RATIONAL CATALAN COMBINATORICS: AN OUTLINE FROM THE AIM WORKSHOP, DECEMBER 2012

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The American Institute of Mathematics (AIM), D. Armstrong, S. Griffeth, V. Reiner, and M. Vazirani arranged for three groups of mathematicians to meet during the week of December 16th for a workshop on *Rational Catalan Combinatorics*. The three areas of study these mathematicians represented were:

- Catalan Combinatorics and Reflection Groups;
- Symmetric Functions, Macdonald Polynomials, and Diagonal Harmonics; and
- Rational Cherednik Algebras (RCAs).

Lectures on the first subject were given by V. Reiner and D. Armstrong; on the second by F. Bergeron, A. Garsia, and M. Haiman; and on the third by R. Bezrukavnikov, I. Gordon, and E. Gorsky.

What follows is an account of the problems proposed in the two moderated problem sessions from the workshop. The brave moderators who led the problem sessions were N. Reading and E. Niese. The problems are broken up into sections according to the above classification with short summmaries of the relevant lectures and some additional background information for context.

1. Catalan Combinatorics and Reflection Groups

It is now generally recognized that there are two different flavors of Catalan objects: nonnesting and noncrossing. C. Stump led a problem session on noncrossing objects from the point of view of the subword complex.

V. Reiner identified four directions of study for these objects:

- Narayana and Kirkman refinements;
- Reflection group generalizations;
- Fuss-Catalan and Rational Catalan generalizations; and
- q-Catalan and (q,t)-Catalan numbers as bigraded Hilbert series.

D. Armstrong gave a presentation on type A rational Catalan combinatorics, and B. Rhoades and N. Williams gave the associated problem session.

Recommended expository references are [1, 27, 44].

1.1. **Uniform Bijections and Enumerations.**

Nonnesting partitions are defined for crystallographic reflection groups. Due to work by P. Cellini and P. Papi, extending work of M. Haiman and J. Shi [22, 38, 47], they are *uniformly* enumerated by the formula

$$\mathrm{Cat}(W) := \prod_{i=1}^{n} \frac{h+d_i}{d_i}, \tag{1}$$

where the d_i are the degrees of W and h is the Coxeter number. Noncrossing partitions are defined in greater generality, but are enumerated *case-by-case*. One approach to uniformly enumerating the noncrossing partitions in the crystallographic case is through a bijection to nonnesting partitions.

D. Armstrong, C. Stump, and H. Thomas gave a uniformly-characterized bijection between noncrossing and nonnesting partitions for Weyl groups [6], proving conjectures of D. Bessis and V. Reiner [16], and D. Panyushev [42]. Their proof remains case-by-case, exploiting a coincidence of two actions: the Kreweras complement on noncrossing partitions and the polyonymous action rowmotion on nonnesting partitions.

Problem 1 (V. Reiner)**.** *Construct a uniform bijection (with a uniform proof) between noncrossing partitions (or c-sortable elements or c-clusters) and nonnesting partitions for Weyl groups.*

It remains a peculiar fact that while noncrossing partitions may be defined for real reflection groups (and even for complex reflection groups), nonnesting partitions are only defined in the crystallographic case (see [1, 23] for attempts to construct the missing root posets). A promising approach to relating the two objects is given in [4], which bypasses the crystallographic hypothesis on the definition of nonnesting partitions by defining an algebraic W-parking space isomorphic as a $\mathbb{C}[W]$-module to the nonnesting parking functions $Q/(h+1)Q$ [16] when W is a Weyl group. This algebraic W-parking space is defined more generally when W is a real irreducible reflection group, and can therefore be conjecturally related to a third parking space defined over the noncrossing partitions. See also [45] for an extension of this theory to the Fuss-Catalan case.

Problem 2 (V. Reiner)**.** *Prove the Parking Space Conjecture (see Section 2.6 in* [4]*).*

Rather than relate noncrossing and nonnesting objects, V. Ripoll suggested a direct strategy to uniformly enumerate Fuss-Catalan noncrossing objects by the formula

$$\mathrm{Cat}^{(k)}(W) := \prod_{i=1}^{n} \frac{kh+d_i}{d_i} \tag{2}$$

for W a well-generated irreducible complex reflection group (note that in the non-well-generated case, $\mathrm{Cat}^{(k)}(W)$ is not even an integer). Note that such a proof would be new, even for $k=1$. A more complete introduction to this problem is given in V. Ripoll's paper Lyashko-Looijenga Morphisms and Submaximal Factorizations of a Coxeter Element [46].

