# PROJECT DESCRIPTION: NEW BIJECTIVE TECHNIQUES IN ALGEBRAIC COMBINATORICS

Research Program

My research exploits the interplay between algebraic combinatorics and other fields, with applications to Macdonald theory, representation theory, and geometry. There are many interesting problems in a wide range of areas of mathematics that can be attacked with bijective methods. I have introduced new techniques to resolve some of these problems, and believe that my original toolkit can continue to yield substantial new progress.

1. Bijections in K-theoretic Schubert Calculus

In 1983, R. Proctor exploited the branching rule from the Lie algebra inclusion $\mathfrak{sp}_{2n}(\mathbb{C}) \hookrightarrow \mathfrak{sl}_{2n}(\mathbb{C})$ to prove the combinatorial identity that there are the same number of plane partitions of heights at most k of rectangular shape and of shifted trapezoidal shape [Kin75, Lit50]. A small example is given by the posets with Hasse diagrams and , which each have six plane partitions of height 1, as illustrated below (a gray vertex cannot lie below a white vertex).
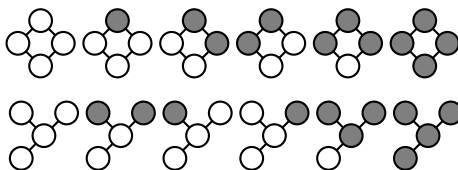

R. Proctor remarks that "the question of a combinatorial correspondence... seems to be a complete mystery." Indeed, the state of the art for over 30 years was limited for the case $k = 1$: J. Stembridge produced a jeu-de-taquin bijection [Ste86] and V. Reiner gave an argument using type B noncrossing partitions [Rei97]. In 2015, S. Elizalde proved the identity bijectively in the language of pairs of lattice paths for $k \leq 2$ [Eli15]. No bijection of any kind was previously known for $k > 2$. In [HPPW16], we found the missing bijection for all k: our solution synthesizes a remark about E_7 by R. Proctor and a beautiful idea of Z. Hamaker to exploit A. Yong and H. Thomas's recent (co)minuscule K-theoretic Schubert calculus techniques.

Theorem 1 ([HPPW16]). *K-theoretic jeu-de-taquin gives a bijection between plane partitions of heights at most k of rectangular shape and of shifted trapezoidal shape.*

We actually proved something substantially more general, placing this specific problem into a robust bijective framework for proving similar identities, based on minuscule K-theoretic Schubert calculus. A *minuscule weight* of a Lie algebra is a dominant weight whose Weyl group orbit contains all the weights in its highest-weight representation; the associated crystal is then a distributive lattice whose underlying poset is called a *minuscule poset*. These posets have many remarkable properties with applications to Schubert calculus, representation theory, and combinatorics [Pro84, Ste94, Ste01b, Kup94, Gre13]. There is a complete classification (three infinite families and two exceptional examples coming from the root systems of types E_6 and E_7), of which a few examples are illustrated in Figure 1.

For G a semisimple complex Lie group and P a parabolic subgroup such that G/P is a minuscule variety (that is, P corresponds to a minuscule weight), we prove the equivalence of a product in the
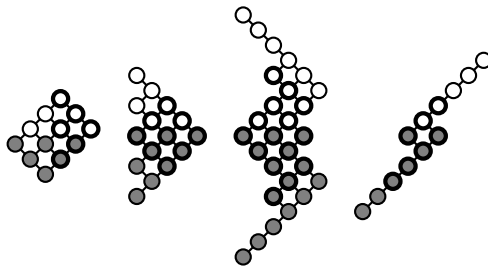
FIGURE 1. Minuscule posets of types A_6, B_5, E_6, and D_7.

Grothendieck ring $K(G/P)$ of algebraic vector bundles over G/P with a bijection between two sets of increasing tableaux. This then yeilds several other theorems of the same flavor as Theorem 1, using the posets in Figure 1. Our arguments are usefully interpreted as statements about rational equivalence of certain generalized Schubert and Richardson subvarieties of minuscule flag varieties—each of the bijections we obtain corresponds to the fact that a certain Richardson variety represents the same element of the Chow ring as a certain Schubert variety.

Combinatorially, our bijections are very simple to describe. We had observed in [Wil13a] that the *coincidental (Cartan) types* A, B, H_3, and $I_2(m)$ were exactly those types whose (fake) root posets $\Phi^+(W)$ satisfied certain related poset-theoretic identities. The crucial observation is that the coincidental types are exactly those types whose root poset is (dual to) the *bottom half* of an "ambient" minuscule poset. As illustrated in Figure 2, our approach was to embed a *second* minuscule into the ambient one as a Richardson variety (the red circles), and then use K-theoretic jeu-de-taquin to degenerate this embedding to a Schubert variety (the blue circles).
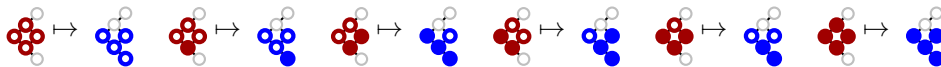

FIGURE 2. The bijection from plane partitions in a rectangle to plane partitions in a shifted trapezoid.

These techniques yield a uniform way to construct bijections using multiplicity-free expansions in the Grothendieck ring, and we expect many further applications. It would be especially fruitful to return to R. Proctor's original Lie-theoretic explanation of the original rectangle/trapezoid identity and understand the representation-theoretic consequences of our bijections on Littelmann's path model [Lit94, Lit95, NS05]. This problem has garnered recent attention by the Littelmann school, as it is a branching rule not arising from the restriction to a Levi subalgebra (but still behaving as if it were) [Tor16, ST16].

Problem 1. *Translate Theorem 1 into the language of Littelmann paths to combinatorially understand the branching rule* $\mathsf{sp}_{2n}(\mathbb{C}) \hookrightarrow \mathsf{sl}_{2n}(\mathbb{C})$.

There is a symplectic cominuscule identity relating plane partitions of height at most k in a staircase to plane partitions of height at most $2k$ in a shifted staircase. This was proven bijectively by J. Sheats, but we suspect that it should also fall into our framework [She99, Pro90]—K. Purbhoo has given a jeu-de-taquin bijection for cohomology (relating standard Young tableaux of staircase and shifted staircase shape) that we have thus far been unable to extend to K-theory [Pur14].

Problem 2. Find a K-theoretic jeu-de-taquin bijection between plane partitions of height at most k in a staircase to plane partitions of height at most $2k$ in a shifted staircase.

R. Proctor's d-complete posets share many properties of minuscule posets, and should be expected to yield further interesting bijections.

Problem 3. *Generalize Theorem 1 to λ-minuscule elements in Kac-Moody groups.*

O. Pechenik and I are currently interested in the project of classifying multiplicity-free K-theoretic minuscule Schubert calculus, using techniques introduced by A. Knutson and generalizing work of J. Stembridge, H. Thomas, and A. Yong [Knu09, Ste01a, TY05, Sni09].

