RESEARCH STATEMENT

NATHAN WILLIAMS

My research is in the field of algebraic combinatorics. Specifically, I am interested in combinatorics arising within the framework of reflection groups or their associated Artin groups, as well as in Garside theory, representation theory, and lattices. I am also committed to undergraduate research, having mentored student projects on original research problems over two summers.

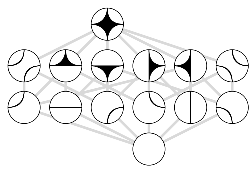
A first course in classical enumerative combinatorics teaches us to breathe life into inanimate integer sequences by interpreting them as counting friendly combinatorial objects. The *Catalan numbers* are the well-loved sequence

$$\mathsf{Cat}(n) := \frac{1}{n+1}\binom{2n}{n} = 1, 1, 2, 5, 14, 42, 132, 429, 1430, \ldots,$$

which are known to count over two hundred different sets of objects [Sta15]. One example is the set of *noncrossing partitions*—those set partitions of $[n] := \{1, 2, \ldots, n\}$ whose blocks have disjoint convex hulls when drawn on a circle. The image to the right of Equation (1) shows the 14 noncrossing partitions of $n = 4$.

These animating combinatorial objects often come with additional structure, which can lead to interesting and beautiful mathematics. Thus, refining $\mathsf{Cat}(n)$ to enumerate noncrossing partitions by their number of blocks gives the identity

$$\mathsf{Cat}(n) = \sum_{k=0}^{n} \frac{1}{k}\binom{n-1}{k-1}\binom{n}{k-1} \qquad (1)$$



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Since the mid-1980s, there has been spectacular growth in the interaction between combinatorics, algebra, and geometry. Modern algebraic combinatorics modifies this elementary approach by finding interpretations of combinatorial objects within naturally occurring algebraic and geometric constructions. This philosophy has been successfully implemented using reflection groups, representation theory, and algebraic geometry, where classical combinatorics often masquerades as "type A" phenomena. Interpreting objects in these contexts often supplies powerful new techniques, suggests generations, and provides motivation and expanatory power.

For example, by thinking of the blocks of a noncrossing partition as the cycles of a permutation, we obtain a special subset of elements of the symmetric group $\mathfrak{S}_n$; letting $\mathfrak{S}_n$ act on $\mathbb{R}^{n-1}$ and counting these elements by the dimension of their fixed space gives Equation (1). Another beautiful algebraic interpretation associates noncrossing partitions with a basis for the highest weight representation $\mathsf{Sp}_{2n}(\omega_n)$ of the symplectic group; the branching rule to GL_N gives Equation (1) again. A third interpretation comes from the diagonal action of $\mathfrak{S}_n$ on the polynomial ring $\mathbb{C}[\mathbf{x}, \mathbf{y}] := \mathbb{C}[x_1, y_1, \ldots, x_n, y_n]$; the bigraded Hilbert polynomial of the alternating part of the quotient ring of diagonal coinvariants $\mathbb{C}[\mathbf{x}, \mathbf{y}]/\mathbb{C}[\mathbf{x}, \mathbf{y}]_+^{\mathfrak{S}_n}$ then gives a two-variable polynomial refinement of $\mathsf{Cat}(n)$.

My research naturally breaks into three related themes that draw from the fruitful interaction between combinatorics, geometry, and algebra:

1. Actions on root-theoretic posets and poset identities;
2. Coxeter-Catalan combinatorics; and
3. Affine Weyl groups and infinite Coxeter groups.

1. Posets.

[1] (with J. Striker) *Promotion and Rowmotion*, European Journal of Combinatorics **33**, 1919–1942, 2012.
[2] (with Z. Hamaker) *Subwords and Plane Partitions*, DMTCS Proceedings, Daejeon, South Korea, 2015.
[3] (with A. Schilling, N. Thiéry, G. White) *Braid moves in commutation classes of the symmetric group*, arXiv preprint, arXiv:1507.00656, 2015, *submitted.*
[4] (with Z. Hamaker, R. Patrias, O. Pechenik) *Coincidental Types and their Minuscule Doppelgängers*, *in progress.*

