STATEMENT OF RECENT WORK

I have chosen to highlight three of my results in three different areas of algebraic combinatorics: K-theoretic Schubert Calculus, diagonal coinvariants, and Coxeter-Catalan combinatorics.

Bijective methods in K-theoretic Schubert Calculus. In 1983, R. Proctor exploited the branching rule from the Lie algebra inclusion $\mathfrak{sp}_{2n}(\mathbb{C}) \hookrightarrow \mathfrak{sl}_{2n}(\mathbb{C})$ to prove the combinatorial identity that there are the same number of plane partitions of heights at most k of rectangular shape and of shifted trapezoidal shape [Pro83]. R. Proctor remarks that "the question of a combinatorial correspondence. . . seems to be a complete mystery." Indeed, the state of the art for over 30 years was limited for the case $k \leq 2$: for $k \leq 1$, J. Stembridge produced a jeu-de-taquin bijection [Ste86] and V. Reiner gave an argument using centrally-symmetric noncrossing partitions [Rei97], while S. Elizalde used the language of lattice paths to describe a bijection for $k \leq 2$ [Eli15]. In [HPPW18], we found a bijection for all k, synthesizing M. Haiman's rectification, a remark about E_7 by R. Proctor, and minuscule K-theoretic Schubert calculus techniques introduced by A. Yong and H. Thomas [Hai92, TY09, BS14].

Theorem 1 ([HPPW18])**.** *There is a bijection using K-theoretic jeu-de-taquin between plane partitions of heights at most k of rectangular shape and of shifted trapezoidal shape.*

Our results are substantially more general, placing this specific problem into the robust framework of minuscule K-theoretic Schubert calculus. For G a semisimple complex Lie group and P a parabolic subgroup such that G/P is a minuscule variety, we prove the equivalence of a product in the Grothendieck ring $K(G/P)$ of algebraic vector bundles over G/P with a bijection between two sets of certain tableaux. Other choices of G/P give similar theorems of the same flavor as Theorem 1.

Our arguments are usefully interpreted as statements about rational equivalence of certain generalized Schubert and Richardson subvarieties of minuscule flag varieties—each of the bijections we obtain corresponds to the fact that a certain Richardson variety represents the same element of the Chow ring as a certain Schubert variety. Our techniques yield a uniform way to construct bijections using multiplicity-free expansions in $K(G/P)$, and we are working on several further applications. It would be especially fruitful to return to R. Proctor's original Lie-theoretic explanation of the original rectangle/trapezoid identity using Littelmann's path model [Lit95, NS05].

Diagonal Coinvariants and the Sweep Map. It is a classical result that the Hilbert series for the space of coinvariants of a Weyl group W may be written as a generating function over elements of W. Motivated by the rich combinatorics for coinvariants, Garsia and Haiman introduced the space of *diagonal coinvariants* [GH96] as the quotient $\mathcal{DH}_n := \mathbb{C}[\mathbf{x}, \mathbf{y}]/\mathbb{C}[\mathbf{x}, \mathbf{y}]_+^{\mathfrak{S}_n}$, where $\mathbb{C}[\mathbf{x}, \mathbf{y}] := \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ and $\mathbb{C}[\mathbf{x}, \mathbf{y}]_+^{\mathfrak{S}_n}$ is the ideal of $\mathfrak{S}_n$-invariant polynomials with no constant term ($\mathfrak{S}_n$ acts diagonally). It turns out that the Hilbert series of the alternating subspace $\mathcal{DH}_n^\epsilon$ of the space of diagonal coinvariants may be expressed as a generating function over certain lattice paths, using a bijection called the *zeta map* [GH02, Hai02, Hag03, GH02].

In more generality, let $\mathcal{D}_{a,b}$ be the set of lattice paths from $(0, 0)$ to (b, a) that stay above the main diagonal. Armstrong, Loehr, and Warrington's *sweep map* is a general form of the zeta map that sends $\mathcal{D}_{a,b}$ to itself by rearranging the steps of a path according to the order in which they are encountered by a line of slope a/b "sweeping" down from above [ALW15] (see Figure 1). The reason for working at this level of generality is that the Borel-Moore homology of affine type A Springer fibers leads to further conjectural expressions of Hilbert series using the lattice paths $\mathcal{D}_{a,b}$ and the sweep map [Hik14]; these same series appear in the study of the triply graded Khovanov-Rozansky homology of (a, b)-torus knots and links [Gor12, EH16]. Proving invertibility of the sweep map (and hence the conjectured Hilbert series) was known as a notoriously difficult problem [ALW15, Xin15, CDH16, GMV16, Thi16, GMV17]. In [TW18], we succeeded in proving that a very general form of the sweep map was invertible.

