

INDEPENDENCE POSETS

INTRODUCTION

Independent sets of graphs are ubiquitous objects in combinatorics with applications to complexity theory and statistical mechanics. The proposed research program will develop new combinatorial structures on independent sets, focusing on problems related to models of classical combinatorial objects, representation theory of finite-dimensional algebras, statistical mechanics, and dynamical algebraic combinatorics.

Central to the proposal is the PI's recent definition of *independence posets* (Section 1, see Figure 1 for an example), which give a new partial ordering on the independence sets of a directed acyclic graph G [TW20b]. When G is the comparability graph of a poset P , the independence poset on G is isomorphic to the distributive lattice of order ideals of P . Thus, an independence poset answers the question:

“What if a distributive lattice weren't a lattice?”

The class of independence posets includes not only all distributive lattices, but also (rational) Tamari lattices, Cambrian lattices of finite type, Fuss-Cambrian lattices [STW15], torsion pairs of finite type tilted Artin algebras, and many other related notions of both classical and current interest. Interesting research in those areas can now be unified and generalized to the setting of independence posets. The specific goals of the proposed research include:

- Extend the definition of independence posets to infinite digraphs, digraphs with oriented cycles, and proper colorings guided by the representation theory of certain classes of algebras, lattice theory, and symmetric function theory [Asa16, RST19] (Section 1);
- Study derived equivalences of independence posets [Lad07]; develop a framework for bijective proofs using independence posets (Section 2);
- Exploit the monotone structure on independent sets provided by independence posets to extend the applicability of the Propp-Wilson coupling from the past algorithm to randomly sample independent sets [PW98] (Section 3);
- Relate independence posets with integral points in dilations of polytopes constructed by Chvátal [Chv75], generalizing P -partitions from distributive lattices to independence posets [Sta86]; generalize independence posets to *any* convex lattice polytopes;
- Find new examples of relevance to Dynamical Algebraic Combinatorics; i.e., combinatorial, piecewise-linear, and birational rowmotion periodicity in the generalized setting of independence posets [Rob16, Str17] (Section 4).

The PI's related work in [SW12] has served as a catalyst for the involvement of undergraduate and beginning graduate students in cutting-edge research at REUs and doctoral programs. There have been many developments motivated by the appearance of [SW12]—to name a few: [CHHM15, EP13, EFG⁺15, Had14, Hop16, GR14, GR15, GR16, PR15, Rob16, RS13, RW15, Rus16, DPS17, Str15, Str16, JR18, MR19, DSV19, Jos19, JR20, Hop20, JR20]. In 2015, the PI, Striker, Propp, and Roby organized an AIM workshop that launched a new field of combinatorics now termed “Dynamical

Algebraic Combinatorics.” This same group is organizing a follow-up BIRS online workshop for Fall 2020 (originally accepted in-person, but delayed due to COVID-19; the PI is preparing his current undergraduate honors reading class to attend this workshop). The PI has additionally organized several successful AMS and JMM special sessions in this field. An integral part of this proposal is to continue supporting the PI’s ongoing and future efforts to involve students in cutting-edge research in algebraic combinatorics and related areas.

The PI has already laid some of the theoretical groundwork underpinning this proposal in the two recent publications [TW19b, TW19a]. Over the course this previous research, the PI has developed an original toolkit and perspective that has yielded substantial new progress in related fields. Based on this new perspective, the PI has created an interconnected library of concrete combinatorial problems especially suitable for early-stage students.

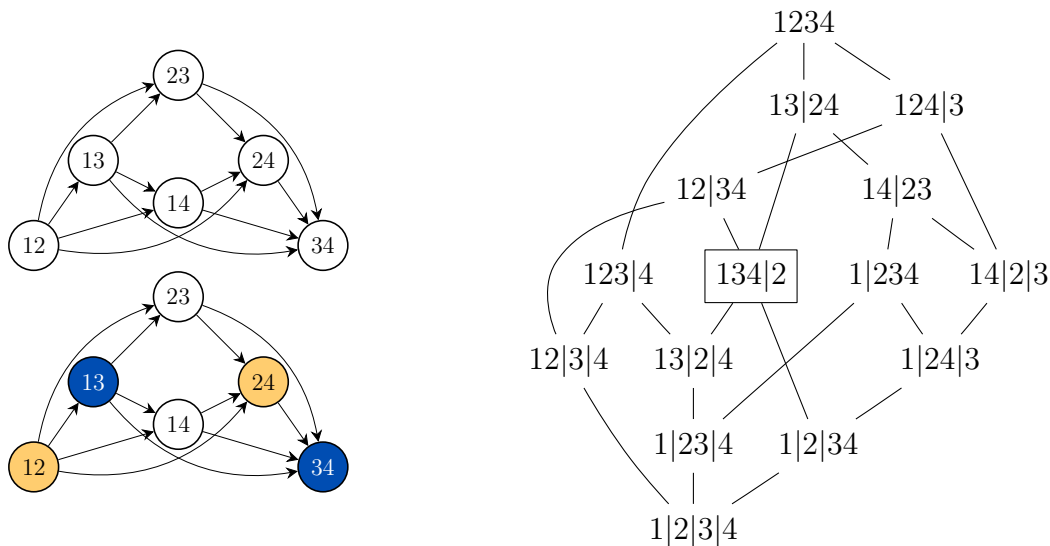


FIGURE 1. *Top left:* acyclic digraph G whose independent sets encode the 15 set partitions of $\{1, 2, 3, 4\}$. *Bottom left:* the tight orthogonal pair in G (Definition 1) corresponding to the set partition $134|2$ (boxed on right). *Right:* the corresponding independence poset on set partitions.