Problem 4. *Classify multiplicity-free K-theoretic Schubert calculus.*

A more ambitious goal is to use our techniques to explain the existence of Little bumps and Edelman-Greene-like bijections between reduced words for the longest element and linear extensions of the root poset, based on the Chevelley rule for minuscule varieties [Sta84, EG87, Lit03, HY14].

Of course, the most celebrated open problem in this area remains to find a bijection between totally symmetric self-complementary plane partitions and alternating sign matrices. We note only that experts like C. Krattenthaler believe that there ought to be a jeu-de-taquin-like bijection between Zeilberger's Gog and Magog triangles, and have suggested that K-theoretic jeu-de-taquin might be applicable.

2. Macdonald Theory, Equivariant Bijections, and Fixed Point Theorems

In this section I describe applications of of my work (with original motivations from Lie theory) to the study of diagonal harmonics. The underlying theme is a new method that often allows one direction of a bijection to be guessed—inverses to these purported bijections are highly sought after and can be difficult to construct. Recent investigations have revealed a relation to fixed point theorems.

2.1. **Symmetry of affine Dynkin diagrams.** In [Sut02, Sut04], R. Suter showed that a subposet of *Young's lattice*—consisting of those integer partitions whose largest part plus number of parts is at most $(a - 1)$, ordered by inclusion—has a cyclic symmetry of order a. Figure 3 illustrates this for $a = 5$.
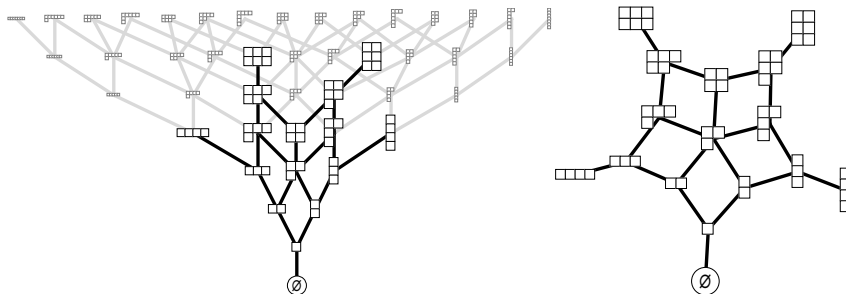

FIGURE 3. Left—the first few ranks of Young's lattice; right—the restriction to those partitions whose largest part plus number of parts is at most four, drawn to emphasize the five-fold symmetry.

R. Suter's result is the specialization to the affine symmetric group $\widetilde{\mathfrak{S}}_a$ of a result of D. Peterson, who proved that the abelian ideals of a Borel subalgebra of a complex simple Lie algebra indexed the two-fold dilation of the fundamental alcove $\mathcal{A}$ in the corresponding affine Weyl group [Kos98]. Celini and Papi later generalized Peterson's bijection to ad-nilpotent ideals and dominant Shi regions [CP02] (we shall return to this in the next section). In general, the natural symmetry Ω of the fundamental alcove (or the affine Dynkin diagram) is given by the center of the corresponding simply-connected compact simple

Lie group [IM65]—the cyclic symmetry thus arises from the fact that the Dynkin diagram for $\widetilde{\mathfrak{S}}_a$ is an a-cycle.

We generalized R. Suter's construction to arbitrary dilations of $\mathcal{A}$ in [BWZ11], combinatorially describing exactly which elements occur. Using the well-known correspondence between highest weights for $\mathfrak{sl}_a$ and integer partitions, we also explain the existence of a-core models for the quotient $\widetilde{\mathfrak{S}}_a/\mathfrak{S}_a$.

In [TW14]—using a novel method to construct bijections—we bijectively characterized the orbits of Suter's cyclic symmetry (illustrated in Figure 4).

Theorem 2 ([TW14]). *There is an equivariant bijection between the elements of $\widetilde{\mathfrak{S}}_a$ contained in the b-fold dilation of $\mathcal{A}$ under its a-fold cyclic symmetry and words of length a on $\mathbb{Z}/b\mathbb{Z}$ with sum $(b-1)$ mod b under rotation.*
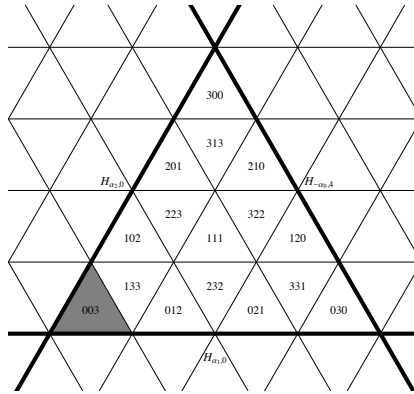

FIGURE 4. The four-fold dilation of the fundamental alcove in $\widetilde{\mathfrak{S}}_3$ labeled by words of length 3 in $\mathbb{Z}/4\mathbb{Z}$ with sum 3 mod 4.

One direction of the bijection in Theorem 2 is easy—the difficulty lies in finding the inverse bijection, which relies on a finite algorithm that "converges" to the correct answer, reminiscent of the Gale-Shapley stable marriage algorithm [GS62]. Motivated by my work to invert zeta for rational parking functions, I have recently understood the nature of this convergence in the context of Tarski's fixed point theorem [Tar55].

Of particular interest, Theorem 2 unexpectedly also resolves a long-standing conjecture related to (q, t)-symmetry of Macdonald polynomials and diagonal harmonics [ALW14, Xin15, CDH16].

2.2. **(Diagonal) Coinvariants and Weyl Groups.** The Hilbert series for the space of coinvariants is the generating function for two important statistics on the $n!$ permutations in $\mathfrak{S}_n$:

$$(1) \qquad \mathrm{Hilb}\left(\mathbb{C}[\mathbf{x}_n]/\langle\mathbb{C}[\mathbf{x}_n]_+^{\mathfrak{S}_n}\rangle; q\right) = \sum_{w\in\mathfrak{S}_n} q^{\mathsf{inv}(w)} = \sum_{w\in\mathfrak{S}_n} q^{\mathsf{maj}(w)},$$

where $\mathbb{C}[\mathbf{x}_n]$ is shorthand for a polynomial ring in n variables and $\langle\mathbb{C}[\mathbf{x}_n]_+^{\mathfrak{S}_n}\rangle$ is the ideal of $\mathbb{C}[\mathbf{x}_n]$ generated by symmetric polynomials with no constant term.

Artin gave a basis for this space using the code of a permutation to reflect the first generating function of Equation (1) [Art44], while Garsia and Stanton found a basis using the descents of a permutation to explain the second [GS84]. A statistic with the same distribution as inv or maj is called *mahonian*, after MacMahon [Mac13], but Foata gave the first bijection sending one statistic to the other [Foa68]. Exploiting the fact that this bijection preserves descents of the inverse permutation, Foata and Schützenberger

later found an involution that *interchanges* inv and maj [FS78], combinatorially proving

$$\sum_{w \in \mathfrak{S}_n} q^{\mathsf{inv}(w)} t^{\mathsf{maj}(w)} = \sum_{w \in \mathfrak{S}_n} t^{\mathsf{inv}(w)} q^{\mathsf{maj}(w)}. \tag{2}$$

Although inv generalizes to all Coxeter groups, there is no satisfactory definition of maj.