The number of order ideals in a product of two chains $\mathcal{P} := [a] \times [b]$ is $\binom{a+b}{b}$; the number of standard Young tableaux and plane partitions of a certain height in $\mathcal{P}$ also have beautiful product formulas (which can be explained, for example, using the Weyl dimension formula [Hum72]). Finite and affine root systems give rise to interesting and natural partial orders that generalize this example. In [1], we combinatorially explain the behavior of a cyclic action—"rowmotion" row—on the order ideals of a wide class of such (sub)posets of positive roots by relating row to promotion of Young tableaux [Sta09, BS74, FDF93, CFDF95, Pan09]. There have been many subsequent developments—notably, the papers [EP13, PR15, GR14, RS13] and a related AIM workshop in 2015 (for which I was one of the organizers). I am interested in further exploring the connections of generalized (birational) rowmotion on minuscule posets and Zamolodchikov periodicity. In an ongoing project [4], we use techniques from K-theoretic Schubert calculus to give bijective proofs of a vast generalization of certain poset identities compiled in [7,8], whose only existing proofs were either computational or representation-theoretic [Pro83, She99, Pur14, Hai92, TY09, BS14].

2. Catalan combinatorics.

[5] *Rational Catalan Combinatorics.* Notes from the Rational Catalan Combinatorics AIM Workshop, 2012.
[6] (with D. Armstrong and B. Rhoades) *Rational associahedra and noncrossing partitions*, Electronic Journal of Combinatorics **20**(3), P54, 2013.
[7] *Cataland.* Ph.D. dissertation, 2013.
[8] *Bijactions in Cataland*, DMTCS Proceedings, Chicago, USA, 597–608, 2014.
[9] (with H. Mühle) *Tamari Lattices for Parabolic Quotients of the Symmetric Group*, DMTCS Proceedings, Daejeon, South Korea, 2015.
[10] (with C. Stump and H. Thomas) *Cataland: Why the Fuss?*, arXiv preprint, 2015, arXiv:1503.00710.
[11] (with T. Gobet) *Noncrossing partitions and Bruhat order*, 2015, *accepted* to European Journal of Combinatorics.
[12] *W-Associahedra are In-Your-Face*, arXiv preprint, arXiv:1502.01405, 2015, *submitted.*
[13] (with C. Stump and H. Thomas) *Cataland: Bijactions*, *in progress.*

The Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$ are associated to the Coxeter group $\mathfrak{S}_n$, and count—for example—the number of triangulations of a convex $(n+2)$-gon and the number of 231-avoiding permutations. Catalan numbers beautifully generalize to all other finite Coxeter groups; triangulations become finite-type clusters [FZ03] and 231-avoiding permutations become sortable elements [BW97, Rea07a, Rea07b, RS11]. In a different direction, the number of $(m+2)$-angulations of a convex $(mn+2)$-gon are counted by the Fuss-Catalan numbers $\frac{1}{mn+1}\binom{(m+1)n}{n}$. This formula also generalizes to other finite Coxeter groups, and we show in [10] that the interval $[e, w_\circ^m]$ (in the positive Artin monoid) is the correct setting for generalizing N. Reading's sortable elements and clusters [Rea07b], obtaining a new lattice structure that generalizes the associahedron. This lattice appears to have deep connections to higher spaces of diagonal harmonics [BPR11, BMFPR12, BMCPR13]. We perform related constructions for parabolic quotients in [7,9], and generalize [STT88, CP16] in our study of geodesics on associahedra in [12]. Continuing work from Section 1, we refine the study of rowmotion in [7,8,13], and using root orders we have produced tantalizingly explicit—but still unproven—conjectural bijections between finite-type clusters and order ideals of the root poset.

3. Affine Weyl groups.

[14] (with H. Thomas) *Cyclic Symmetry of the Scaled Simplex*, Journal of Algebraic Combinatorics **39**(2), 225–246, 2012.

[15] (with M. Visontai) *Stable multivariate W-Eulerian polynomials*, Journal of Combinatorial Theory Series A **120**(7), 1929–1945, 2013.

[16] (with C. Berg and M. Zabrocki) *Symmetries of the k-bounded partition lattice*, accepted to Annals of Combinatorics, 2015.

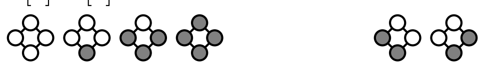
[17] (with M. Thiel) *Strange Expectations*, arXiv preprint, 2015, arXiv:1508.05293.

[18] (with P. Nadeau and C. Hohlweg) *Automata, reduced words, and Garside shadows in Coxeter groups*, 2015.