1. DEFINITION AND MOTIVATION

1.1. Definition: Independence Posets. Let G be a finite acyclic directed graph (without oriented cycles, loops, or multiple edges) so that the transitive closure of G admits a partial order on its vertices called G -order. An *independent set* $\mathcal{F} \subseteq G$ is a set of pairwise non-adjacent vertices of G . *Independence posets*, introduced by the PI in [TW19a], are a certain partial ordering on the independent sets of G —depending on the orientation of G —whose cover relations are given by a novel definition of (non-local) *flips*. For example, there are 126 acyclic orientations of the graph in Figure 1, giving 126 different independence poset structures on the set partitions of $\{1, 2, 3, 4\}$.

The notion of “independence poset” is a natural generalization of that of “distributive lattice,” but where the lattice requirement is eliminated. An independence poset that is a graded lattice is a distributive lattice. Many other well-known posets (such as Tamari and Cambrian lattices) turn out to be special cases of independence posets. The definitions below can be explored using the PI’s FPSAC 2020 online interactive poster [TW20b].

Definition 1 (Tight Orthogonal Pairs). A pair $(\mathcal{D}, \mathcal{U})$ of disjoint independent sets of G is called *orthogonal* if there is no edge in G from an element of \mathcal{D} to an element of \mathcal{U} . An orthogonal pair of independent sets $(\mathcal{D}, \mathcal{U})$ is called *tight* if whenever any element of \mathcal{D} is increased (removed and replaced by a larger element with respect to G -order) or any element of \mathcal{U} is decreased, or a new element is added to either \mathcal{D} or \mathcal{U} , then the result is no longer an orthogonal pair of independent sets. We write $\text{top}(G)$ for the set of all **tight orthogonal pairs** of G .

One can show that for any independent set \mathcal{F} , there is a unique $(\mathcal{F}, \mathcal{U}) \in \text{top}(G)$ and a unique $(\mathcal{D}, \mathcal{F}) \in \text{top}(G)$. Tight orthogonal pairs now allow us to define a non-local *flip* operation, which generate the cover relations of a partial order we call the *independence poset*.

Definition 2 (Flips). The *flip* of $(\mathcal{D}, \mathcal{U}) \in \text{top}(G)$ at an element $g \in G$ is the tight orthogonal pair $\text{flip}_g(\mathcal{D}, \mathcal{U})$ defined as follows: if $g \notin \mathcal{D}$ and $g \notin \mathcal{U}$, the flip does nothing. Otherwise, preserve all elements of \mathcal{D} that are not less than g and all elements of \mathcal{U} that are not greater than g (and delete all other elements); after switching the set to which g belongs, then greedily add elements to \mathcal{D} and \mathcal{U} (respecting the conditions to form an orthogonal pair) in reverse G -order and G -order, respectively.

Figure 2 illustrates a flip on a tight orthogonal pair in an orientation of $[7] \times [7]$. For G an acyclic directed graph, the *independence relations* on $\text{top}(G)$ are the reflexive and transitive closure of the relations $(\mathcal{D}, \mathcal{U}) < (\mathcal{D}', \mathcal{U}')$ if there is some $g \in \mathcal{U}$ such that $\text{flip}_g(\mathcal{D}, \mathcal{U}) = (\mathcal{D}', \mathcal{U}')$.

Theorem 3 (Independence Posets). *Independence relations are antisymmetric, and hence define an independence poset, denoted $\text{top}(G)$. Flips and cover relations of $\text{top}(G)$ coincide.*

There is an obvious partial order $\mathcal{I}(G)$ on independent sets of G defined by simple *inclusion*: $\mathcal{F} \leq_{\mathcal{I}(G)} \mathcal{F}'$ iff $\mathcal{F} \subseteq \mathcal{F}'$. In [TW19a], we left open the following natural question relating $\mathcal{I}(G)$ and $\text{top}(G)$ (generalizing the corresponding property from noncrossing partition lattices and Cambrian lattices; see [Rea11, BB09]).

Problem 1. Show that the map $\mathcal{I}(G) \rightarrow \text{top}(G)$ defined by $\mathcal{F} \mapsto (\mathcal{F}, \mathcal{U})$ is order preserving: if $\mathcal{F} \leq_{\mathcal{I}(G)} \mathcal{F}'$, then $(\mathcal{F}, \mathcal{U}) \leq_{\text{top}(G)} (\mathcal{F}', \mathcal{U})$.

1.2. Motivation I: Representation Theory. The representation theory of finite-dimensional algebras provides a first motivation for independence posets. Let k be a field, and A a finite-dimensional k -algebra such that the module category $\text{mod } A$ has no cycles. Examples include Dynkin path algebras (here, a module assigns a vector space to each vertex of the Dynkin quiver, and a linear map to each edge; by Gabriel’s theorem, the indecomposable modules are in bijection with positive roots); as well as all quotients of Dynkin path algebras.

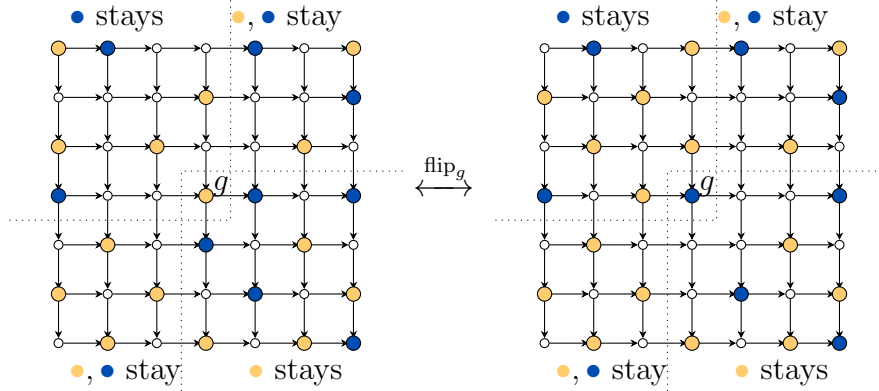


FIGURE 2. A flip on a top $(\mathcal{D}, \mathcal{U})$ in the 7×7 grid oriented from top left to bottom right. The blue vertices correspond to the elements of \mathcal{D} , while the orange vertices correspond to the elements of \mathcal{U} . Flipping at the vertex g changes its color, and divides the grid into 5 connected regions (delineated by the dotted lines): the blue vertices not less than g (i.e., not in the bottom right) and the orange vertices not greater than g (i.e., not in the top left) are preserved by the flip. The orange vertices in the top left are filled in greedily from bottom right to top left; the blue vertices in the bottom right are filled in greedily from top left to bottom right.