Problem 5. *Find definitions of* major index *for all Cartan types satisfying Equation* (2).

Motivated by the rich combinatorics of coinvariant spaces for Weyl groups, Garsia and Haiman introduced the space of *diagonal* coinvariants [Hai94, GH96], which has since been an extremely active area of research. Write

$$\mathbb{C}[\mathbf{x}, \mathbf{y}] := \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n].$$

The ring of diagonal invariants $\mathbb{C}[\mathbf{x}, \mathbf{y}]_+^{\mathfrak{S}_n}$ is the ring of $\mathfrak{S}_n$-invariant polynomials (with no constant term) in two sets of commuting variables, where $\mathfrak{S}_n$ acts diagonally (permuting the x and y variables simultaneously). The space of *diagonal coinvariants* is the quotient

$$\mathcal{DH}_n := \mathbb{C}[\mathbf{x}, \mathbf{y}]/\mathbb{C}[\mathbf{x}, \mathbf{y}]_+^{\mathfrak{S}_n}.$$

The most general *rational* (m, n) version of this theory comes from Hikita's study of the Borel-Moore homology of affine type A Springer fibers, which has a natural basis indexed by the m^{n-1} elements of the affine symmetric group $\widetilde{\mathfrak{S}}_n$ lying inside an m-fold dilation of the fundamental alcove [Hik14, Che03, Shi87, CP02, Hai94, Som05, GMV16a, Thi16]. Thus, while the space of coinvariants $\mathbb{C}[\mathbf{x}_n]/\langle \mathbb{C}[\mathbf{x}_n]_+^{\mathfrak{S}_n} \rangle$ is related to the symmetric group $\mathfrak{S}_n$, the diagonal coinvariants are related to the affine symmetric group $\widetilde{\mathfrak{S}}_n$.

2.3. **Zeta and sweep maps on lattice paths.** Perhaps due to the relative complexity of the underlying combinatorial objects, the combinatorics of diagonal coinvariants was first understood, and generalized for the alternating subspace $\mathcal{DH}_n^\epsilon$ of the space of diagonal coinvariants [KOP02, Hag03, GH02, ALW15, TW15].

Let $\mathcal{D}_{a,b}$ be the set of lattice paths from $(0,0)$ to (b, a) that stay above the main diagonal; write $\mathcal{D}_n = \mathcal{D}_{n+1,n}$. The classical *zeta map* ζ is a bijection from $\mathcal{D}_n$ to itself developed by Garsia, Haglund, and Haiman to explain the equidistribution of area with Haglund's statistic bounce and Haiman's statistic dinv in the combinatorial expansion of the Hilbert series of $\mathcal{DH}_n^\epsilon$ [GH02, Hai02, CM15, HX17]:

$$\mathrm{Hilb}\left(\mathcal{DH}_n^\epsilon; q, t\right) = \sum_{\mathsf{d} \in \mathcal{D}_n} q^{\mathsf{dinv}(\mathsf{d})} t^{\mathsf{area}(\mathsf{d})} = \sum_{\mathsf{d} \in \mathcal{D}_n} q^{\mathsf{area}(\mathsf{d})} t^{\mathsf{bounce}(\mathsf{d})},$$

where q records the degree of the variables $\mathbf{x}$ and t the degree of $\mathbf{y}$. Specifically, ζ has the pleasant property of translating Haglund's and Haiman's (inspired) statistics into the simple statistic area, so that [AKOP02, Hag03]:

$$\mathrm{Hilb}\left(\mathcal{DH}_n^\epsilon; q, t\right) = \sum_{w \in \mathcal{D}_n} q^{\overbrace{\mathsf{area}(\zeta^{-1}(w))}^{\mathsf{dinv}}} t^{\mathsf{area}(w)} = \sum_{w \in \mathcal{D}_n} q^{\mathsf{area}(w)} t^{\overbrace{\mathsf{area}(\zeta(w))}^{\mathsf{bounce}}}.$$

As Dyck paths have been generalized, so too have these zeta maps [Loe03, Egg03, GM14, LLL14, ALW14]—but proving invertibility of these generalized zeta maps has been a traditionally difficult problem [Xin15, CDH16]. We note that the zeta map has been rediscovered many times (often by accident)—perhaps most recently, it appeared as an answer to a question on MathOverflow [Vat13, Stu14].

To state one reasonably general version, the *sweep map* from $\mathcal{D}_{a,b} \to \mathcal{D}_{a,b}$ rearranges the steps of a path in $\mathcal{D}_{a,b}$ according to the order in which they are encountered by a line of slope a/b sweeping down from above [ALW14, Section 3.4]. Figure 5 computes the sweep map on a lattice path in $\mathcal{D}_{4,7}$. Unexpectedly, it turns out that our the bijection in Theorem 2 is a generalization of these sweep maps—from which we obtained the following theorem.

Theorem 3 ([TW15]). *For $a, b \in \mathbb{N}$, the sweep map is a bijection on $\mathcal{D}_{a,b}$.*

In fact, we prove a substantially more general form of the theorem above, which has already inspired several related papers, including [GX16a, GX16b]. In particular, our theorem covers the traditional case when m and n are coprime, but also the more recently considered (and more difficult) case when $\gcd(m, n) > 1$ [GMV17].
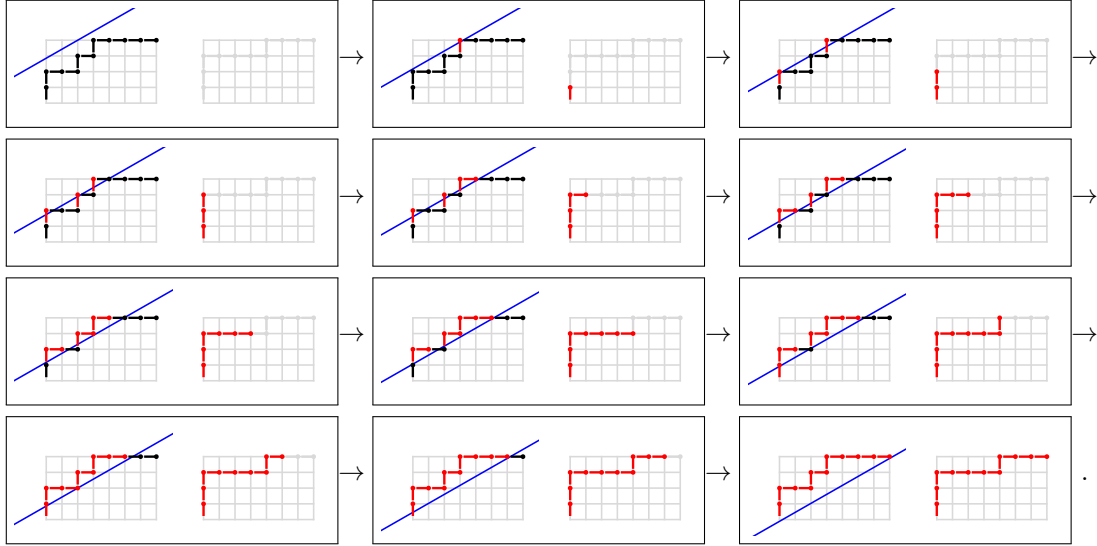

FIGURE 5. An illustration of the geometric interpretation of sweep. To form the right path, the steps of the left path are rearranged according to the order in which they are encountered by a line of slope 4/7 sweeping down from above.