Work of P. Edelman, F. Chapoton, V. Reiner, and D. Armstrong (for an account of the history that traces as far back as L. Euler, see the discussion before Theorem 3.5.2 of [1]) established case-by-case that $\mathrm{Cat}^{(k)}(W)$ counts multichains $w_1 \leq w_2 \leq \cdots \leq w_k \leq c$ of length k in $NC(W,c)$ (see, for example, [1]). Using standard zeta polynomial computations, we can relate multichains in $[1,c]_T$ to strictly increasing chains in $[1,c]_T$, which we then interpret as reduced factorizations of the Coxeter element into nontrivial factors. Interestingly, even though the formula for multichains has a beautiful uniform expression, the formula for strict chains does not seem nice in general.

A j-block factorization of c is a factorization of c into j factors

$$c = u_1 u_2 \cdots u_j,$$

such that $\sum_{i=1}^{j} \ell_T(u_i) = \ell_T(c) = n$ and $u_i \neq 1$.

Let $\mathrm{Fact}_j(c)$ denote the number of j-block factorizations of c. Using the relation between multichains and strict chains, we obtain the identity

$$\prod_{i=1}^{n} \frac{kh+d_i}{d_i} = \sum_{j=1}^{n} \binom{k+1}{j} \mathrm{Fact}_j(c). \tag{3}$$

Even though $\mathrm{Fact}_j(c)$ does not in general seem to have a nice uniform formula, we can understand it geometrically—$\mathrm{Fact}_j(c)$ counts the cardinality of fibers of the Lyashko-Looijenga map. Specifically, D. Bessis showed case-by-case that the generic fibers of this map correspond to n-block factorizations, or reduced decompositions in the reflections T [15]. Though the proof is case-by-case, the resulting formula is uniform:

$$\mathrm{Fact}_n(c) = \frac{n!h^n}{|W|} = \frac{(h)(2h)(3h)\cdots(nh)}{d_1 d_2 \cdots d_n}.$$

V. Ripoll gave a similar formula for the almost generic fibers.

Problem 3 (V. Ripoll)**.** *Uniformly prove Equation 3 by interpreting the right-hand side as fibers of the Lyashko-Looijenga map.*

1.2. **Rational Catalan Objects.** *Nonnesting* rational Catalan objects of crystallographic type W—the parking functions and nonnesting partitions—are uniformly constructed through integral dilations of the fundamental alcove of $\widetilde{W}$ [3]. On the other hand, the construction of rational *noncrossing* Catalan objects is currently constrained to type A, where it is possible to use the combinatorics of (a,b)-Dyck paths to extend many classical Catalan objects to the rational case [5].

From this combinatorial point of view, C. Ballantine, S. Fishel, A. Hicks, and R. Orellana studied rotation of the (a,b)-homogeneous noncrossing partitions (which generalize the k-divisible noncrossing partitions) and properties of the (a,b)-noncrossing partition poset.

One would expect rational algebraic constructions to extend existing Fuss-Catalan constructions and yield definitions for rational objects beyond type A. As one possible approach, recall that the k-divisible noncrossing partitions are defined as k-multichains of elements under a Coxeter element c in the absolute order [1].

Problem 4 (C. Stump)**.** *Define rational noncrossing partitions by restricting parabolic support on multichains in the absolute order.*

Remark 4. *When $(a,b)=(n,kn-1)$, the condition on (a,b)-noncrossing partitions is known. They correspond to the positive part of the k-cluster complex* [26].

Given the algebraic definition of any one rational noncrossing object, it would almost surely be possible to bijectively translate this definition into a construction of the other noncrossing objects at the rational level of generality.

Problem 5. *Define any rational noncrossing object algebraically.*

1.3. **Statistics on Reflection Groups.** A more complete introduction to this problem is given in Bimahonian Distributions by H. Barcelo, V. Reiner, and D. Stanton [10].

It has been difficult to define a major index statistic for Weyl groups with certain desirable properties. P. MacMahon showed that inversions and major index are equidistributed on the symmetric group $\mathfrak{S}_n$, so that

$$\sum_{w\in\mathfrak{S}_n} q^{\mathrm{inv}(w)} = \sum_{w\in\mathfrak{S}_n} q^{\mathrm{maj}(w)} = \mathrm{Hilb}\left(\frac{\mathbb{C}[V]}{\mathbb{C}[V]_+^{\mathfrak{S}_n}}, q\right).$$

In general, the mahonian distribution for a complex reflection group W with degrees $d_1,\ldots,d_n$ can be defined by

$$W(q) = \prod_{i=1}^{n} \frac{1-q^{d_i}}{1-q} = \mathrm{Hilb}\left(\frac{\mathbb{C}[V]}{\mathbb{C}[V]_+^{W}}, q\right).$$