Define a directed graph G with vertices indexed by the indecomposable A -modules and an arrow from M to N if and only if $\text{Hom}(M, N) \neq 0$. A *simple-minded collection* for A is a collection of objects X_1, \dots, X_r in the derived category of $D^b(A)$ such that [KY14]:

- (1) $\text{Hom}(X_i, X_j[m]) = 0$ for $m < 0$,
- (2) $\text{End}(X_i)$ is a division algebra and $\text{Hom}(X_i, X_j) = 0$ unless $i = j$,
- (3) X_1, \dots, X_r generate $D^b(A)$ in the sense that the smallest thick subcategory containing all of them is $D^b(A)$ itself,

A *2-simple-minded collection* additionally satisfies $H^j(X_i) = 0$ for $j \neq 0, -1$ and for each X_i . It follows from work of Asai [Asa16] that tight orthogonal pairs on the directed graph G defined above (with vertices the indecomposable A -modules) are in bijection with 2-simple-minded collections.

Theorem 4. *Let A be a representation-finite k -algebra with no cycles in $\text{mod } A$. There is a bijection from tight orthogonal pairs to 2-simple-minded collections, sending $(\mathcal{D}, \mathcal{U})$ to $\mathcal{D} \cup \mathcal{U}[1]$.*

Relaxing the property that the module category is finite suggests the following problem regarding the definition of infinite independence posets and independence posets associated to directed graphs with oriented cycles. Any such definition should be guided by developing a combinatorial understanding of the special case when G arises from the representation theory of more general module categories (the first example to work out would be affine quivers), following similar arguments as in [TW19a, Section

7] (see also [RST19]), and should have connections to the combinatorics of cluster algebras of infinite type.

Problem 2. Extend the construction of independence posets to *infinite* acyclic digraphs and digraphs with oriented cycles.

1.3. Motivation II: Lattice Theory. Birkhoff’s well-known *fundamental theorem of finite distributive lattices* proves that finite distributive lattices are parametrized by finite posets P (as the lattice $J(P)$ of order ideals under inclusion). Markowsky (a student of Birkhoff) generalized Birkhoff’s theorem to a lesser-known representation theorem for finite *extremal lattices*—that is, lattices whose longest chain is equal to both the number of join irreducible elements and meet irreducible elements—showing that finite extremal lattices are parametrized by (finite) acyclic graphs G (as the lattice of maximal orthogonal pairs of G under inclusion) [Mar92].¹

Independence posets are a **different** generalization of Birkhoff’s theorem: although independence posets (like extremal lattices) are still parametrized by acyclic directed graphs, their elements are the ubiquitous independent sets, rather than the more technical “maximal orthogonal pairs”. When an independence set happens to be a lattice, then it is a special kind of extremal lattice called a *trim lattice* (which then admit a canonical labeling of cover relations by join and meet irreducibles [TW19b]).

Theorem 5. *If $\text{top}(G)$ is a lattice, then it is a trim lattice. Every trim lattice can be realized as $\text{top}(G)$ for a unique (up to isomorphism) acyclic directed graph G .*

The recently renewed theory of lattice quotients (with substantial applications to cluster algebras and quiver representation theory [Rea06]) suggests that there out to be a procedure to “blow up” a trim lattice into a *graded* (possibly semidistributive) lattice. A motivating problem in this direction is that of reconstructing weak order from its Cambrian lattice (see Figure 3)—in type A , for example, it is known how to reconstruct the collapsed intervals between sortable and anti-sortable elements using trees [PP18]. This is closely related to the problem of extending the construction of trim lattices or independence posets to digraphs that contain cycles from Section 1.2.

Problem 3.

- (1) Use the representation of trim lattices as maximal orthogonal pairs (X, Y) to recursively “blow up” an element (X, Y) using maximal orthogonal pairs on $G \setminus (X \cup Y)$. The result should be a graded lattice. (See Figure 3.)
- (2) Extend the construction of trim lattices and independence posets to general digraphs. The result should be a theory that unifies semidistributive and trim lattices (see also [RST19]).

2. PROBLEMS IN REPRESENTATION THEORY AND COMBINATORICS

2.1. Derived Equivalences of Posets. Ladkani has defined an interesting notion of *derived equivalence* of posets [Lad07, Lad08] (this technology has recently been used by

¹In fact, Markowsky’s work included a representation theorem for *any* finite lattice, previously anticipated by Barbut [Bar65]; these ideas were later developed by Wille in the guise of *Formal Concept Analysis*. [Wil82].

