Bayesian Methods

Nicholas Ruozzi
University of Texas at Dallas

based on the slides of Vibhav Gogate

Binary Variables

- Coin flipping: heads=1, tails=0 with bias μ

$$p(X = 1|\mu) = \mu$$

- Bernoulli Distribution

$$Bern(x|\mu) = \mu^x \cdot (1 - \mu)^{1-x}$$

$$E[X] = \mu$$

$$var(X) = \mu \cdot (1 - \mu)$$

Binary Variables

- N coin flips: $X_1, \ldots, X_N$

$$p(\textstyle\sum_i X_i = m | N, \mu) = \binom{N}{m} \mu^m (1-\mu)^{N-m}$$

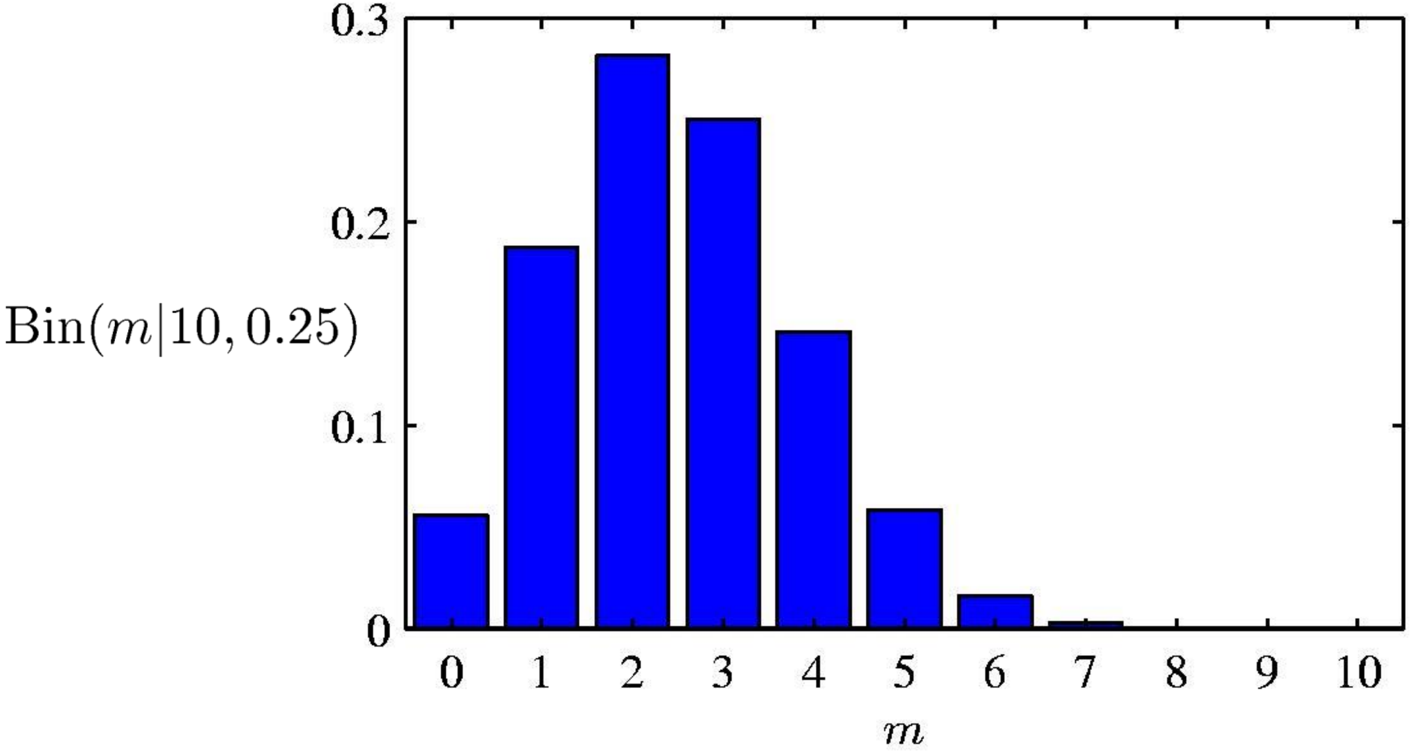
- Binomial Distribution

$$Bin(m|N,\mu) = \binom{N}{m} \mu^m (1-\mu)^{N-m}$$

$$E\left[\sum_i X_i\right] = N\mu$$

$$var\left[\sum_i X_i\right] = N\mu(1-\mu)$$

Binomial Distribution



Estimating the Bias of a Coin

- Suppose that we have a coin, and we would like to figure out what the probability is that it will flip up heads
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Estimating the Bias of a Coin

- Suppose that we have a coin, and we would like to figure out what the probability is that it will flip up heads
  - How should we estimate the bias?