In type A, the mahonian distribution was extended by D. Foata and M.-P. Schützenberger to a bivariate distribution, whereby

$$(4)\qquad \underbrace{\sum_{w\in W} q^{\mathrm{maj}(w)} t^{\mathrm{maj}(w^{-1})}}_{\text{bimahonian distribution}} = \underbrace{\sum_{\lambda\in Irr(W)} f^{\lambda}(q) f^{\lambda}(t)}_{\text{fake degree polynomial for } \lambda} = \underbrace{\mathrm{Hilb}\left(\frac{\mathbb{C}[V\oplus V^*]^{\Delta(W)}}{\langle \mathbb{C}[V\oplus V^*]_+^{W\times W}\rangle}, q, t\right)}_{\text{Hilbert series of ``diagonal coinvariants''}}.$$

But the same string of equalities also holds in type B; as in type A, the first equality follows from Robinson-Schensted:

$$\begin{aligned} w &\mapsto (P, Q) \\ \mathrm{maj}(w) &= \mathrm{maj}(Q) \\ \mathrm{maj}(w^{-1}) &= \mathrm{maj}(P). \end{aligned}$$

For general W, one could therefore ask for a relation to cellular algebra structure in the sense of J. Graham and G. Lehrer [33].

Problem 6 (V. Reiner)**.** *Is there a major index for Weyl groups with the properties in Equation 4, analogous to S. Griffeth's approach for $G(r, 1, n)$ [34]?*

Remark 6. *F. Bergeron suggested a different strategy to find the major index. Let q be a primitive nth root of unity and define the Klyachko idempotent to be*

$$\frac{1}{n} \sum_{w \in \mathfrak{S}_n} q^{maj(w)} w.$$

When we act on the symmetric group algebra by the Klyachko idempotent, it projects onto the Lie component. One method for finding the correct analogue of major index would be to determine analogous idempotents for other known reflection group analogues of free Lie algebras, such as in [14].

2. Symmetric Functions, Macdonald Polynomials, and Diagonal Harmonics

A. Garsia gave an introduction to parking functions and Macdonald polynomials, covering topics from the indispensability of pleythistic notation to the miracle of (q, t)-symmetry to the discovery of ∇. In particular, the story is now succinctly summarized by defining the symmetric function operator ∇ satisfying

$$\nabla \widetilde{H}_\mu = q^{\nu(\mu')} t^{\nu(\mu)} \widetilde{H}_\mu, \tag{5}$$

where $\widetilde{H}_\mu = \widetilde{H}_\mu[X; q, t]$ is a modified Macdonald polynomial (or, more accurately, where H_μ is a modified Haiman polynomial). Define

$$DH_n := \{f \in \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n] \; : \; \sum_{i=1}^{n} \frac{\partial^h}{x_i^h} \frac{\partial^k}{x_i^k} f = 0, \forall h + k > 0\}.$$

Then it was a conjecture of A. Garsia and M. Haiman and is now a theorem of M. Haiman that

$$\nabla e_n = \mathrm{Frob}(DH_n; q, t), \tag{6}$$

where e_n is an elementary symmetric functions and $\mathrm{Frob}(DH_n; q, t)$ is the bigraded Frobenius series of the space of diagonal harmonics. Furthermore,

$$\mathrm{Cat}_n(q, t) := \langle \nabla e_n, s_{1^n} \rangle \tag{7}$$

is the bigraded Hilbert series of the subspace of alternating elements.

2.1. **Diagonal Harmonics.**

Problem 7 (S. Assaf)**.** *Prove the Shuffle Conjecture (Conjecture 3.1.2 of* [36]*). See also* [37] *for the Compositional Shuffle Conjecture, which refines the Shuffle Conjecture by specifying the points where Dyck paths touch the diagonal.*

A. Garsia and M. Haiman conjectured the existence of a statistic on $(n, n+1)$-Dyck paths so that

$$\mathrm{Cat}_n(q,t) = \sum_{\pi \text{ an } (n,n+1)-\text{Dyck path}} q^{\mathrm{area}(\pi)} t^{\mathrm{bounce}(\pi)}.$$

A construction of this statistic in the form of *bounce* was conjectured by J. Haglund "after a prolonged study of tables of" $\mathrm{Cat}_n(q,t)$, [35] and was shortly thereafter proven by A. Garsia and J. Haglund [28]. After J. Haglund unveiled his discovery to A. Garsia, A. Garsia informed M. Haiman that the search for the statistic was over (without giving the construction of bounce). M. Haiman proceeded to quickly and independently develop a statistic *dinv* with the same property that

$$\mathrm{Cat}_n(q,t) = \sum_{\pi \text{ an } (n,n+1)-\text{Dyck path}} q^{\mathrm{dinv}(\pi)} t^{\mathrm{area}(\pi)}.$$