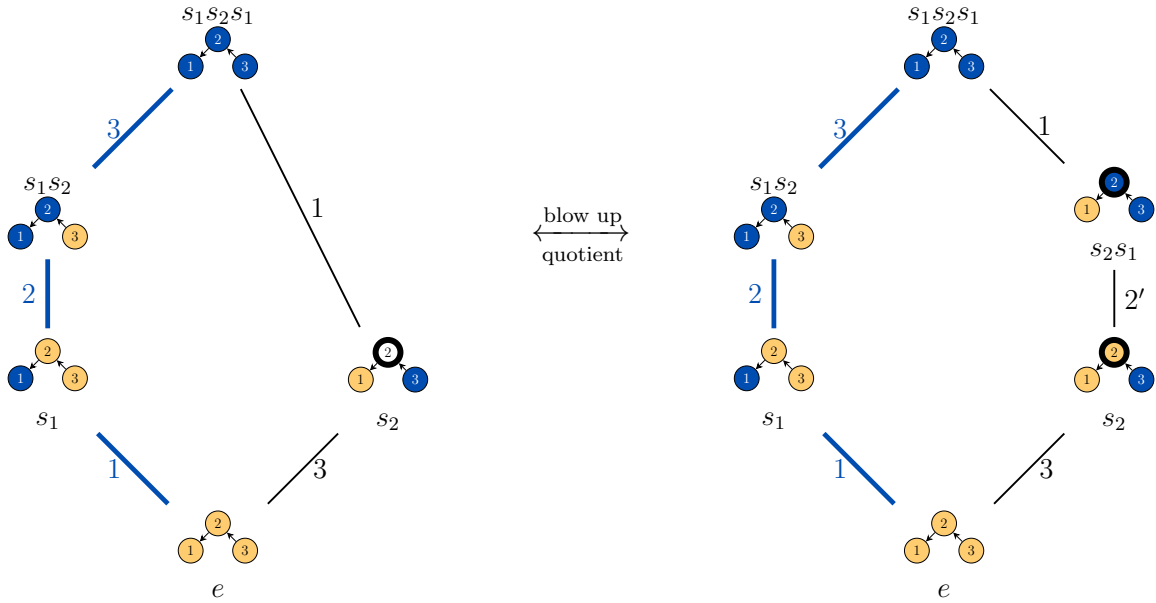



FIGURE 3. *Left* : the c -Cambrian lattice $\text{Camb}_c(A_2)$ for $c = s_1s_2$, which recovers the Tamari lattice on five elements. *Right*: weak order for the Weyl group $W(A_2)$, obtained by “blowing up” the restriction of the graph to the single vertex labelled by 2, as in Problem 3(1). This construction can also be obtained by “piecing together” the independence posets that arise from the two linear orientations of the path of length three as , as in Problem 3(2).

Chapoton and coauthors, specifically in the case of Cambrian lattices [SMHP12]). Let X be a poset and k a field. Translating the usual quiver-theoretic definitions to posets, the *incidence algebra* kX of X over k is the algebra spanned by e_{xy} for $x \leq y \in X$, with multiplication

$$e_{xy}e_{zw} = \begin{cases} e_{xw} & \text{if } y = z, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Let $D^b(X)$ denote the bounded derived category of finite dimensional (right) modules over the incidence algebra kX (which assign to each vertex of X a vector space, and to each cover relation a linear map). Two posets X and Y are called *derived equivalent* if $D^b(X)$ and $D^b(Y)$ are equivalent (as triangulated categories).

We say that $g \in G$ is *extremal* if it is a minimal or maximal element with respect to G -order. Define the *mutation* of an acyclic directed graph G at an extremal vertex g of G to be the acyclic directed graph $\text{tog}_g(G)$ obtained by reversing all edges incident to g [TW19a]. This notion is similar to the definition of BGP reflection functors arising in cluster combinatorics and quiver representation theory [BGP73], and we say that two acyclic graphs are in the same *mutation class* if they can be related by a sequence of mutations. We propose the following relationship between mutation and derived equivalence.

Problem 4. Show that if G and H are two acyclic directed graphs in the same mutation class, then the independence posets $\text{top}(G)$ and $\text{top}(H)$ are (universally) derived equivalent.

The proposed method of proof is to follow Ladkani’s *flip-flop* technology from [Lad07] (requiring a partition of $\text{top}(G)$ into two pieces X, Y , along with an order preserving function $f : X \rightarrow Y$), and then the PI’s structural recurrences on independence posets from [TW19a] along with the definition of flip in Definition 2. More precisely, writing the independence poset intervals

$$\text{top}_g(G) := [\hat{0}, (\mathcal{D}, \{g\})] \text{ and } \text{top}^g(G) := [(\{g\}, \mathcal{U}), \hat{1}],$$

if g is an extremal element of G we have the decomposition

$$\text{top}(G) = \text{top}_g(G) \sqcup \text{top}^g(G).$$

Furthermore, if g is minimal, $(\mathcal{D}, \mathcal{U}) \in \text{top}_g(G)$ if and only if $g \in \mathcal{U}$. Ladkani’s flip-flop technology seems likely to apply with only light modifications using the order-preserving map

$$\text{flip}_g : X = \text{top}_g(G) \mapsto \text{top}^g(G) = Y$$

defined by flipping at g . Generalized mutation operations on independence posets (see Problem 6) lead to generalizations of Problem 4.

2.2. Combinatorial Models and Bijections. Since distributive lattices $J(P)$ are recovered by independence posets when $G = \text{Comp}(P)$ is the comparability graph of the poset P (antichains in P become independent sets of $\text{Comp}(P)$), many classical combinatorial objects (to name a few: integer partitions in a box, various classes of plane partitions, domino tilings, stable marriages, alternating sign matrices, and minuscule lattices) can all be represented using independence posets. Placing different acyclic orientations on the comparability graph (for example, using the mutations of Section 2.1) gives new partial orders—different from the one obtained from their distributive lattice structure—on these classical objects.



The real interest is that **many objects in combinatorics can be encoded as independent sets of particular graphs**. Using the framework of independence posets, these objects can now be endowed with a wide variety of new partial orders. For example, we can now obtain new orientations on Coxeter–Catalan objects (such as trees or triangulations), including both noncrossing and nonnesting objects (generalizing Cambrian lattices), as well as on the intersection lattice of a real central hyperplane arrangement (including, for example, set partitions as the intersections of the type A braid arrangement, as in Figure 1 for the the 15 set partitions of $\{1, 2, 3, 4\}$ with 126 possible independence posets depending on the acyclic orientation chosen).

Efficient procedures for navigating these independence posets (see Section 3) allow for fast random sampling of these objects. Surprisingly, despite many surveys on independent sets, there is a dearth of attempts in the literature at compiling a list of exactly which combinatorial objects can be represented as independent sets (for a surprising example, see the two graphs just below Problem 6). The PI has written and made publically available some code for dealing with independence posets [TW20a], but the following project suggests itself as worthwhile to the community.