  - With these coin flips, our estimate of the bias is: 3/5
    - Why is this a good estimate of the bias?

Coin Flipping – Binomial Distribution

- $P(Heads) = \theta,\ P(Tails) = 1 - \theta$
- Flips are i.i.d.
  – Independent events
  – Identically distributed according to Binomial distribution
- Our training data consists of α_H heads and α_T tails

$$p(D|\theta) = \theta^{\alpha_H} \cdot (1-\theta)^{\alpha_T}$$

Maximum Likelihood Estimation (MLE)

- **Data:** Observed set of α_H heads and α_T tails
- **Hypothesis:** Coin flips follow a binomial distribution
- **Learning:** Find the "best" θ
- **MLE:** Choose θ to maximize probability of D given θ

$$
\begin{aligned}
\widehat{\theta} &= \arg\max_{\theta} \; P(\mathcal{D} \mid \theta) \\
&= \arg\max_{\theta} \; \ln P(\mathcal{D} \mid \theta)
\end{aligned}
$$

First Parameter Learning Algorithm

$$
\begin{aligned}
\hat{\theta} &= \arg\max_{\theta} \quad \ln P(\mathcal{D} \mid \theta) \\
&= \arg\max_{\theta} \quad \ln \theta^{\alpha_H}(1-\theta)^{\alpha_T}
\end{aligned}
$$

Set derivative to zero, and solve!

$$
\begin{aligned}
\frac{d}{d\theta} \ln P(\mathcal{D} \mid \theta) &= \frac{d}{d\theta}\left[\ln \theta^{\alpha_H}(1-\theta)^{\alpha_T}\right] \\
&= \frac{d}{d\theta}\left[\alpha_H \ln \theta + \alpha_T \ln(1-\theta)\right] \\
&= \alpha_H \frac{d}{d\theta} \ln \theta + \alpha_T \frac{d}{d\theta} \ln(1-\theta) \\
&= \frac{\alpha_H}{\theta} - \frac{\alpha_T}{1-\theta} \quad = 0
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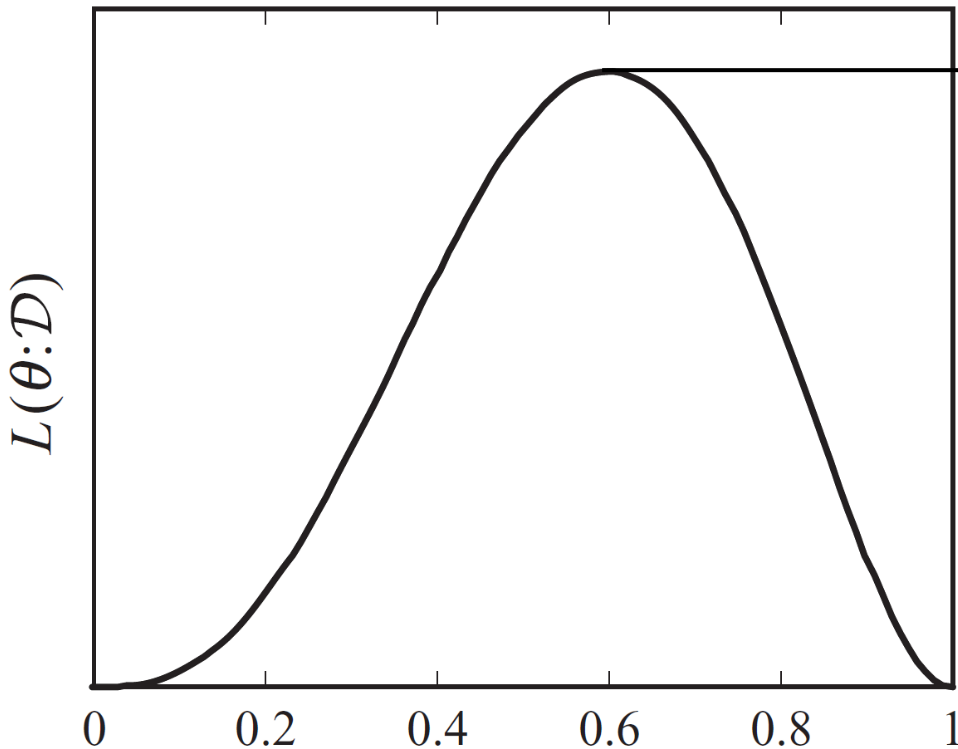
$$
\boxed{\widehat{\theta}_{MLE} \quad = \quad \frac{\alpha_H}{\alpha_H + \alpha_T}}
$$