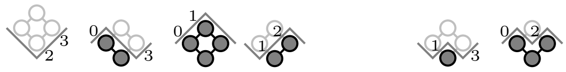
In [14,16], we study the combinatorics of the b-fold dilations of an a-dimensional simplex associated to the affine symmetric group $\widetilde{\mathfrak{S}}_a$, which is closely linked to parking functions [Shi87, Sta96, Sut02, Sut04, CP02]. The affine hyperplanes naturally subdivide this simplex into b^a smaller similar simplices, which may be indexed by certain integer partitions. We characterize and explain this labeling in [16], and we describe its symmetries in [14]. The (coroot) lattice points inside the simplex are indexed by integer partitions that are simultaneously an a-core and a b-core; in [17], generalizing to all simply-laced types, we use Ehrhart theory to give uniform formulas for the maximum and expected "size" of such a point [Mac71, GKS90, Joh15]. This polytope generalizes to any Coxeter group using the notion of small roots, while the combinatorics of the enclosed region is captured with the algebraic definition of a Garside shadow [DDH15, Deh15]. In [18], we study automata recognizing reduced words for infinite Coxeter groups using Garside shadows, and we are interested in further studying the minimal such automaton.

1. Root Systems, Posets, Promotion, and Rowmotion

Fix a finite poset $\mathcal{P}$, and let $\mathcal{J}(\mathcal{P})$ be its set of *order ideals*—("down-closed") subsets I of $\mathcal{P}$ such that if $q \preceq p$ and $p \in I$, then $q \in I$. Given $I \in \mathcal{J}(\mathcal{P})$, we define $\mathsf{row}(I)$ to be the order ideal generated by the minimal elements of $\mathcal{P}$ *not* in I [Duc74, DF90, Pan09]. Because row is invertible, $\mathcal{J}(\mathcal{P})$ under row decomposes into cyclic orbits. This operation has motivations from matroid theory, and it often exhibits fascinating orbit structure. The illustration below gives the two orbits of the six order ideals of $[2] \times [2]$ under row.



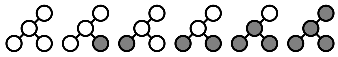
For general posets $\mathcal{P}$, the orbit structure of $\mathcal{J}(\mathcal{P})$ under row defies characterization. Remarkably, there are several families of posets (coming from root systems) on which row has been observed to be "well-behaved," in the sense that row acts as a cyclic group of much smaller order than $|\mathcal{J}(\mathcal{P})|$. For example, there are $\binom{a+b}{b}$ order ideals when $\mathcal{P} = [a] \times [b]$ is the product of two chains, but the order of row is just $a + b$. In [1], we explain this coincidence in many interesting cases by proving that row (*rowmotion*) is conjugate—inside the "toggle group"—to an action pro (*promotion*) that is more obviously well-behaved. The illustration below gives the orbits of $[2] \times [2]$ under pro—an order ideal is traced out by a path, and the positions of the north-west steps define a bijection to subsets of $\{0, 1, 2, 3\}$ of size two under the cyclic action $i \mapsto i + 1 \pmod 4$.



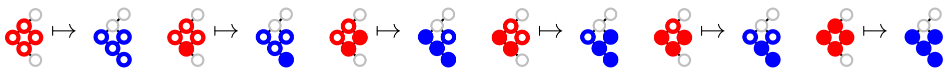
Theorem 1 ([1]). *There is an (explicit) equivariant bijection between $\mathcal{J}(\mathcal{P})$ under* pro *and $\mathcal{J}(\mathcal{P})$ under* row*.*

By connecting $\mathcal{J}(\mathcal{P})$ to standard Young tableaux, we simultaneously generalize results of R. Stanley [Sta09] and D. Armstrong, C. Stump, and H. Thomas [AST13].

In [7,8], I observed that the *coincidental (Cartan) types* A, B, H_3, and $I_2(m)$ were exactly those types whose root posets $\Phi^+(W)$ satisfied certain poset-theoretic identities: for example, it is a result of R. Proctor that there are the same number of plane partitions of height k in $\Phi^+(B_n)$ and in $[n] \times [n]$. The six plane partitions of height 1 for the $\Phi^+(B_2)$ root poset are illustrated below.