Problem 5. Compile and make available a detailed digital database of combinatorial objects that can be encoded as independent sets. Integrate this library into Sage for widespread use.

There are also bijective consequences of compiling such a database.

Problem 6. Generalize the mutations of [Section 2.1](#) to a wider set of “local moves” on digraphs (preserving the number of independent sets) to give a framework for producing useful new bijections between combinatorial objects.

For example, the graphs  and  have the same number of independent sets (one graph encodes the seven 3×3 alternating sign matrices, the other the corresponding set of totally symmetric self-complementary plane partitions). Useful places to search for such generalized moves from Auslander-Reiten quivers, derived categories, and the theory of heaps.

3. STATISTICAL MECHANICS AND RANDOM SAMPLING

In this section, we propose using independence posets to extend Propp and Wilson’s *coupling from the past algorithm* from distributive lattice theory to independence sets of any graph.

The random sampling of independent sets (weighted by the number of vertices in the set) is termed the *hard-core model* in statistical mechanics. Given a graph G with independent sets $I(G)$ and a *fugacity* $\lambda > 0$, define the partition function

$$P_G(\lambda) = \sum_{a \in I(G)} \lambda^{|a|}.$$

Efficiently sampling independent sets according to this measure is a well-known problem that has only been solved in certain special cases (certain types of graphs, ex. comparability graphs of posets; graphs with low maximum vertex degree; etc.)—note that it is known to be NP-complete to determine the maximum size of an independent set of a graph G , which forces this problem to be intractable in general (take $\lambda \rightarrow \infty$).

Let δ denote the maximum degree of a graph G . We briefly summarize some of the known results. It is already #P-complete to count independent sets in graphs with $\delta = 3$; Glauber dynamics has a mixing time $O(n \ln n)$ when $\lambda < 2\delta - 2$; for $\lambda = 1$ and all $\Delta \geq 6$, Dyer, Frieze and Jerrum proved there exists a bipartite graph for which the mixing time of any Markov chain making only “local moves” is exponential (but recall that our flips from [Definition 2](#) are highly *non-local*, so that this result does not apply!); Luby and Vigoda describe a Markov chain that approximately counts independent sets in graphs with $\delta \leq 4$ in polynomial time[LV97, LV99, HN98].

Propp and Wilson’s *coupling from the past* (CFTP) algorithm allows for uniform sampling without knowing the mixing time of the underlying Markov chain. In general, CFTP requires as many instances to be run as states; in practice, an additional monotonicity assumption reduces the number of concurrent running instances to just two. Applications are numerous, but the most well-known example is to distributive lattices (which allows, for example, random sampling of domino tilings of an Aztec diamond). It is known that CFTP cannot be fast in general—the simple example of a complete bipartite graph already produces a bottleneck that forces the sampling to

take exponential time—but the theoretical guarantees provided by CFTP make it an attractive method for sampling.

A trick due to Shor and Winkler encodes independent sets of *bipartite* graphs as order ideals in a corresponding distributive lattice, but no such trick is known in general: “For general (non-bipartite) graphs G there is no monotone structure which would allow one to use monotone CFTP” [LP17, 22.4]. But independence posets would seem to now provide such a structure, and it is worth at least running experiments to see if this structure provides an improvement over traditional Glauber/heat-bath dynamics.

Problem 7. Use independence posets to extend CFTP and its theoretical implications to the independent sets of any graph. Produce computational and experimental evidence for the efficacy of this method.

4. INTEGER POINTS IN POLYTOPES

In this section we discuss the potential application of using a generalization of independence posets to efficiently generate all lattice points inside convex lattice polytopes. We then give a brief treatment of the study of those lattice points from the point of view of dynamical algebraic combinatorics.

4.1. Chvátal’s perfect graph polytopes. Recall that the independence poset $\text{top}(G)$ recovers the lattice of order ideals $J(P)$ when $G = \text{Comp}(P)$ is the comparability graph of a poset P . Order ideals of a poset have a natural generalization to the theory of P -partitions [GHL⁺16], which can be interpreted as the lattice points inside of a certain polytope called the *order polytope*. In this section we propose to develop piecewise-linear generalizations of independence posets, thereby providing a generalization of the construction of P -partitions.

Given a set $X = \{x_1, \dots, x_n\}$, we write \mathbb{R}^X for the set of functions $f : X \rightarrow \mathbb{R}$. For a poset P , Stanley proved that the *chain polytope* in \mathbb{R}^P defined as the set of points $f \in \mathbb{R}^P$ satisfying the inequalities

$$0 \leq f(p) \text{ for all } p \in P \text{ and}$$

$$\sum_{i=1}^k f(p_i) \leq 1 \text{ for any chain } p_1 < \dots < p_k \text{ in } P$$

is the convex hull of the characteristic functions of antichains of P [Sta86]. In fact (as Stanley remarks), this is a special case of a beautiful construction of Chvátal [Chv75]:

Theorem 6. *Let $G = \{g_1, \dots, g_n\}$ be a perfect graph. Then the convex hull of the indicator functions of the independent sets of G matches the polytope $\mathcal{C}(G)$ defined as the set of points $f \in \mathbb{R}^G$ satisfying the inequalities*

$$0 \leq f(g) \text{ for all } g \in G \text{ and}$$

$$\sum_{g \in C} f(g) \leq 1 \text{ for any clique } C \text{ of } G.$$

Stanley’s result follows from Chvátal’s by taking $G = \text{Comp}(P)$, which is necessarily a perfect graph. (It turns out that Chvátal’s result is actually a *characterization* of perfect graphs!)

