time given the rings of a tree
  - In the missile tracking example, this corresponds to finding the position of the missile at a particular time
- Want to compute $p(y_T|x, \theta)$

Prediction in HMMs

- Want to compute $p(y_T|x,\theta) = p(x,y_T|\theta)/p(x|\theta)$

- Direct approach:

$$p(x, Y_T = i|\theta) = \sum_{y_1,\dots,y_{T-1}} p(x, y_1, \dots, y_{T-1}, Y_T = i|\theta)$$

- Dynamic programming approach:

$$\begin{aligned} p(x, Y_T = i|\theta) &= \sum_j p(x, Y_T = i, Y_{T-1} = j) \\ &= \sum_j p(x_1, \dots, x_{T-1}, Y_{T-1} = j) p(x_T, Y_T = i|x_1, \dots, x_{T-1}, Y_{T-1} = j) \\ &= \sum_j p(x_1, \dots, x_{T-1}, Y_{T-1} = j) p(x_T|Y_T = i) p(Y_T = i|Y_{T-1} = j) \end{aligned}$$

Prediction in HMMs

- Want to compute $p(y_T|x,\theta) = p(x,y_T|\theta)/p(x|\theta)$

- Direct approach:

$$p(x, Y_T = i|\theta) = \sum_{y_1,\dots,y_{T-1}} p(x, y_1, \dots, y_{T-1}, Y_T = i|\theta)$$

- Dynamic programming approach:

Called **filtering**: easy to implement using dynamic programming

$$p(x, Y_T = i|\theta) = \sum_j p(x, Y_T = i, Y_{T-1} = j)$$

$$= \sum_j p(x_1, \dots, x_{T-1}, Y_{T-1} = j) p(x_T, Y_T = i|x_1, \dots, x_{T-1}, Y_{T-1} = j)$$

$$= \sum_j p(x_1, \dots, x_{T-1}, Y_{T-1} = j) p(x_T|Y_T = i) p(Y_T = i|Y_{T-1} = j)$$

Latent Variables & EM

- Previous updates derived for a single observation (to simplify)
    - Can get the general updates for multiple sequences by adding sums in the appropriate places
- Same principle as EM for mixture models
    - Also suffers from the existence of lots of local optima