Coin Flip MLE



$$\hat{\theta}_{MLE} = \frac{\alpha_H}{\alpha_H + \alpha_T}$$

Priors

- Suppose we have 5 coin flips all of which are heads
  - Our estimate of the bias is?

Priors

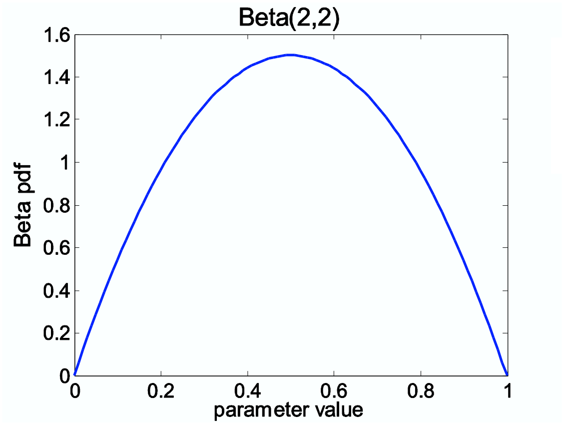
- Suppose we have 5 coin flips all of which are heads
  - MLE would give $\theta_{MLE} = 1$
  - This event occurs with probability $\frac{1}{2^5} = \frac{1}{32}$ for a fair coin
  - Are we willing to commit to such a strong conclusion with such little evidence?

Priors

- Priors are a Bayesian mechanism that allow us to take into account "prior" knowledge about our belief in the outcome

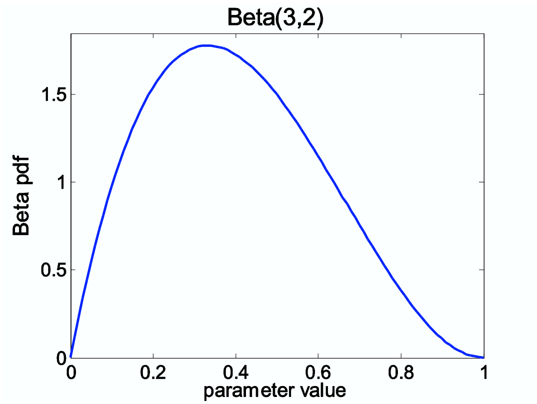
- Rather than estimating a single θ, consider a distribution over possible values of θ given the data
  - Update our prior after seeing data

Our best guess in the absence of any data



Observe flips
e.g.: {tails, tails}

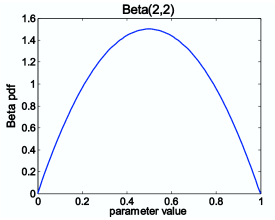
Our estimate after we see some data



Bayesian Learning

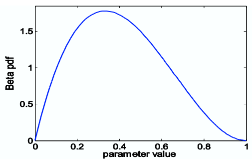
Apply Bayes rule:

Prior



Data Likelihood

Posterior



$$p(\theta|D) = \frac{p(D|\theta)p(\theta)}{p(D)}$$

Normalization

- Or equivalently: $p(\theta|D) \propto p(D|\theta)p(\theta)$
- For uniform priors this reduces to the MLE objective

$$p(\theta) \propto 1 \qquad \Rightarrow \qquad p(\theta|D) \propto p(D|\theta)$$

Picking Priors

- How do we pick a good prior distribution?
  - Could represent expert domain knowledge
  - Statisticians choose them to make the posterior distribution "nice" (conjugate priors)
- What is a good prior for the bias in the coin flipping problem?