Stanley defined a second polytope called the *order polytope* $\mathfrak{O}(P)$, again using inequalities coming from the ordering on the poset, by

$$\begin{aligned} 0 \leq f(p) \leq 1 \text{ for all } p \in P \text{ and} \\ f(p) \leq f(q) \text{ if } p \leq q \text{ in } P. \end{aligned}$$

Stanley gives a simple piecewise-linear transfer map between $\mathfrak{O}(P)$ and $\mathfrak{C}(P)$, which restricts to a bijection on integer points in their dilations. On the combinatorial side again, the number of lattice points inside the m -fold dilation $m\mathfrak{O}(P)$ (counted by the *Ehrhart polynomial* for $\mathfrak{O}(P)$) is given by the number of multichains of order ideals $\emptyset = I_0 \subseteq I_1 \subseteq \dots \subseteq I_{m+1} = P$ in $J(P)$, or equivalently by $J(P \times [m])$. It is therefore of interest to determine a *combinatorial* generalization of what an independence poset “ $\text{top}(G \times [m])$ ” might look like, with the idea that when G is a perfect graph then the integer points in the dilation $m\mathfrak{C}(G)$ agrees with the number of vertices of the generalized independence set (note that it is not immediately clear what an “independent set” ought to mean in this context). The PI has recently constructed what appears to be the correct generalization (having checked many examples by computer), and the task remains to prove these results.

Namely, given $\mathcal{D} \in \mathbb{R}^G$, we define piecewise-linear *toggle* operators

$$\begin{aligned} \text{tog}_g^{(m)} : \mathbb{R}^G &\rightarrow \mathbb{R}^G \\ \mathcal{D}(x) &\mapsto \begin{cases} \mathcal{D}(x) & \text{if } x \neq g \\ m - \max_{\substack{C \text{ a clique} \\ g \in C}} \sum_{h \in C} \mathcal{D}(h) & \text{otherwise} \end{cases} \end{aligned}$$

We now define *rowmotion* to be the operator

$$\begin{aligned} \text{row}^{(m)} : \mathbb{R}^G &\rightarrow \mathbb{R}^G \\ \mathcal{D} &\mapsto \prod_{g \in G} \text{tog}_g^{(m)}(\mathcal{D}), \end{aligned}$$

where the product in the definition is in G -order (this matches the combinatorial and piecewise-linear definitions of rowmotion for $G = \text{Comp}(P)$ [SW12, EP13, Jos19, JR20]). Finally, the elements of $\text{top}^{(m)}(G)$ will be pairs $(\mathcal{D}, \mathcal{U})$ with $\mathcal{U} = \text{row}^{(m)}(\mathcal{D})$, generated starting from $(\emptyset, \text{row}^{(m)}(\emptyset))$ using the piece-wise generalization of *flip*: subtract 1 from $f^\top(g)$ (if possible) add one to $f(g)$, and fill in the remainder of f above and below g as in the combinatorial flip definition (but using the new piecewise-linear toggles).

Problem 8. Show that this combinatorial construction recovers all integer points in Chvátal’s polytope $\mathfrak{C}(G)$ (defined by inequalities on the cliques).

As a special case when $G = \text{Comp}(P)$, this recovers Stanley’s theory of order polytopes and P -partitions.

Problem 9. Generalize central properties of P -partitions and Stanley’s chain polytopes to independence posets and $\mathfrak{C}(G)$. For example, a triangulation of $\mathfrak{C}(G)$ should

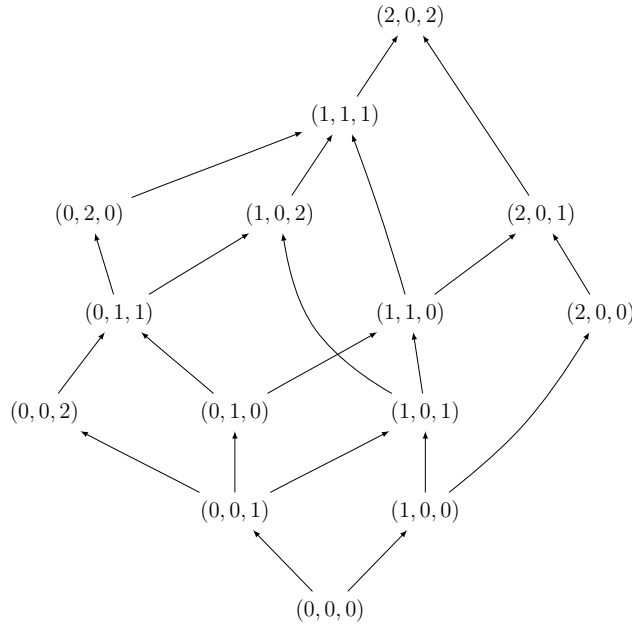


FIGURE 4. The 14 integer points in $2\mathcal{C}(G)$ for G the directed graph $1 \rightarrow 2 \rightarrow 3$, arranged in generalized independent set order (generated using flips from the point $(0,0,0)$).

suggest a notion of “linear extensions” for independence posets, and then ought to give a formula for the Ehrhart generating function of $\mathcal{C}(G)$.

A corollary to the solution of [Problem 8](#) is that flips give what ought to be a particularly efficient way to generate all lattice points inside $m\mathcal{C}(G)$. This suggests that, given a (positive) lattice polytope

$$P = \{x \in \mathbb{R}^d : \langle c_i, x \rangle \leq a_i \text{ for } i = 1, \dots, m\} \text{ with } c_i \in \mathbb{N}^d \text{ and } a_i \in \mathbb{N}$$

we can further generalize the construction to place constraints *separately* on each clique, thereby generalizing independence posets to counting lattice points in *any* convex lattice polytope.

- Problem 10.**
- Show that the polytopal interpretation of independence posets gives an efficient algorithm to generate the lattice points in $m\mathcal{C}(G)$.
 - Generalize the definition of flip to accommodate separate conditions on each clique, and show that the analogue of [Problem 8](#) holds.
 - Show that this gives an explicit and efficient algorithm to generate lattice points in *any* convex lattice polytope.