It came as a surprise when *bounce* and *dinv* turned out to be *different*; J. Haglund and M. Haiman found a bijection on $(n, n+1)$-Dyck paths—the ζ map—that interchanged the pair (*area*, *bounce*) with (*dinv*, *area*). Combinatorially exchanging *area* and *bounce* (or *area* and *dinv*) is still an open problem.

Problem 8 (S. Assaf)**.** *Prove the* (q,t)*-symmetry of* $\mathrm{Cat}_n(q,t)$ *combinatorially.*

In the Fuss-Catalan direction, M. Haiman defined (q,t)-Fuss-Catalan numbers by

$$\mathrm{Cat}_n^{(k)}(q,t) := \langle \nabla^k e_n, s_{1^n} \rangle \tag{8}$$

(see the discussion leading up to Conjecture 7.1.3 in Combinatorics, Symmetric Functions, and Hilbert Schemes [39] and Section 4 of A Remarkable q,t-Catalan Sequence and q-Lagrange Inversion [29]).

Problem 9. *Prove the k-Shuffle Conjecture (Conjecture 6.2.2 of* [36]*).*

In his thesis, N. Loehr extended the combinatorics of $(n, n+1)$-parking functions to the $(n, kn+1)$ case by defining generalizations of *dinv* and *bounce*. With these definitions, a version of the ζ map again interchanges the pairs of statistics

$$\mathrm{Cat}_n^{(k)}(q,t) = \sum_{\pi \text{ an } (n,kn+1)-\text{Dyck path}} q^{\mathrm{area}(\pi)} t^{\mathrm{bounce}_k(\pi)} = \sum_{\pi \text{ an } (n,kn+1)-\text{Dyck path}} q^{\mathrm{dinv}_k(\pi)} t^{\mathrm{area}(\pi)}.$$

D. Armstrong, N. Loehr and G. Warrington, and E. Gorsky and M. Mazin have independently constructed conjectural extensions of the ζ map for (a,b)-Catalan numbers.

- D. Armstrong's version passes through J. Anderson's bijection between (a,b)-Dyck paths and (a,b)-cores and seems to provide the most structure by extending the notion of Shi tableau;
- N. Loehr and G. Warrington's version is elegantly defined directly on (a,b)-Dyck paths; and
- E. Gorsky and M. Mazin's version uses their construction of the theory in the language of semigroups.

While D. Armstrong, N. Loehr, and G. Warrington have shown in a preprint that all three definitions are equivalent, it remains to prove that any one of these maps is bijective.

Problem 10 (D. Armstrong)**.** *Prove that the extended* ζ *map for* (a,b)*-Dyck paths is bijective.*

Remark 10. *This is known in the Fuss-Catalan case $(a, b) = (n, kn + 1)$.*

During the workshop, D. Armstrong, S. Fishel, E. Gorsky, M. Mazin, M. Vazirani, and N. Williams worked with (a, b)-Dyck paths, (a, b)-cores, and (a, b)-parking functions. They conjectured that ζ arises from taking a minimal alcove from a dominant chamber in the k-Shi arrangement, applying J. Anderson's bijection to pass from $(n, kn + 1)$-cores to $(n, kn + 1)$-Dyck paths, and interpreting the resulting Dyck path as an increasing parking function under the Stanley-Pak labeling. This appears to be closely related to D. Armstrong's observation arising from his work in [2] that ζ arises from the study of the Stanley-Pak labeling [49, 50] and the Athanasiadis labeling [7]. Additional recommended references are [25, 48], and [47].

2.2. **Parallelogram Polyominoes.** M. Dukes and Y. Le Borgne connected recurrent configurations of the sandpile model on the complete bipartite graph $K_{m,n}$ with parallelogram polyominoes and (q, t)-Narayana numbers [24]. A more complete introduction to parallelogram polyominoes is given in their paper, with recent progress made in [8] and [9] extending the ζ map to parallelogram polyominoes.

Given two lattice paths R and G from $(0, 0)$ to (m, n), so that R and G are non-intersecting (except at the endpoints) with R above G, the shape for which R and G form the boundary is called a *parallelogram polyomino*.