2.4. **Zeta map on rational parking functions.** For $m, n \in \mathbb{N}$, the *(m,n)-parking functions* $\mathcal{P}_m^n$ are those words $\mathsf{p} = \mathsf{p}_0 \cdots \mathsf{p}_{n-1} \in [m]^n = \{0, 1, \ldots, m-1\}^n$ such that

$$(3) \qquad \left| \{ j : \mathsf{p}_j < i \} \right| \geq \frac{in}{m} \text{ for } 1 \leq i \leq m.$$

Write $\mathcal{P}_n = \mathcal{P}_{n+1}^n$. Just as Dyck paths encoding the Hilbert series of the alternating subspace of the space of diagonal coinvariants, the full Hilbert series of $\mathcal{DH}_n$ is encoded by parking functions.

$$(4) \qquad \mathrm{Hilb}\left(\mathcal{DH}_n; q, t\right) = \sum_{\mathsf{p} \in \mathcal{P}_n} q^{\mathsf{dinv}(\mathsf{p})} t^{\mathsf{area}(\mathsf{p})} = \sum_{\mathsf{p} \in \mathcal{P}_n} q^{\mathsf{area}(\mathsf{p})} t^{\mathsf{dinv}(\mathsf{p})},$$

where area and dinv are certain statistics on parking functions.

We recently came up with extremely simple combinatorics governing these statistics on parking functions [MTW17]. The previous state-of-the art was work of Gorsky, Mazin, and Vazirani, who used the affine symmetric group to define the zeta map on $\mathcal{P}_m^n$, which takes area to dinv. They conjectured that it was a bijection by providing what they believed to be an inverse map [GMV16b]. In [MTW17], we invert their zeta by having parking functions act on $\mathbb{1}_m \backslash \mathbb{R}^m / \mathfrak{S}_m$ (that is, $\mathbb{R}^m$ up to permutation of coordinates and addition of multiples of the all-ones vector) and applying the Brouwer fixed point theorem—a letter $i \in [m]$ acts on $\mathbf{x} \in \mathbb{1}_m \backslash \mathbb{R}^m / \mathfrak{S}_m$ by adding m to the ith smallest coordinate of $\mathbf{x}$, and a word $\mathsf{w} \in [m]^n$ acts on $\mathbf{x} \in \mathbb{1}_m \backslash \mathbb{R}^m / \mathfrak{S}_m$ by acting by its letters from left to right.

Theorem 4 ([MTW17])**.** *The action of* $\mathsf{w} \in [m]^n$ *on* $\mathbb{1}_m \backslash \mathbb{R}^m / \mathfrak{S}_m$:
- *has a* **unique** *fixed point iff* $\mathsf{w} \in \mathcal{P}_m^n$ *and* $\gcd(m,n) = 1$;
- *has* **infinitely many** *fixed points iff* $\mathsf{w} \in \mathcal{P}_m^n$ *and* $\gcd(m,n) > 1$; *and*
- *has* **no** *fixed points iff* $\mathsf{w} \in [m]^n \setminus \mathcal{P}_m^n$.

As a corollary of Theorem 4, we show that dinv and area are equidistributed on coprime (m,n)-parking functions.

Theorem 5. *For m and n relatively prime,*

$$\sum_{\mathsf{p} \in \mathcal{P}_m^n} q^{\mathsf{dinv}(\mathsf{p})} = \sum_{\mathsf{p} \in \mathcal{P}_m^n} q^{\mathsf{area}(\mathsf{p})}.$$

We have recently begun work on extending this theorem to understand what happens when $\gcd(m,n) > 1$, generalizing the setup in [GMV17] to parking functions (our Theorem 2 already provides the inverse to the zeta map on Dyck paths in the non-coprime case). Interestingly, the regions of the Shi arrangement and its Fuss generalizations may be described as the points fixed by some (n, kn)-parking function; more generally, we have a $\gcd(m,n)$-dimensional collection of fixed points living in $\mathbb{R}^m$ that warrants further investigation.

Problem 6. *Extend Theorem 5 to $\gcd(m,n) > 1$. Generalize the Shi arrangement by describing the points in $\mathbb{1}_m \backslash \mathbb{R}^m / \mathfrak{S}_m$ fixed by some element of $\mathcal{P}_m^n$.*

Loehr and Warrington's sweep maps on lattice paths are quite general, while the zeta maps on parking functions (thought of as labeled Dyck paths) seem rather more specialized—for example, sweep maps have no restriction on the number of different directions for steps, while zeta maps only allow two so that the paths must lie in a plane.

Problem 7. *Find a common generalization of sweep maps on lattice paths, and the zeta map on rational parking functions.*

Although it has gained a reputation as being intractable, it would be worth trying to apply our techniques and perspective to the problem of combinatorially explaining (q,t)-symmetry.

Problem 8. *Combinatorially prove (q,t)-symmetry of* $\mathrm{Hilb}\,(\mathcal{DH}_n; q, t)$.

There are numerous other problems that our methods might shed light on—for example, the q,t-Kostka numbers have still not been explained combinatorially.

2.5. **Other Cartan Types.** The definition of rational parking functions as the b^{a-1} alcoves in the b-fold dilation of the fundamental alcove in $\widetilde{\mathfrak{S}}_a$ easily extends to other Cartan types. And yet, the combinatorics of parking functions for other root systems is almost completely undeveloped (see also Problem 5).

Problem 9. *Find statistics to explain (q,t)-Catalan numbers in other Cartan types.*

A project detailed in Section 4.2 has given me some experience with finding interesting statistics in other types, and our Theorem 4 shows that rational parking functions in type A may also be characterized as those words of length $a - 1$ whose action on $\mathbb{R}^b$ has a fixed point. This new characterization suggests a novel way to approach Problem 9—by finding the right set of steps and the right space in which to act. Since $\widetilde{\mathfrak{S}}_n$ is interchanged with $\widetilde{\mathfrak{S}}_m$, there may be some sort of Howe duality involved. More generally, one might expect a generalization of Theorem 4 to hold.