All general existing proofs of these identities were computational and representation-theoretic, and—in particular—*non-bijective* [8]. An exciting ongoing project with O. Pechenik, R. Patrias, and Z. Hamaker synthesizes a remark about E_7 by R. Proctor and a beautiful idea of Z. Hamaker to use the (co)minuscule *K-theoretic Schubert calculus* techniques of A. Yong and H. Thomas to give *bijective* proofs of these identities [4]. The crucial observation is that the coincidental types are exactly those types whose root poset is the *bottom half* of a minuscule poset. This minuscule poset corresponds to a minuscule variety—as illustrated below, our approach is to embed a *second* minuscule into the first (all circles, representing $\mathsf{OGr}(4)$) as a Richardson variety (the red circles, representing $\mathsf{Gr}(2, 4)$), and then use K-theoretic jeu-de-taquin to degenerate it to a Schubert variety (the blue circles, representing the dual of $\Phi^+(B_2)$).



Theorem 2 ([4]). *K-theoretic jeu-de-taquin gives bijections from standard and increasing fillings of embeddings $\mathsf{LG}(n, 2n) \hookrightarrow \mathsf{Gr}(n, \mathbb{C}^{2n})$, $\mathsf{Gr}(n, \mathbb{C}^{2n-1}) \hookrightarrow \mathsf{OG}(2n+1, 4n+2)$, $\mathsf{OG}(7, 14) \hookrightarrow \mathsf{G}_\omega(\mathbb{O}^3, \mathbb{O}^6)$, and $\mathbb{Q}^{2n-2} \hookrightarrow \mathbb{Q}^{4(n-1)-2}$ to corresponding fillings of root posets of coincidental type.*

This embedding technique yields many other highly nontrivial bijections, which we are in the process of classifying using multiplicity-free Schubert calculus.

There have been several subsequent developments motivated by the appearance of our paper [1]. D. Einstein and J. Propp generalized toggles to a piecewise linear action on the order polytope $\mathcal{O}(\mathcal{P})$, and then even further to a birational setting evocative of cluster algebras [EP13, PR15]. T. Roby and D. Grinberg proved that the order of birational rowmotion on $[a] \times [b]$ remains $a + b$ [GR14]. In a different direction, by relating row on minuscule posets to the action of a Coxeter element on certain parabolic quotients W^J, the University of Minnesota REU students D. Rush and X. Shi gave a uniform proof that the order of row is the Coxeter number h [RS13].

Problem 3. *Prove that the order of birational rowmotion on minuscule posets is h. What about (birational) rowmotion on R. Proctor's D-complete posets?*

In 2015, J. Striker, J. Propp, T. Roby and I organized an AIM workshop on dynamical algebraic combinatorics, which resulted in the preprint [3]. At this workshop, a group consisting of M. Glick, G. Musiker, H. Thomas, D. Griberg, and A. Berenstein showed that birational rowmotion on the product of two chains was a special case of Zamolodchikov periodicity, thus crystallizing the connection to cluster algebras. This presents an interesting dichotomy: ought we to think of the product of two chains as a product of Dynkin diagrams (as in Zamolodchikov periodicity), or as a minuscule poset? I expect that understanding the correct algebraic and geometric context for these birational actions will lead to new combinatorics.

Problem 4. *Use Zamolodchikov periodicity to produce new examples of posets on which rowmotion is well-behaved.*

2. Coxeter-Catalan combinatorics

Let (W, S) be a finite Coxeter system of rank $|S| = n$, and let c be a Coxeter element. W-Catalan numbers come in three levels of generality[1]:

$$\mathsf{Cat}(W) := \prod_{i=1}^{n} \frac{h+d_i}{d_i}, \mathsf{Cat}^{(m)}(W) := \prod_{i=1}^{n} \frac{mh+d_i}{d_i}, \text{ and } \mathsf{Cat}^{[b]}(W) := \prod_{i=1}^{n} \frac{b+d_i-1}{d_i},$$

where $d_1 < d_2 < \cdots < d_n$ are the degrees of W, $h := d_n$ is its Coxeter number, m is a positive integer, and b is a positive integer coprime to h.

At the first level of generality, there are three main families of noncrossing objects—associated to a finite Coxeter group W and Coxeter element c—counted by $\mathsf{Cat}(W)$: *noncrossing partitions*, *cluster algebras*, and *sortable elements*. Through work of D. Armstrong [?], and N. Reading and S. Fomin [FR05], the noncrossing partitions and clusters admit *Fuss–Catalan* extensions to the second level of generality, whereby an additional integral parameter m is introduced and the resulting objects are counted by $\mathsf{Cat}^{(m)}(W)$.