In particular, one can define a notion of *area* and *bounce* for these parallelogram polyominoes [24], so that if

$$\mathrm{Nara}_{m,n}(q, t) = \sum_{P \in \mathrm{Polyo}_{m,n}} q^{\mathrm{area}(P)} t^{\mathrm{bounce}(P)}, \tag{9}$$

then

$$\mathrm{Nara}_{m,n}(1, 1) = \frac{1}{m + n - 1} \binom{m + n - 1}{m} \binom{m + n - 1}{m - 1}.$$

Moreover, it was shown in [9] that

$$\mathrm{Nara}_{m,n}(q, t) = (qt)^{m+n-1} \langle \nabla e_{m+n-2}, h_{m-1} h_{n-1} \rangle,$$

from which it follows that

$$\mathrm{Nara}_{m,n}(q, t) = \mathrm{Nara}_{m,n}(t, q) \text{ and } \mathrm{Nara}_{m,n}(q, t) = \mathrm{Nara}_{n,m}(q, t).$$

If we treat the higher lattice path R as a parking function (leaving G as a lattice path), and label its n steps north with the "cars" $\{1, 2, \ldots, n\}$ so that vertical runs are given increasing labels, we call the resulting object a *parking parallelogram polyomino* P. Let $P(n, m)$ be the set of all such parking parallelogram polyominoes.

Given an parking parallelogram polyomino P, we next construct a permutation by reading the "cars" of P from northeast to southwest along the diagonals $y = x + i$ for $i = n-1, n-2, \ldots, -m+1$. Let $\mathrm{ides}(P)$ be the inverse descent set (that is, those i for which $i+1$ lies to the left) of the resulting permutation. Let $\mathrm{area}(P)$ be the number of boxes between the R and G associated to P.

Let F_s be the Gessel quasisymmetric function. Then M. D'Adderio has experimentally made the following conjecture:

$$\sum_{P \in P(n,m)} F_{\mathrm{ides}(P)}(X) t^{\mathrm{area}(P)} = \Delta_{h_{m-1}} e_n \Big|_{q=1}. \tag{10}$$

Problem 11 (A. Garsia)**.** *What is the corresponding* dinv *statistic to make Equation 10 true after removing the specialization to $q = 1$ from the right-hand side?*

The centrality of ∇ in the theory of diagonal harmonics suggests that we define a wider class of operators on symmetric functions. To this end, let f, g be symmetric functions. Then Δ_f is defined to be an eigenoperator for $\widetilde{H}_\mu(X; q, t)$ so that

$$\Delta_f \widetilde{H}_\mu = f[B_\mu]\widetilde{H}_\mu, \quad \text{where} \tag{11}$$

$$B_\mu := B_\mu(q,t) = \sum_{\text{cells } c \in \mu} t^{\text{coleg}(c)} q^{\text{coarm}(c)}. \tag{12}$$

Note that we recover ∇ as a special case of this construction: $\nabla = \Delta_{e_n}$, when applied to a degree n symmetric function.

Problem 12 (F. Bergeron)**.** *What are good choices for f and g so that $\Delta_f(g)$ gives a (q,t)-Schur-positive expansion? Find "nice" spaces that explain and characterize this, similar to the coinvariant ring.*

Returning now to parallelogram polyominoes, it is a theorem of A. Hicks and M. D'Adderio that

$$\langle \Delta_{h_{m-1}} e_n, e_n \rangle = \langle \Delta e_{n+m-1}, h_{m-1} h_{n-1} \rangle. \tag{13}$$

In particular, the right-hand side of Equation 13 relates to a partition with two parts, while the left-hand side should be the character of some module.

Problem 13 (A. Garsia, F. Bergeron)**.** $\Delta_{h_{m-1}}(e_n)$ *should be the multiplicity of the alternating component of the bigraded Frobenius characteristic for a space with dimension $m^{n-1}\binom{n+m-2}{m-1}$. A candidate could be the decorated parallelogram polyominoes, which contain the parking functions in several different ways. What is the space?*

2.3. **The k-Tamari Lattice and Multivariate Diagonal Harmonics.** F. Bergeron gave an exposition of multivariate diagonal harmonics, which are intimately related to intervals and labeled intervals in his k-Tamari lattice.

The k-Tamari lattice is defined combinatorially in Section 5 of F. Bergeron and L.F. Préville-Ratelle's "Higher Trivarariate Diagonal Harmonics via Generalized Tamari Posets" [13] (see also F. Bergeron's paper [11]): the vertices are the $(n, kn+1)$-Dyck paths, and covering relations are given by simple operations on these paths. additional work on these lattices, establishing formulas for the number of labeled and unlabeled intervals may be found in [19], [20], and [21].