One can also produce m -eralizations of trim lattices arising from representation theory using derived categories, as we did in [[STW15](#)] for Cambrian lattices. It would be interesting to study generalizations at the level of independence posets.

Problem 11. Extend the construction of independence posets using representation-theoretic m -eralizations as a guide.

It would also be interesting to extend the construction of independence posets to (proper) *colorings*.

5. DYNAMICAL ALGEBRAIC COMBINATORICS

The PI's generation of the *rowmotion* map on independent sets in [TW19a] is given by the map $\mathcal{F} \mapsto \mathcal{F}'$ when $(\mathcal{F}, \mathcal{F}')$ is the unique top with \mathcal{F} as its left set. This recovers other, more classical, notions on distributive lattices. In fact, given two different acyclic orientations on the same underlying graph G, G' , we obtain an interesting generalization of this bijection on independent sets: if $(\mathcal{F}, \mathcal{U}) \in \text{top}(G)$ and $(\mathcal{F}, \mathcal{U}') \in \text{top}(G')$, then we obtain the map on independent sets $\mathcal{F} \mapsto \mathcal{U}'$. In the special case when $G = G'$ are the same orientation, this recovers the above rowmotion.

Problem 12. Find examples of directed graphs for which the generalized rowmotion operator has interesting properties (from the point of view of dynamical algebraic combinatorics: simple order, homomesy, etc.).

Section 4.1 gives us the generalization of independence posets to the piecewise linear realm, and so defines a rowmotion operator (we can further define birational rowmotion by modifying the toggle operation above). Also related is the study of a linear operator coming from the Grothendieck group $K_0(\text{mod}P)$ (see, for example, the recent work in [Yil19] for the case of cominuscule posets).

Problem 13. Study (piecewise-linear/birational) rowmotion and the Coxeter transformation $\theta = -C(C^{-1})^\top$ for independence posets, where $C = [C_{xy}]_{x,y \in \text{top}(G)}$ is the incidence matrix of $\text{top}(G)$ defined by $C_{xy} = 1$ if $x \leq y$ and 0 otherwise.

In general, we expect that directed graphs coming from representation theory ought to have interesting behavior under these actions, including possible connections with cluster algebras, R -systems, and various generalizations of periodicity.

6. PRIOR SUPPORT: NOT APPLICABLE

The PI has not held a personal NSF grant before.

7. INTELLECTUAL MERIT

The PI's research is in algebraic combinatorics, with a broad interest in motivation from other areas of mathematics such as Lie theory, geometric group theory, and reflection groups. The PI has a strong record of solving long-standing problems using an original toolkit and perspective: he has been selected to give six talks at FPSAC and will be an invited speaker at the 2020 Triangle Lectures in Combinatorics as well as Open Problems in Algebraic Combinatorics 2021 at the University of Minnesota.

The PI's related work in [SW12] has served as a catalyst for the involvement of undergraduate and beginning graduate students in cutting-edge research at REUs and doctoral programs. There have been many developments motivated by the appearance of [SW12]—to name a few: [CHHM15, EP13, EFG⁺15, Had14, Hop16, GR14, GR15, GR16, PR15, Rob16, RS13, RW15, Rus16, DPS17, Str15, Str16, JR18, MR19, DSV19, Jos19, JR20, Hop20, JR20]. In 2015, the PI, Striker, Propp, and Roby organized an AIM workshop that launched a new field of combinatorics now termed “Dynamical Algebraic Combinatorics.” This same group is organizing a follow-up BIRS online workshop for Fall 2020 (originally accepted in-person, but delayed due to COVID-19; the PI is preparing his current undergraduate honors reading class to attend this

workshop). The PI has additionally organized several successful AMS and JMM special sessions in this field. An integral part of this proposal is to continue supporting the PI's ongoing and future efforts to involve students in cutting-edge research in algebraic combinatorics and related areas.

The PI has already laid some of the theoretical groundwork underpinning this proposal in the two recent publications [TW19b, TW19a]. Over the course of this previous research, the PI has developed an original toolkit and perspective that has yielded substantial new progress in related fields. Based on this new perspective, the PI has created an interconnected library of concrete combinatorial problems especially suitable for early-stage students.

8. BROADER IMPACTS

The PI has substantial past experience in involving students and underrepresented students in research: he has mentored undergraduate research over six different summers (at UTD, LaCIM, and UMN), supervised three honors theses at UTD, and he currently has two Ph.D. students (Amit Kaushal and Priyojit Palit) pursuing their thesis research in areas related to this proposal. The PI's related work in [SW12] has served as a catalyst for the involvement of undergraduate and beginning graduate students in cutting-edge research at REUs and doctoral programs. There have been many developments motivated by the appearance of [SW12]—to name a few: [CHHM15, EP13, EFG⁺15, Had14, Hop16, GR14, GR15, GR16, PR15, Rob16, RS13, RW15, Rus16, DPS17, Str15, Str16, JR18, MR19, DSV19, Jos19, JR20, Hop20, JR20]. In 2015, the PI, Striker, Propp, and Roby organized an AIM workshop that launched a new field of combinatorics now termed “Dynamical Algebraic Combinatorics.” This same group is organizing a follow-up BIRS online workshop for Fall 2020 (originally accepted in-person, but delayed due to COVID-19; the PI is preparing his current undergraduate honors reading class to attend this workshop). The PI has additionally organized several successful AMS and JMM special sessions in this field. An integral part of this proposal is to continue supporting the PI's ongoing and future efforts to involve students in cutting-edge research in algebraic combinatorics and related areas.