The most important idea we advance in [10] is that the correct setting for the Fuss–Catalan numbers is provided by the Artin monoid corresponding to W.

Definition-Theorem 5 ([10])**.** *The* (W, m, c)-sortable elements *are a subset of elements in the interval* $[e, w_\circ^m]$ *in the positive Artin monoid. Their cardinality is* $\mathsf{Cat}^{(m)}(W)$.

Building on work with B. Rhoades and D. Armstrong, our framework appears to scale to the third ("rational" [5,6]) level of generality in the classical types, providing combinatorial objects counted by $\mathsf{Cat}^{[b]}(W)$. This philosophy of studying Fuss-Catalan combinatorics in the positive Artin monoid suggests the existence of a connection between the Artin group of a finite Weyl group and its corresponding affine group.

Problem 6. *Extend our Fuss-Catalan constructions to infinite Coxeter groups.*

The restriction of the weak order on $[e, w_\circ^m]$ to our m-sortable elements allows us to construct the *Fuss-Cambrian lattices*, which had previously evaded definition. In type A, G. Chapuy and L.-F. Préville-Ratelle have expressed significant interest in the many tantalizing similarities of our Fuss-Cambrian lattices to F. Bergeron's m-Tamari lattices.

[1] For simplicity, our formula for $\mathsf{Cat}^{[b]}(W)$ is valid only for *crystallographic* Coxeter groups.

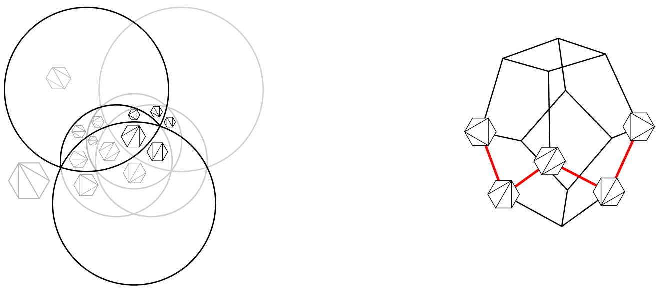
Problem 7 ([10])**.** *Prove that our Fuss-Cambrian lattices for the linear orientation in type* A_{n-1} *have* $\frac{m+1}{n(mn+1)}\binom{(m+1)^2n+m}{n-1}$ *intervals.*

The combinatorics of these intervals is believed to have deep connections to certain generalized spaces of diagonal harmonics [BPR11, BMFPR12, BMCPR13].

In [12], I give a geometric interpretation of a normalization map due to D. Sleator, R. Tarjan, and W. Thurston [STT88] (see also [CP16]) as a projection to a parabolic subgroup. This is illustrated for $\mathfrak{S}_4$ on the left-hand side of the figure below, where a triangulation is mapped to the unique black triangulation in the same region. My interpretation, combined with work of N. Reading and D. Speyer [RS11], results in a uniform proof that W-permutahedra and W-associahedra have the "non-face-leaving" property.

Theorem 8 ([12])**.** *For any two vertices* v *and* v' *of a* W*-associahedron, any geodesic from* v *to* v' *stays in the minimal face containing both.*

This method should carry over to infinite Coxeter groups and also—despite the absence of geometry—to Fuss-Catalan associahedra.



I have also pursued a novel direction of generalization of W-Catalan combinatorics to parabolic quotients. For $J \subseteq S$, the parabolic subgroup W_J is the subgroup of W generated by J; the corresponding parabolic quotient W^J is W/W_J. In [7], I define *aligned elements* for W^J and give theoretical and computational evidence that suggests that many (though certainly not all) properties of W-Catalan combinatorics carry over to this generalized setting. In [9], H. Mühle and I explicitly work out combinatorial models and bijections for all parabolic quotients of the symmetric group. It remains to work out the combinatorics in other types.

The noncrossing Catalan objects have three natural actions: the *Kreweras complement* Krew_c, the *positive Kreweras complement* Krew_c^+, and *Cambrian rotation* Camb_c. These actions have been defined in the literature in a way that may be seen as "global" [Arm09, FZ03]. In [7,13], we develop natural "local" methods to compute all three actions as *walks* on the $\mathsf{Cat}(W)$ vertices of the associahedron. For example, clusters for $\mathfrak{S}_{n+1}$ correspond to triangulations of an $(n+3)$-gon; in this language, we describe a sequence of flips of diagonals that rotates a triangulation. Such a sequence is illustrated for $\mathfrak{S}_4$ by the red path on the right-hand side of the figure above. Our walks reveal an unexpected relation between the three actions.