S. Assaf, N. Reading, F. Bergeron, H. Thomas, and C. Stump studied these k-Tamari lattices during the problem sessions.

Problem 14 (S. Assaf, N. Reading)**.** *Generalize and study the k-Tamari lattice in the rational direction using (a,b)-Dyck paths. In particular, many properties of the type A Fuss-Catalan $(n, kn+1)$ case are known, but the $(n, kn-1)$ case is already new.*

Problem 15 (H. Thomas, C. Stump)**.** *Are the k-Tamari lattices geometrically realizable? A graphic from F. Bergeron's talk strongly suggests that they are (see Figure 6 of* [11]*).*

Remark 15. *By embedding the k-Tamari lattice in the usual Tamari lattice of size nk, H. Thomas and C. Stump found that a simple function of the resulting coordinates gave a realization of of k-Tamari lattice.*

Problem 16 (N. Reading)**.** *Generalize and study the rational (a,b)-Tamari lattice in the Coxeter direction. What are the combinatorial models—one approach may be through a theory of (a,b)-c-sortable elements* [43].

Remark 16. *Note that this is closely related with Problem 5.*

Problem 17 (F. Bergeron). *Find an appropriate space to attach these (decorated) objects, as in Conjecture 1 of* [13].

Remark 17. *I. Gordon mentioned that Triple Affine Hecke Algebras could play a role; F. Bergeron suggested that his paper "Science Fiction and Macdonald's Polynomials"* [12] *with A. Garsia may be a good place to start.*

Problem 18 (H. Thomas). *The Zamolodchikov Periodicity Conjecture takes as input two Dynkin diagrams* [40]*, where the classical case corresponds to $W \times A_1$ (this accounts for the increase of the Coxeter number from h to $h+2$; for example, A_n clusters are modeled by triangulations of an $((n+1)+2)$-gon). Do rational Catalan objects also have an additional parameter? What would be the resulting combinatorics?*

F. Bergeron's lecture culminated with his Trivariate Shuffle Conjecture, which expresses the graded Frobenius characteristic of the space of trivariate diagonal harmonics as a sum over labeled intervals in the k-Tamari lattice. While two of the statistics on these intervals are direct analogues of *area* and *dinv*, there is no description yet of the third statistic.

Problem 19. *Prove F. Bergeron's Trivariate Shuffle Conjecture (Conjecture 3 and Equation (88) of* [11]*). Find a definition for the missing third statistic.*

2.4. **Wreath Products.** M. Haiman lectured on extending the theory of diagonal harmonics to the wreath products $G(r, 1, n)$. See Section 7 of M. Haiman's Combinatorics, Symmetric Functions, and Hilbert Schemes [39] and also R. Bezrukavnikov and M. Finkelberg's "Wreath Macdonald Polynomials and Categorical McKay Correspondence" [17].

Problem 20 (R. Bezrukavnikov). *Generalize ∇ to the $G(r, 1, n)$ case by finding an analogue of Macdonald polynomials that are its eigenfunctions.*

L. Lapointe suggested a different generalization of M. Haiman's type B_n (q, t)-analogues. A more complete introduction to this problem is given in [18].

Let

$$\widetilde{H}_{\lambda,\mu}(x, y, q, t) := \widetilde{H}_\lambda \left[x + qy; q, \frac{t}{q}\right] \widetilde{H}_\mu \left[\frac{x}{t} + y; \frac{q}{t}, t\right].$$

This definition arises in a super-symmetric model. When we expand the definition above as a product of Schur functions, we obtain B_n analogues of (q, t)-Kostska numbers. We can make the following first guess for ∇.

$$\nabla^n \widetilde{H}_{\lambda,\mu} = q^{|\lambda|} t^{|\mu|} \widetilde{H}_{\lambda,\mu}$$

$$\langle \nabla e_n(y), p_{(1^n)}(x + y)\rangle_{B_n} = \left(\frac{[n]_{q,t} + [n + 1]_{q,t}}{(qt)^{\frac{n-1}{2}}}\right)^n.$$

Problem 21 (L. Lapointe). *Is there a natural grading of an RCA that gives this (q, t)-analogue? If not, is there a better definition for ∇?*

3. Rational Cherednik Algebras

3.1. **Representations.** In his first lecture, I. Gordon gave an overview of Rational Cherednik Algebras and obtained q-Fuss-Catalan numbers as the Poincaré series of the representation $L_c(\mathrm{triv})^W$. I. Gordon discussed S. Griffeth's work on the combinatorics of representations of Cherednik algebras in his second talk.