Conjecture 6. *Fix a complex simple Lie algebra $\mathfrak{g}$ with weight lattice $\Lambda \subset V$. Let $(p_1, \ldots, p_k) \in \Lambda^k$ be a path in V, and write $\mathrm{wt}(p) = \sum p_i$. The path p acts on a dominant point $x \in V$: for $1 \leq i \leq k$, add p_i to x and reflect whenever a simple hyperplane is crossed. Then p has a fixed point if and only if $(\mathrm{wt}(p), \lambda_i) \leq 0$ for all fundamental weights λ_i.*

A second approach comes from a different characterization of rational parking fucntions I have found. In ongoing work, I explain the classical cycle lemma in combinatorics and generalize it to other Cartan types using Ω; this application was apparently unknown to the experts. In more detail, let Λ_{min} be the set of fundamental weights in the orbit of 0 (the *minuscule weights*). Define the usual dot action of Ω on V by $g \cdot x = g(x + \rho/h) - \rho/h$, where ρ is the half sum of the positive roots and h is the Coxeter number.

Theorem 7 (The Cycle Lemma). *A fundamental domain for the dot action of Ω on V is given by* $\{x \in V : (x, \omega) \geq 0, \omega \in \Lambda_{min}\}$.

By applying this theorem to the natural Weyl group action on crystals, I am able to give a unified framework for many combinatorial results in the literature. For example, let V_{ω_1} be the fundamental representation for $\mathfrak{sl}_b$. Then for a coprime to b, $\mathfrak{sl}_b(\omega_1)^{\otimes a}$ has a tableau model is in bijection with arbitrary words of length a with entries in $[b]$. The action of Ω on $\mathfrak{sl}_b(\omega_1)^{\otimes a}$ has free orbits and each orbit contains exactly one of the b^{a-1} rational parking functions. It is reasonable to wonder if this construction can be extended to give a definition of parking functions in other Cartan types.

3. New Directions in Coxeter-Catalan Combinatorics

3.1. **Dual Pure Artin Groups.** Let W be a finite Coxeter Group. The braid group

$$B(W) = \pi_1 \left(\mathbb{C}^n \setminus \bigcup_{\alpha \in \Phi^+} \mathcal{H}_\alpha / W \right) = \left\langle S : (s_i s_j)^{m_{ij}} = (s_j s_i)^{m_{ij}} \right\rangle$$

has a standard presentation using the simple reflections S, built from the weak order with Garside element the image of the longest element $w_\circ$. On the other hand, building on the work in the work of Birman, Ko, and Lee in 1998 for $\mathfrak{S}_n$, in the early 2000s Bessis and—independently—Brady and Watt defined the *dual Artin group* $B_c(W) \simeq B(W)$ with a presentation using *all* reflections, now depending on the choice of a Coxeter element c:

$$B_c(W) = \left\langle T : rt = t^r r \text{ for } r, t \in T \text{ such that } rt \leq_T c \right\rangle,$$

built from the *noncrossing partition lattice* $[e, c]_T$ with Garside element the image of the Coxeter element c itself. Note that this is *not* the group obtained by taking all relations satisfied by the reflections in W.

The pure braid group

$$P(W) = \pi_1 \left(\mathbb{C}^n \setminus \bigcup_{\alpha \in \Phi^+} \mathcal{H}_\alpha \right)$$

is the subgroup of $B(W)$ whose image in W is the identity e so that the following short exact sequence relates W, its Artin group $B(W)$, and the pure Artin group $P(W)$:

$$1 \to P(W) \to B(W) \to W \to 1.$$

This group is generated by the squares of the elements of T—for $t \in T$, we will abbreviate its square by $\Bbbk = t^2$.

Presciently, the problem of giving a presentation for $P(\mathfrak{S}_n)$ was first solved by Artin in 1925 with a non-positive presentation also involving all reflections. Starting with Artin's presentation, in 2006 Margalit and McCammond found simpler *positive* presentations of $P(\mathfrak{S}_n)$. In particular, these allow the definition of the corresponding pure monoid, as later studied by Lee; these all correspond to the choice of linear Coxeter element in type A.

In general, we denote the copy of $P(W)$ inside $B_c(W)$ by $P_c(W)$; we write $\mathrm{Red}_c(\Bbbw)$ be the set of reduced words in squares of reflections in the dual pure braid *monoid* $B^+(W, c)$ for the element $\Bbbw \in P_c(W)$. It is natural to suspect that there should be positive presentations for $P_c(W)$ using all reflections, again depending on a Coxeter element c. The hope is then—just as with the braid and dual braid groups—to apply Garside theory to $P_c(W)$, with the role of Garside element played by the *full twist* $\Bbbc = w_\circ^2 = c^h$.

Unfortunately, this naive hope cannot work because the interval $[e, \mathbb{c}]$ in $B^+(W, c)$ is not generally a lattice. Figure 6 gives some (small) data on the interval $[e, \mathbb{c}]$.

Cartan type	# Elements	# Max Chains
A_1	2	1
A_2	8	3
A_3	62	48
A_4	882	5150
B_2	14	4
B_3	290	234
D_4	2798	21312

FIGURE 6. Number of elements in the interval $[e, \mathbb{c}]$, and the number of reduced words for the full twist.

Problem 10. *Give simple presentations for $P_c(W)$.*

It strikes us as remarkable that this problem does not appear to have been solved in any explicit way in the literature, since in some ways the pure braid groups are more fundamental than the braid groups themselves. A presentation for type B was obtained by Digne and Gomi in [DG01], but their methods were strongly based on inclusions of groups and iterated free products, do not appear to extend in any sort of uniform way, and give rather unwieldy presentations. Even though the full twist does not give us the desired Garside theory, one might still hope that all relations in the pure braid group come from words for the full twist. (Although they contain a huge amount of redundant information, such presentations do hold for the braid group with the long element and the dual braid group with a Coxeter element.)

Conjecture 8.
$$P_c(W) = \left\langle \mathbb{t} \text{ for } t \in T : \mathbb{c}_1 = \mathbb{c}_2 \text{ for any } \mathbb{c}_1, \mathbb{c}_2 \in \mathrm{Red}_c(\mathbb{c}) \right\rangle.$$

What makes such a presentation even more enticing is that we found an elegant conjectural description of the reduced words for the full twist, inspired by Bessis's proof that the complements of complexified arrangements are $K(\pi, 1)$. To state this description, recall that any reduced word for the full twist in $\mathrm{Red}_c(\mathbb{c})$ defines a total ordering on the reflections of W. On the other hand, as a portion of the Cayley graph of W with respect to T—the noncrossing partition lattice comes equipped with a natural labeling by reflections and is EL-shellable with this labeling. We define EL_c as the set of total orderings of T such that for every nonsingleton interval $[u, v]$ in the noncrossing partition lattice, there is a *unique* maximal chain in $[u, v]$ whose edge labels increase.