Theorem 9 ([13])**.** $\mathsf{Camb}_c = \mathsf{Krew}_c \circ \mathsf{Krew}_c^+$.

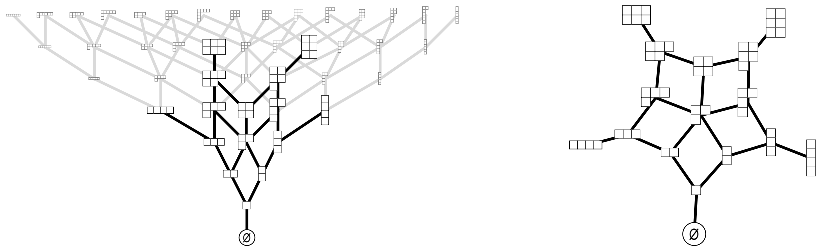
The *nonnesting partitions*—defined to be the order ideals in the root poset—are the fourth member of the family of W-Catalan objects. They have a significantly different flavor from the noncrossing objects: they are only naturally defined for Weyl groups, and they do not depend on a Coxeter element c. One of the most promising avenues towards connecting nonnesting partitions with noncrossing objects comes from a remarkable relationship between row on nonnesting partitions (see Section 1) and the Kreweras complement on noncrossing partitions [Pan09, BR11, AST13]. By modifying row to accommodate the input of a Coxeter element c—mimicking our walks on the W-associahedron using the distributive lattice of nonnesting partitions—we produce remarkable conjectural (compatible) bijections between nonnesting partitions and clusters, as well as between nonnesting partitions and noncrossing partitions [7,8,13].

Problem 10 ([7,8,13])**.** *Show that these maps are bijections.*

This has been checked up to rank eight for all Coxeter elements. The resolution of this exciting and explicit open problem would have a deep impact on our understanding of Catalan combinatorics.

3. Affine Weyl groups and Coxeter Groups

In [Sut02, Sut04], R. Suter proved that a subposet of *Young's lattice*—consisting of those integer partitions whose largest part plus number of parts is less than $(a-1)$—has a cyclic symmetry of order a. The illustration below is for $a = 5$.



This subposet is isomorphic to the weak order on the elements contained in the two-fold dilation of the fundamental alcove $\mathcal{A}$ in affine type $\widetilde{\mathfrak{S}}_a$; the cyclic symmetry arises from the fact that the Dynkin diagram for $\widetilde{\mathfrak{S}}_a$ is an a-cycle. In [16], we generalize this construction to arbitrary dilations of $\mathcal{A}$, combinatorially characterizing exactly which elements occur in the dilation. Using the well-known correspondence between highest weights for $\mathfrak{sl}_a$ and integer partitions, we also explain the existence of a-core models for the quotient $\widetilde{\mathfrak{S}}_a/\mathfrak{S}_a$. In [14], we characterize the orbits of this cyclic symmetry, which gives an instance of the cyclic sieving phenomenon.

Theorem 11 ([14])**.** *There is an equivariant bijection between the elements of $\widetilde{\mathfrak{S}}_a$ contained in the b-fold dilation of $\mathcal{A}$ under its a-fold cyclic symmetry and words of length a on $\mathbb{Z}/b\mathbb{Z}$ with sum $(b-1) \mod b$ under rotation.*

For relatively prime a and b, the coroot points inside the b-fold dilation of $\mathcal{A}$ are in bijection with (simultaneous) *(a,b)-cores*—integer partitions whose Ferrers diagram contains no box whose hook-length is divisible by either a or b. Recently, there has been a surge of interest on statistics for simultaneous cores [Nat08, AKS09, Fay11, AL14, YZZ14, Nat14, CHW14, Agg14, Fay14, Xio14, Agg15, Fay15]. Results of J. Olsson and D. Stanton [OS07], and P. Johnson [Joh15] (confirming a conjecture of D. Armstrong [Arm15, AHJ14]), prove that the maximum number and the expected number of boxes in an (a,b)-core are

$$\max_{\lambda \in \mathsf{core}(a,b)} (\mathsf{size}(\lambda)) = \frac{(a^2-1)(b^2-1)}{24}, \quad \mathop{\mathbb{E}}_{\lambda \in \mathsf{core}(a,b)} (\mathsf{size}(\lambda) = \frac{(a-1)(b-1)(a+b+1)}{24}.$$

For b relatively prime to the Coxeter number h, by extending the definitions of "simultaneous core" and "number of boxes" to all affine Weyl groups $\widetilde{W}$, we use Ehrhart theory in [17] to give uniform generalizations to simply-laced affine types.