Picking Priors

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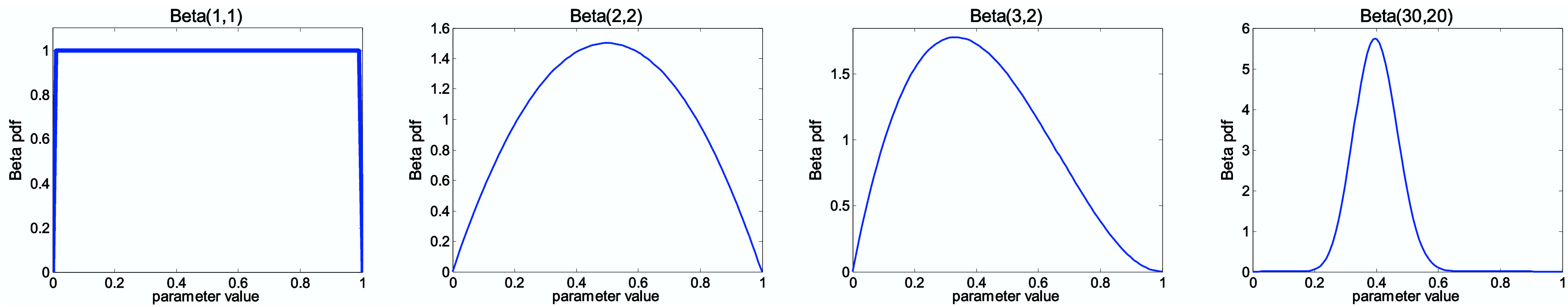
- What is a good prior for the bias in the coin flipping problem?
  - Truncated Gaussian (tough to work with)
  - Beta distribution (works well for binary random variables)

Coin Flips with Beta Distribution

Likelihood function: $$P(\mathcal{D} \mid \theta) = \theta^{\alpha_H}(1-\theta)^{\alpha_T}$$

Prior: $$P(\theta) = \frac{\theta^{\beta_H - 1}(1-\theta)^{\beta_T - 1}}{B(\beta_H, \beta_T)} \sim Beta(\beta_H, \beta_T)$$



Posterior:

$$\begin{aligned} P(\theta \mid \mathcal{D}) &\propto \theta^{\alpha_H}(1-\theta)^{\alpha_T}\ \theta^{\beta_H - 1}(1-\theta)^{\beta_T - 1} \\ &= \theta^{\alpha_H + \beta_H - 1}(1-\theta)^{\alpha_T + \beta_T - 1} \\ &= Beta(\alpha_H + \beta_H, \alpha_T + \beta_T) \end{aligned}$$

MAP Estimation

- Choosing θ to maximize the posterior distribution is called maximum a posteriori (MAP) estimation

$$\theta_{MAP} = \arg\max_{\theta} p(\theta|D)$$

- The only difference between θ_{MLE} and θ_{MAP} is that one assumes a uniform prior (MLE) and the other allows an arbitrary prior

Priors

- Suppose we have 5 coin flips all of which are heads
  - MLE would give $\theta_{MLE} = 1$
  - MLE with a $Beta(2,2)$ prior gives $\theta_{MAP} = \frac{6}{7} \approx .857$
  - As we see more data, the effect of the prior diminishes
    - $\theta_{MAP} = \frac{\alpha_H + \beta_H - 1}{\alpha_H + \beta_H + \alpha_T + \beta_T - 2} \approx \frac{\alpha_H}{\alpha_H + \alpha_T}$ for large # of observations

Sample Complexity

- How many coin flips do we need in order to guarantee that our learned parameter does not differ too much from the true parameter (with high probability)?

- Can use Chernoff bound (again!)
  - Suppose $Y_1, \ldots, Y_N$ are i.i.d. random variables taking values in $\{0, 1\}$ such that $E_p[Y_i] = y$. For $\epsilon > 0$,

$$p\left(\left|y - \frac{1}{N}\sum_i Y_i\right| \geq \epsilon\right) \leq 2e^{-2N\epsilon^2}$$

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  - For the coin flipping problem with $X_1, \ldots, X_n$ iid coin flips and $\epsilon > 0$,

$$p\left(\left|\theta_{true} - \frac{1}{N}\sum_i X_i\right| \geq \epsilon\right) \leq 2e^{-2N\epsilon^2}$$

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$$\delta \geq 2e^{-2N\epsilon^2} \Rightarrow N \geq \frac{1}{2\epsilon^2} \ln \frac{2}{\delta}$$