Conjecture 9. *The orderings of T defined by the reduced words for the full twist in $\mathrm{Red}_c(\mathbb{c})$ are exactly the orderings of T in EL_c.*

The method of proof ought to be geometric: an element of EL_c coincides with a shelling of the order complex of the noncrossing partition lattice; but this complex embeds in $\mathbb{C}^n$, and the statement then becomes that a shelling coincides with a homotopy class of loop representing the full twist. We have another—completely combinatorial—characterization of EL_c that is useful for performing computations.

We might hope to be even more surgical in our presentation: for both the braid and dual braid groups, the relations are completely determined by rank two parabolic subgroups. Similarly, there are some obvious *dual pure braid relations* in $P_c(W)$ satisfied by products of consecutive reflections of a dihedral noncrossing parabolic subgroup—but there are also somewhat surprising relations, not of these obvious forms. For example, in $P_{(1234)}(\mathfrak{S}_4)$, we have the relation

$$(\mathbb{13})(\mathbb{12})(\mathbb{24})(\mathbb{23}) = (\mathbb{12})(\mathbb{24})(\mathbb{23})(\mathbb{13}).$$

There is a natural generalization of such relations to any dihedral parabolic subgroup, which we have termed *broken relations*.

Conjecture 10.

$$P_c(W) = \left\langle \,\natural \text{ for } t \in T : \quad \begin{array}{c} \textit{dual pure braid relations} \\ \textit{broken relations} \end{array} \quad \right\rangle.$$

This conjecture is true for type A, and we have checked it in B_3 and D_4.

3.2. **Noncrossing and nonnesting.** In Coxeter-Catalan combinatorics, the Catalan numbers $\mathsf{Cat}(n) = \mathsf{Cat}(\mathfrak{S}_n)$ are associated to the symmetric group, and count various objects, including: the noncrossing partitions, the triangulations of a convex $(n{+}2)$-gon and the 231-avoiding permutations (the 14 triangulations of a hexagon are drawn on the right-hand side of Figure 8). In fact, Catalan numbers beautifully generalize to all other finite Coxeter groups in terms of invariants of the group: triangulations become finite-type clusters [FZ03] and 231-avoiding permutations become sortable elements [BW97, Rea07a, Rea07b, RS11]. Despite having uniform definitions, there are only type-by-type proofs (using recursions or combinatorial models) that the noncrossing partitions, clusters, and sortable elements are counted by $\mathsf{Cat}(W)$. As such the biggest open problem in Coxeter-Catalan theory is the following:

Problem 11. *Uniformly prove that* any *of these families are counted by* $\mathsf{Cat}(W)$.

The most important idea we advance in [STW15] is that the correct setting for a generalization of $\mathsf{Cat}(W)$ called the *Fuss–Catalan numbers* is provided by the *Artin monoid* $B^+(W)$. This allows us to not only give a uniform treatment of previous work, but also supply a missing definition of sortable elements (a generalization of 231-avoiding permutations) to the Fuss level of generality.

Definition-Theorem 11 ([STW15])**.** *The* Fuss c-sortable elements *are a certain subset of the interval* $[e, w_\circ^m]$ *in the positive Artin monoid. Their cardinality is* $\mathsf{Cat}^{(m)}(W)$.

The entire interval $[e, w_\circ^2]$ for $B^+(\mathfrak{S}_3)$ is illustrated on the left-hand side of Figure 7, with the c-sortable elements shaded in gray.

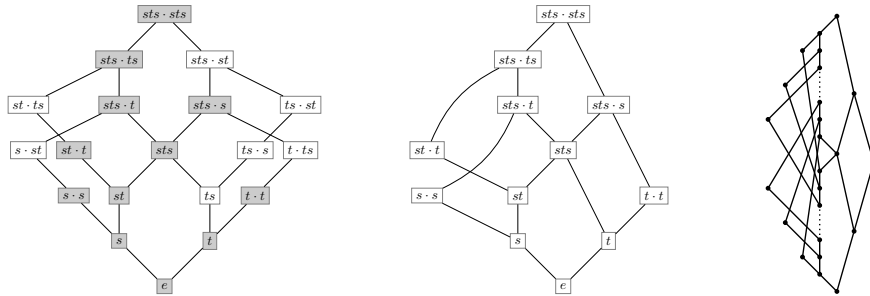FIGURE 7. Left—the inverval $[e, w_\circ^2]$ in $B^+(\mathfrak{S}_3)$; center—the restriction to c-sortable elements; right—the exchange graph for $\widetilde{A}_1$.

Building on my work with B. Rhoades and D. Armstrong [ARW13], the framework inside the positive braid monoid that C. Stump, H. Thomas, and I proposed for noncrossing partitions, sortable elements, and clusters appears to scale a "rational" level of generality in the classical types [ARW13]. Unfortunately, our methods are entirely ad-hoc, and there is much that we don't understand—for example, despite exhaustive computer searches, we are unable to give any reasonable construction of clusters for type F_4.

Problem 12. *Give uniform constructions of rational noncrossing Catalan objects.*

Catalan numbers naturally appear in a markedly different context—in the study of affine Weyl groups and rational Cherednik algebras (this is related to the Macdonald theory of the previous section). For well-generated finite complex reflection groups, $\mathsf{Cat}^{[b]}(W)$ is defined as the dimension of the finite-dimensional irreducible representation $eL_{b/h}(\mathsf{triv})$ of the rational Cherednik algebra at the parameter b/h. Specializing to crystallographic Coxeter groups, $\mathsf{Cat}^{[b]}(W)$ (uniformly) counts the number of coroot points inside a b-fold dilation of the fundamental alcove in the corresponding affine Weyl group [Hai94, Sut98]. For $b = h + 1$, these coroot points are called *nonnesting partitions*, and are in bijection with order ideals in the root poset (or, equivalently, ad-nilpotent ideals in a Borel subalgebra of the corresponding complex simple Lie algebra). Although nonnesting and noncrossing partitions have many similarities, finding a uniform bijection between the two sets has been an active and motivating area of research since the late 1990s [Rei97, Ath98]. Since nonnesting partitions are uniformly enumerated, such a bijection would answer Problem 11.

In [Wil13b], we conjectured exactly such a bijection between nonnesting and noncrossing objects for any Coxeter element and any finite Weyl group, suggesting that the root poset encodes a remarkable amount of information related to the corresponding Weyl group (compare with the duality between the heights of roots and the degrees). Recently, J. Michel found a uniform proof for the number of factorizations of a Coxeter element for Weyl groups using Deligne-Lusztig theory [Mic14]. We are interested in how these two methods are related.

In more detail, our method is based on an original analogy between noncrossing and nonnesting partitions. Noncrossing partitions have a cyclic action called the *Kreweras complement* Krew_c—which may be slightly modified to form a positive version, Krew_c^+—while clusters have a natural *Cambrian rotation* Camb_c. These actions have been defined in the literature in a way that may be seen as "global" [Arm09, FZ03]. In [Wil13a, STW17], we develop natural "local" methods to compute all three actions as *walks* on the $\mathsf{Cat}(W)$ vertices of the associahedron. For example, clusters for $\mathfrak{S}_{n+1}$ correspond to triangulations of an $(n + 3)$-gon; in this language, Camb_c is described as a sequence of flips of diagonals that rotates a triangulation. Such a sequence is illustrated for $\mathfrak{S}_4$ by the red path on the left-hand side of Figure 8. Our walks reveal an unexpected relation between the three actions.