Theorem 12 ([17])**.** *For $\widetilde{W}$ a simply-laced affine Weyl group,*

$$\max_{\lambda \in \mathsf{core}(\widetilde{W},b)} (\mathsf{size}(\lambda)) = \frac{n(h+1)(b^2-1)}{24}, \quad \mathop{\mathbb{E}}_{\lambda \in \mathsf{core}(\widetilde{W},b)} (\mathsf{size}(\lambda) = \frac{n(b-1)(h+b+1)}{24}.$$

By setting $a = h = n+1$, we recover the formulas for $\widetilde{\mathfrak{S}}_a$. We further explain the appearance of the number 24 using the "strange formula" of H. Freudenthal and H. de Vries. We compute the variance for all simply-laced affine Weyl groups and third moment for $\widetilde{\mathfrak{S}}_{n+1}$ (see also S. B. Ekhad and D. Zeilberger's subsequent preprint [EZ15]). There is a conjectural weighted version of Theorem 12, which involves summing over *all* weights inside $\check{Q}/b\check{Q}$, rather than just the coroots.

Problem 13. *For $\widetilde{W}$ a simply-laced affine Weyl group,*

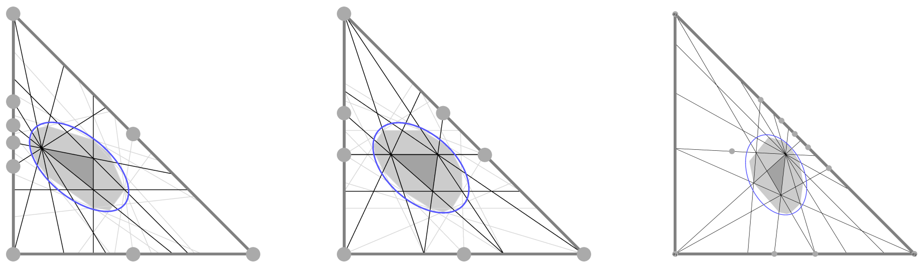
$$\mathop{\mathbb{E}}_{w \in b\mathcal{A}}(\mathsf{size}(w)) = \frac{n(b^2-1)}{24}.$$

It is natural to consider these problems on more exotic lattices.

Problem 14. *Fix L a lattice with automorphism group $\mathsf{Aut}(L)$, and let W be a subgroup of $\mathsf{Aut}(L)$. Determine the number of W-orbits in L/bL.*

It would be especially interesting to study this problem for G. Nebe's primitive root lattices for complex reflection groups W under the action of W—as well as for the Leech lattice under the Conway groups Co_0 and Co_1. This is a natural way to assign a "Catalan number" to any group that stabilizes a lattice.

There is a sense in which B. Brink and R. Howlett's notion of *small roots* [BH93, DS91] generalize to all Coxeter systems (W, S) the dilations of the fundamental alcove in affine type. Suggestive rank-three illustrations are provided below, in which the inverses of those elements (in gray) with only small lower bounding hyperplanes have coalesced into a convex polyhedron.



Problem 15 ([18])**.** *Show that these polyhedra are always convex.*

A *Garside shadow* is a subset $B \subseteq W$ containing S and closed under weak-order join and suffixes [DDH15, DH15]; the polyhedra above are examples of a general construction of (finite) Garside shadows using small roots. In [18], C. Hohlweg, P. Nadeau and I study projections to an arbitrary Garside shadow.

Theorem 16 ([18])**.** *Any finite Garside shadow produces a finite deterministic automata recognizing the language $\mathsf{Red}(W, S)$ of reduced words for (W, S).*

Garside shadows are closed under intersection; we believe that projecting onto the *smallest* Garside shadow produces the *minimal* automata recognizing $\mathsf{Red}(W, S)$.

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