As the only combinatorialist at UTD, the PI has designed new undergraduate and graduate courses in combinatorics; due to the success of his undergraduate Discrete Math and Combinatorics class, the PI was asked by the honors college to teach honors reading courses in Fall 2019 and Fall 2020. When the PI's BIRS workshop on “Dynamical Algebraic Combinatorics” was moved online for Fall 2020 due to COVID-19 (the in-person workshop was postponed to 2021), the PI decided to focus his Fall 2020 undergraduate honors reading course around preparing the students to participate in this two-week online BIRS conference. **The PI intends to use his past experience in conference organization to set up a yearly online workshop, with the goal of bringing together early graduate and undergraduate students (including the honors students in his current reading course, as well as Ph.D. student of the PI's collaborators).**

The PI has a history of service to the combinatorial community: he has refereed for over twenty journals, became an editor for *Annals of Combinatorics* in 2019, served on the program committee of FPSAC in 2019 (and is grant coordinator for

2021–2023), and has organized many conferences, workshops, and special sessions. He has represented the larger mathematical community to the public by appearing as a mathematical consultant in a 2018 nationally televised report (WFAA) regarding the NCAA basketball bracket, and hosting mathematical events at UTD (such as setting up a freshman orientation table for potential math majors, and serving as a faculty speaker at a MATHCOUNTS competition).

The PI has a record of producing problems and research areas accessible to beginning researchers, including the now-active area of dynamical algebraic combinatorics. At least three of the PI’s papers have independently led to Research Experience for Undergraduates (REU) projects at three different institutions.

8.1. Mentoring and REUs. The PI has substantial past experience in involving students and underrepresented students in research: this past Spring 2020, the PI supervised two undergraduate honors theses (one already submitted for publication, the other nearing completion), and this past Summer 2020, the PI supervised two graduate students on a research project. He currently has two Ph.D. students (Amit Kaushal and Priyojit Palit) pursuing their thesis research in areas related to this proposal. The PI will continue to seek out such opportunities with the goal to eventually build a strong combinatorics program at UT Dallas, include applying for REU funding.

While at UT Dallas the PI has worked with **graduate students** in the following ways:

- Currently the **thesis advisor** of Amit Kaushal (since Fall 2018);
- Currently the **thesis advisor** of Priyojit Palit (since Spring 2019);
- Organized the 2018 **Graduate Student Combinatorics Conference**;
- Supervised several **independent study** courses with graduate students (Fall 2017, Spring 2019, Summer 2020).
- Organized a **research group** with graduate students Charlie Preston and Erin Pierce on the problem of which type A crystals are lattices (Summer 2020; end result was a conjectured complete classification).

While at UT Dallas the PI has worked with **undergraduates** in the following ways:

- Supervised Kevin Zimmer’s **senior honors thesis** in Spring 2018;
- Mentored rising senior Robert Hubbard for eight weeks in Summer 2018 as part of the **Pioneer REU program** (he has since begun his Ph.D. studies at UNC Chapel Hill);
- Supervised **independent research** with junior Joshua Marsh in Spring 2019;
- Supervised **independent research** with undergraduates Christian Kondor and Michelle Patten in the Spring and Summer 2019;
- Due to the success of the Discrete Math and Combinatorics course I designed for the new BS in Data Science program, I was asked by the honors college to teach a **reading course** in Fall 2019 for 10 students. I was asked to teach an honors reading course again in Fall 2020 for 15 students.
- Supervised Joshua Marsh’s **senior honors thesis** in Spring 2020 (he has since begun his Ph.D. studies at GA tech; will submit for publication);
- Supervised Benjamin Cotton’s **senior honors thesis** in Spring 2020 (submitted for publication);

- In Fall 2020, preparing an **honors reading class** to take part in the BIRS Dynamical Algebraic Combinatorics conference.

Further past experience involving undergraduate students in research includes:

- In 2016, co-mentored Florence Maas-Gariepy on a research/study project involving finite reflection groups, which led to her project report (in French) being featured on the funding agency’s website [MG16].
- In 2014, mentored Stephanie Schanack, Fatiha Djermane, and Sarah Ouahib on an original research problem involving the characterization of the fixed points of a certain combinatorial set under a cyclic group action. Guided them through a case-by-case analyses which the three wrote up (in French) [SSD14].
- At the 2011 University of Minnesota REU, provided support to David B Rush and XiaoLin Shi [RS13], who found a generalization of the PI’s work in [SW12].
- For the 2010 Minnesota REU, helped direct Gaku Liu’s research in partition identities [Liu] and helped a second group formulate and computationally test conjectures on a combinatorial reformulation of the four-color theorem [CSS14].

8.2. Conferences and Workshops Organized. The PI has also been very active in **organizing conferences and workshops**:

- Organized a week-long workshop at the American Institute of Mathematics;
- Organized the 2018 Graduate Student Combinatorics Conference at UT Dallas, with over 75 attendees (also obtaining \$20,000 of NSF funding);
- Served on the program committee for FPSAC in 2019.
- Organized four AMS special sessions. With UTD colleague M. Arnold, organized a special session in Hawaii in 2019, and another special session at the 2020 Joint Mathematical Meetings in Denver, both relating to the interactions between dynamical systems and combinatorics;
- In October 2018, organized a two-week “research-in-pairs” program at Oberwolfach, resulting in a 132-page preprint;
- Organized two minisymposia on “Coinvariant Spaces and Parking Functions” for the 2019 SIAM Texas Louisiana Section at Southern Methodist University under the meta-organization of F. Sottile;
- Organizing two-week BIRS workshop with J. Propp, T. Roby, and J. Striker on “Dynamical Algebraic Combinatorics” will take place online in Fall 2020 (with in-person delayed to 2021); and
- Organizing FPSAC’s NSF and NSA funding proposals (2021–2023).

8.3. Referee Activities. As a member of the **mathematical community**, the PI became an **editor** for Annals of Combinatorics in 2019. The PI has **refereed** for over twenty journals, including Proceedings of the American Mathematical Society, Selecta Mathematica, and Transactions of the AMS, as well as for undergraduate journals. The PI has refereed a few mathematical grants, including for the state of Texas (ConTex) and for the French National Research Agency.

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