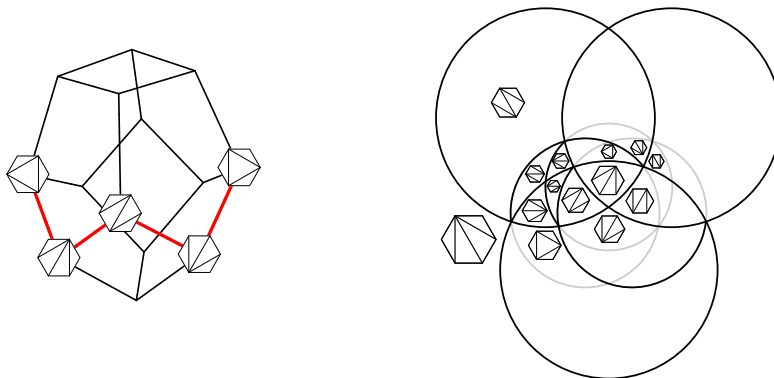**Theorem 12** ([STW17]). $\mathsf{Camb}_c = \mathsf{Krew}_c \circ \mathsf{Krew}_c^+$.

FIGURE 8. On the left is the three dimensional associahedron; on the right is the stereographic projection of the $\mathfrak{S}_4$ hyperplane arrangement with certain regions labeled by triangulations.

Our conjectural bijection between noncrossing and nonnesting objects comes from mimicking our walks on the W-associahedron in Theorem 12—drawing inspiration from [Pan09, BR11, AST13], our methods

produce remarkable conjectural (compatible) bijections from nonnesting partitions to clusters and noncrossing partitions which have been exhaustively checked up to rank eight [Wil13a, Wil14, STW17].

Problem 13 ([Wil13a, Wil14, STW17])**.** *Show that these maps are bijections. Extend them to the Fuss and rational levels of generality.*

A first step would be to restrict from Cellini and Papi's bijection to Peterson's abelian ideals and find uniform support-preserving bijections between the abelian ideals of a Borel subalgebra and the longest elements of parabolic subgroups.

3.3. Infinite Coxeter-Catalan combinatorics.

Noncrossing partitions and clusters generalize readily to infinite Coxeter groups, but sortable elements no longer recover the entire cluster exchange graph beyond finite type—as elements of the Coxeter group, they are limited to the Tits cone. A general research direction championed by N. Reading has been the following.

Problem 14. *Extend the definition of sortable elements to recover the full cluster exchange graph in infinite type.*

We have some new ideas in this direction. Although noncrossing partitions (and even Catalan numbers) generalize to infinite Coxeter groups, it seems unreasonable to expect corresponding notions of clusters or sortable elements in the absence of a natural simple system of generators and "chamber geometry." On the other hand, Markowsky's generalization to extremal lattices of Birkoff's representation theorem of distributive lattices seems a reasonable candidate to capture the full structure. We have recently understood finite-type (Fuss-)Cambrian lattices in terms of Markowsky's representation theorem, and such combinatorics appears to be related to M. Dyer's biclosed sets [Dye11].

Many of our constructions of Fuss-Catalan objects extend easily to infinite Coxeter groups (see the right of Figure 7 for a depiction of the $m = 2$ exchange graph in affine type $\widetilde{A}_1$), but—as in the case for $m = 1$—we again run into a limitation for the definition of c-sortable elements in infinite type.

Problem 15. *Extend our Fuss constructions from [STW15] to infinite Coxeter groups. Do our Fuss-Cambrian lattices arise from some generalization of a cluster algebra?*

4. Other projects

4.1. Garside Shadows.

Each reflection in a Coxeter group W is associated to a positive root, which we collect in a set Φ^+. For $\alpha, \beta \in \Phi^+$, we say α *dominates* β if $-w(\alpha) \in \Phi^+$ implies $-w(\beta) \in \Phi^+$ for all $w \in W$. A *small root* is a positive root that only dominates itself [DS91, BH93, DH15]. Surprisingly, there are always only a *finite* number of small roots—and these suffice to understand much of the structure of W (including the word problem). Finally, an element $w \in W$ is *low* if all of its lower bounding hyperplanes are small.

Problem 16 ([HNW16])**.** *Show that the inverses of the low elements are a convex set.*

Suggestive rank-three illustrations are provided in Figure 9, in which the inverses of the low elements (in gray) have coalesced into a convex polyhedron.

A *Garside shadow* is a subset $B \subseteq W$ containing S and closed under weak-order join and suffixes [DDH15, DH15]; the polyhedra in Figure 9 are examples of a general construction of (finite) Garside shadows using small roots. In [HNW16], C. Hohlweg, P. Nadeau and I study projections to an arbitrary Garside shadow.

Theorem 13 ([HNW16])**.** *Any finite Garside shadow produces a finite deterministic automaton recognizing the language* Red(W, S) *of reduced words for* (W, S).

Since Garside shadows are closed under intersection, we have the following conjecture describing the minimal such automaton.
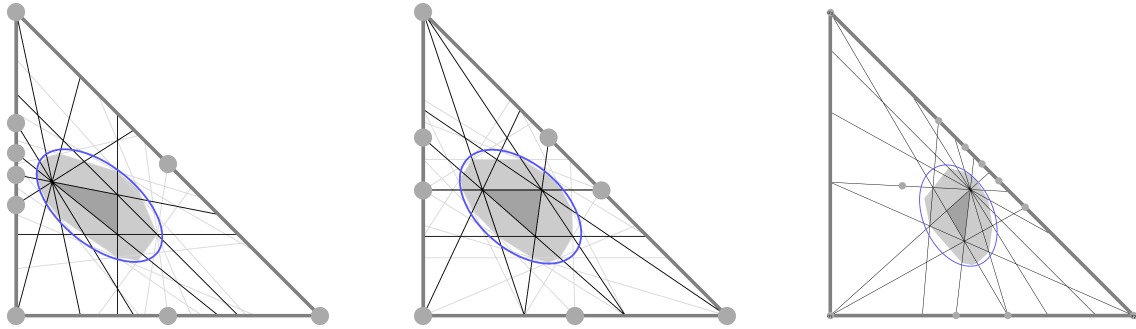
FIGURE 9. The inverses of the low elements in the triangle groups $(3, 3, 6)$, $(3, 4, 4)$, and $(4, 7, 2)$ form convex sets.

Problem 17. *Prove that projecting onto the* smallest *Garside shadow produces the* minimal *automaton recognizing* Red(W, S).