Fix a complex reflection group W acting on a vector space V. The Coxeter number of W is

$$h := \frac{(\#\text{ of pseudo-reflections}) + (\#\text{ of reflecting hyperplanes})}{\dim(V)} \in \mathbb{N}.$$

Let $m \in \mathbb{N}$, and set $c = m + \frac{1}{h}$ constant on all conjugacy classes. Then the q-Fuss-Catalan number is defined to be

$$\mathrm{Cat}_W^{(m)}(q) := \prod_{i=1}^{\dim(V)} \frac{[mh + 1 + e_i(\Psi^m(V^*)^*)]_q}{[d_i]_q},$$

where Ψ is a permutation which is the identity for well-generated groups.

If $H_c(W)$ is the RCA corresponding to G and $\mathcal{O}_c(G)$ is the category of finitely generated modules on which the Dunkl operators act locally and nilpotently, then $\mathrm{Cat}_W^{(m)}(q)$ is the Poincaré series of an irreducible module $L_c(\mathrm{triv})^G$ in $\mathcal{O}_c(W)$. By combining the grading with a filtration, it is also possible to obtain W-(q,t)-Fuss-Catalan numbers.

In fact, for well-generated groups and any positive integer p coprime to h, the graded character of $L_{p/h}(\mathrm{triv})^G$ recovers the rational q-Catalan numbers

$$\prod_{i=1}^{\dim(V)} \frac{[p + e_i({}^g(V))]_q}{[d_i]_q},$$

where g is a Galois twist on $\mathbb{C}$. See [30] for full details.

The associated problem session was given by G. Bellamy.

Problem 22 (C. Stump). *Give an algorithmic approach to construct rational W-(q,t)-Catalan numbers for low rank (say, rank 5). In particular, what is this number for H_4?*

Problem 23 (V. Reiner). *When is $L_c(triv)$ (for $H_c(W)$ with equal parameters c) for a well-generated complex reflection group W finite dimensional?*

V. Reiner made the following "naive" guess, extending Etingof's result in the case of real reflection groups. Define

$$P_W(q) = \prod_{i=1}^{n} [d_i]_q$$

to be the Poincare polynomial for the reflection group W, where the d_i are the degrees of $f_1, \ldots, f_n$, where $\mathbb{C}[V]^W = \mathbb{C}[f_1, \ldots, f_n]$.

Conjecture 23. *$L_c(triv)$ for the well-generated complex reflection group W is finite dimensional iff*

$$\left.\frac{P_W(q)}{P_{W'}(q)}\right|_{q=e^{e\pi i c}} = 0$$

for all proper parabolic subgroups W' of W.

Parabolic here means that W' is the stabilizer of a point in $\mathbb{C}^n$. Note that $\frac{P_W(q)}{P_{W'}(q)}$ is a polynomial, since it is the Hilbert series

$$Hilb\left(\frac{\mathbb{C}[V]^{W'}}{\langle \mathbb{C}[V]_+^W \rangle}, q\right).$$

This characterization does not hold in the non-well-generated case. This has been pointed out by M. Balagovic, who has done calculations for the group non-well-generated group G_{12}, exhibiting other values of the parameter c for which $L_c(triv)$ is finite-dimensional, beyond those predicted in Conjecture 23.

3.2. **Semigroups.** Based on his joint work with M. Mazin [32, 31], E. Gorsky gave a bijection between (a, b)-Dyck paths and affine cells of the moduli spaces

$$J_{a,b} = \{V \subset C[[t]] \ : \ t^a V \subset V,\ t^b V \subset V,\ V \text{ normalized}\}.$$

He defined the $\mathrm{Cat}_{a,b}(q, t)$ numbers in this context, spoke about the algebraic relation with RCAs, and drew connections to the HOMFLY polynomial of (a, b)-torus knots.

More specifically, let a, b be coprime positive integers and let

$$S_{a,b} := \{\Delta \subseteq \mathbb{N} \ : \ \Delta + a \subset \Delta,\ \Delta + b \subset \Delta,\ 0 \in \Delta\}.$$

The elements of $S_{a,b}$ are in bijection with the functions of $J_{a,b}$ under the map Δ that associates to V the set containing the minimum orders of functions in V. Under this bijection, the classical statistic *dinv* arises from the dimensions of the cells

$$C_\Delta := \{V : \Delta(V) = \Delta\}.$$

For example, let $(a, b) = (3, 4)$ so that $S_{a,b}$ contains the $\frac{1}{3+4}\binom{3+4}{3} = 5$ sets

$$S_{3,4} = \{\{0, 3, 4, 6, 7, \ldots\}, \{0, 3, 4, 5, \ldots\}, \{0, 2, 3, 4, 5, \ldots\}, \{0, 1, 3, 4, 5, \ldots\}, \text{ and } \{0, 1, 2, 3, 4, \ldots\}\}.$$