4.2. **Strange Expectations.** For relatively prime a and b, the coroot points inside the b-fold dilation of the fundamental alcove $\mathcal{A}$ are in bijection with (simultaneous) (a, b)-*cores*—integer partitions whose Ferrers diagram contains no box whose hook-length is divisible by either a or b. Recently, there has been a surge of interest on statistics for simultaneous cores [Nat08, AKS09, Fay11, AL14, YZZ14, Nat14, CHW14, Agg14, Fay14, Xio14, Agg15, Fay15]. Results of J. Olsson and D. Stanton [OS07] and of P. Johnson [Joh15] (confirming a conjecture of D. Armstrong [Arm15, AHJ14]) prove that the maximum number and expected number of boxes in an (a, b)-core are

$$\max_{\lambda \in \mathsf{core}(a,b)}(\mathsf{size}(\lambda)) = \frac{(a^2-1)(b^2-1)}{24}, \quad \mathbb{E}_{\lambda \in \mathsf{core}(a,b)}(\mathsf{size}(\lambda)) = \frac{(a-1)(b-1)(a+b+1)}{24}.$$

For b relatively prime to the Coxeter number h, by extending the definitions of "simultaneous core" and "number of boxes" to all affine Weyl groups $\widetilde{W}$, we use Ehrhart theory in [TW17] to give uniform generalizations to simply-laced affine types.

Theorem 14 ([TW17]). *For $\widetilde{W}$ a simply-laced affine Weyl group,*

$$\max_{\lambda \in \mathsf{core}(\widetilde{W},b)}(\mathsf{size}(\lambda)) = \frac{n(h+1)(b^2-1)}{24}, \quad \mathbb{E}_{\lambda \in \mathsf{core}(\widetilde{W},b)}(\mathsf{size}(\lambda)) = \frac{n(b-1)(h+b+1)}{24}.$$

By setting $a = h = n + 1$, we recover the formulas for $\widetilde{\mathfrak{S}}_a$. We further explain the appearance of the number 24 using the "strange formula" of H. Freudenthal and H. de Vries. We compute the variance for all simply-laced affine Weyl groups and third moment for $\widetilde{\mathfrak{S}}_{n+1}$ (see also S. B. Ekhad and D. Zeilberger's subsequent preprint [EZ15]). It would be especially interesting to study this class of problem on more exotic lattices—for example, G. Nebe's primitive root lattices for complex reflection groups [Neb99]. A first step would be to address non-simply-laced types.

Problem 18. *Extend Theorem 14 to non-simply-laced (and twisted) affine types.*

We have a conjectural weight function—the difficulty being to guess exactly which point should be considered the "centroid" of an alcove. There is a conjectural weighted version of Theorem 14, which involves summing over *all* weights inside $\check{Q}/b\check{Q}$, rather than just the coroots.

Problem 19. *For $\widetilde{W}$ a simply-laced affine Weyl group, prove that*

$$\underset{w\in b\mathcal{A}}{\mathbb{E}}(\mathsf{size}(w)) = \frac{n(b^2-1)}{24}.$$

By analogy, we were recently led to consider the expected norm of a weight in a highest weight representation V_λ of a complex semisimple Lie algebra $\mathfrak{g}$. By relating this to the "Winnie-the-Pooh problem" of decomposing $\mathfrak{g}$ into a direct sum of mutually orthogonal Cartan subalgebras, we give a proof that this expectation is $1/(h+1)(\lambda+2\rho,\lambda)$. Our proof works for all types except A and C; the same formula holds in these two types, but we are forced to provide a direct computation. We are very interested in studying various extensions of this problem.

5. Prior Support: Not Applicable

I have not held an NSF grant before.

6. Broader Impacts

6.1. **Dynamical Algebraic Combinatorics.** I believe that my research has had a positive effect on the combinatorics community, and I have a record of producing problems and research areas accessible to beginning researchers.

My work with J. Striker in [SW12] has served as a catalyst for the involvement of undergraduate and young graduate students in cutting-edge research at REUs and doctoral programs—there were many developments motivated by the appearance of our paper [SW12], including a flurry of related projects, REU topics, publications, and theses: [CHHM15, EP13, EFG+15, Had14, Hop16, GR14, GR15, GR16, PR15, Rob16, RS13, RW15, Rus16, DPS15, Str15, Str16]. Similarly, my work with Z. Hamaker, R. Patrias, and O. Pechenik led to at least two separate REU projects over the last two years: one at S. Billey's REU at the University of Washington, and one supervised by O. Pechenik.

In 2015, J. Striker, J. Propp, T. Roby and I organized an AIM workshop. This workshop launched a new field of combinatorics that J. Propp has termed "Dynamical Algebraic Combinatorics", and many papers have resulted from and been inspired by our workshop, including [DPS17, EFG+15, JR17, STWW17, HMP16, GHMP17b, GHMP17a, GP17]. We are organizing a session at the Joint Mathematics Meetings this year for further progress reports in this nascent area.

6.2. **Graduate Education.** Although I have just started my current position at the University of Texas at Dallas, I am already supervising one graduate student (Austin Marstaller) in an independent study course, will be supervising the honors thesis of an undergraduate (Kevin Zimmer) next semester, and I am one of two faculty organizers (along with Andras Farago) for the Graduate Student Combinatorics Conference to be held at UT Dallas in 2018. Funding will allow me to support the research program of a graduate student past the initial two years of coursework.

As the only combinatorialist at UT Dallas, I will be designing a new undergraduate and graduate course in combinatorics to train students in fundamental ideas and concepts of the discipline.

6.3. **Mentoring.** Because of its many elementary problems, combinatorics is a discipline in which undergraduate and graduate students can immediately become involved in research-level mathematics. Furthermore, I have substantial past experience in involving underrepresented students in research, and would continue to seek out such opportunities with the goal to eventually build an REU program at UT Dallas:

- In 2016, I co-mentored Florence Maas-Gariepy on a research/study project involving finite reflection groups, which led to her detailed project report (in French) [MG16]. This report was featured on the funding agency's website.

- In 2014, I mentored Stephanie Schanack, Fatiha Djermane, and Sarah Ouahib on an original research problem involving the characterization of the fixed points of a certain combinatorial set under a cyclic group action. I guided them through an intricate network of case-by-case analyses which the three wrote up in a well-crafted report (in French) [SSD14].

With S. Shin in the statistics department at UT Dallas, I recently proposed a prototype mentoring program for funding by the Women Achieving through Community Hubs, and I am interested in increasing the visibility of women in mathematics at UT Dallas by establishing an AWM chapter here. I was also involved with the very successful combinatorics Research Experience for Undergraduates (REU) at the University of Minnesota:

- At the 2011 REU, I provided support to David B Rush and XiaoLin Shi [RS13], who found a generalization of my work in [SW12].
- For the 2010 Minnesota REU, I helped direct Gaku Liu's research in partition identities [Liu] and helped a second group formulate and computationally test conjectures on a combinatorial reformulation of the four-color theorem [CSS14].