To connect this to with the (a, b)-Catalan numbers, we may either use J. Anderson's bijection to directly interpret the sets as (a, b)-Dyck paths, or interpret these sets as corresponding to the boundary of a partition (where elements in the set correspond to south steps and elements not in the set correspond to west steps). Continuing the example, the sets in $S_{3,4}$ give the boundaries

$$SWWSSWSS\ldots, SWWSS\ldots, SWSSSS\ldots, SSWSSS\ldots, \text{ and } SSSSS\ldots,$$

which trace out the five simultaneous $(3, 4)$-cores

$$(3, 1, 1), (2), (1), (1, 1), \text{ and } \emptyset.$$

Define the (a, b)-(q, t)-Catalan numbers by

$$\mathrm{Cat}_{a,b}(q, t) := \sum_{\Delta \in S_{a,b}} q^{|\mathbb{N}/\Delta|} t^{\frac{(a-1)(b-1)}{2} - \dim C_\Delta}. \tag{14}$$

Problem 24 (E. Gorsky)**.** *Prove that* $Cat_{a,b}(q, t) = Cat_{a,b}(t, q)$*.*

Remark 24. *This is known for* $min(a, b) \le 4$*. This generalizes Problem 8*

Now form the generating function

$$S_{a,b}(t) := \sum_{\Delta \in S(a,b)} t^{\frac{(a-1)(b-1)}{2} - |\mathbb{N}/\Delta|} \sum_{i \in \Delta} t^i.$$

Problem 25 (V. Shende, communicated by M. Mazin)**.** *Prove that the coefficients of* $S_{a,b}(t)$ *are monotone.*

To guarantee integrality of $\mathrm{Cat}_{a,b}$, we have been assuming that a and b are coprime. I. Losev was interested in the non-coprime case; when $\gcd(a, b) \neq 1$, he suggested that the simultaneous (a, b)-cores might be related to infinite-dimensional representations of $L_{\frac{b}{a}}$. M. Mazin and D. Armstrong noted that when studying invariants of $x^a = y^b$, the number of branches is $\gcd(a, b)$. Further connections were implied between knot and torus links and colored Jones polynomials.

Macdonald polynomials take as input (a, b, λ). Write

$$s_\lambda(x_1^a, x_2^a, \ldots, x_n^a) = \sum c^a_{\lambda,\mu} s_\mu = p_\mu[s_\lambda]$$

Now

$$\sum_{\mu} q^{b/a(n(\mu)-n(\mu'))} c^{a}_{\lambda,\mu} s_{\mu}$$

is the λ-colored HOMFLY invariant of the (a, b) torus knot. This is the λ-modified dinv statistic—if λ is just one box, we recover the formula for the character of the (a, b)-parking functions.

Problem 26 (E. Gorsky)**.** *Is there a combinatorial object that has this behavior at $q = t^{-1}$?*

Problem 27 (I. Losev)**.** *Is there a connection to Cherednik algebras? (number of variables is $b|\lambda|$) the characters for the graded Frobenius characteristic?*

3.3. **Shifts and ∇.** R. Bezrukavnikov lectured on a construction of a basis $S_{I,\lambda}$ of $\mathbb{S}_n[t_1^{\pm 1}, t_2^{\pm 1}]$, dependent on a partition λ and an interval I. These $S_{I,\lambda}$ have the property that

$$\widetilde{H}_{\lambda} = \sum_{\mu \vdash |\lambda|} K_{I,\lambda,\mu} S_{I,\mu},$$

where:

- The $K_{I,\lambda,\mu}$ share properties with the usual K;
- $S_{(-\frac{1}{n},\frac{1}{n}),\lambda} = S_{\lambda}$ conjecturally; and
- $\{S_{I+1,\lambda}\} = \{\nabla S_{I,\lambda}\}$.

R. Bezrukavnikov further described a conjectural change of basis to pass between $S_{I,\lambda}$ and $S_{I',\lambda}$, where I and I' are two neighboring intervals.

Problem 28 (M. Mazin)**.** *Geometrically, this is a change of basis in homology. In* [41]*, N. Loehr and G. Warrington have involutions on the level of Dyck paths that explain this symmetry. Are these the same?*

Remark 28 (F. Bergeron)**.** *F. Bergeron pointed to "Science Fiction and Macdonald's Polynomials"* [12] *for more special decompositions of Macdonald polynomials and changes of basis.*

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