Theorem 2 ([TW18]). *For $a, b \in \mathbb{N}$, the sweep map is a bijection on the set of lattice paths $\mathcal{D}_{a,b}$.*

Our theorem has already inspired several related papers [GX16a, GX16b], and has found applications in studying the irreducible components of minuscule affine Deligne-Lusztig varieties.

Turning to the full Hilbert series, write $[a] = \{0, 1, \ldots, a-1\}$. For $a, b \in \mathbb{N}$, the *(a, b)-parking functions* $\mathcal{P}_a^b$ are those words $\mathsf{p} = \mathsf{p}_0 \cdots \mathsf{p}_{b-1} \in [a]^b$ such that $|\{j : \mathsf{p}_j < i\}| \geq \frac{ib}{a}$ for $1 \leq i \leq a$. $\mathcal{P}_a^b$ can be interpreted as a *labeled* version of $\mathcal{D}_a^b$—just as the Hilbert series of the alternating subspace $\mathcal{DH}_n^\epsilon$ (for $a-b=1$) may be written as a generating function for $\mathcal{D}_a^b$ using the sweep map, the full Hilbert series of $\mathcal{DH}_n$ is encoded by $\mathcal{P}_a^b$. Recently, we inverted a zeta map on $\mathcal{P}_a^b$ defined by Gorsky, Mazin, and Vazirani [GMV16] using the following construction. Let V^a be defined as $\mathbb{R}^a$ up to permutation of coordinates and addition of multiples of the all-ones vector. We define an action of a letter $i \in [a]$ on points in V^a by adding a to the ith smallest coordinate; a word $\mathsf{w} \in [a]^b$ then acts by its letters from left to right. The following result extends our results from [TW18] from $\mathcal{D}_a^b$ to $\mathcal{P}_a^b$, resolving several open conjectures.

Theorem 3 ([MTW]). *The action of $\mathsf{w} \in [a]^b$ on V^a has a fixed point iff $\mathsf{w} \in \mathcal{P}_a^b$.*

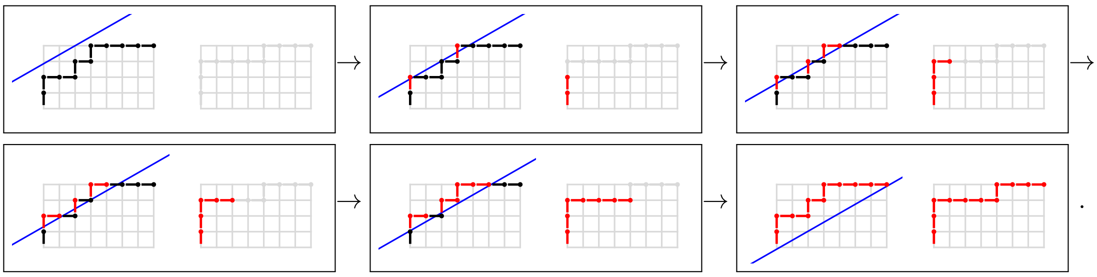

FIGURE 1. An illustration of the geometric interpretation of sweep. To form the right path, the steps of the left path are rearranged according to the order in which they are encountered by a line of slope 4/7 sweeping down from above.

Coxeter-Catalan Combinatorics. In Coxeter-Catalan combinatorics, the usual Catalan numbers are associated to the symmetric group, and count noncrossing partitions, triangulations of a convex $(n+2)$-gon and 231-avoiding permutations. These three Catalan objects beautifully generalize to all other finite Coxeter groups W: noncrossing partitions are interpreted as an interval in the absolute order of W, triangulations become clusters in a finite-type cluster algebras, and 231-avoiding permutations are generalized to Reading's sortable elements [Rea07a, Rea07b].

Our perspective in [STW15] is that the correct setting for extending Coxeter-Catalan combinatorics to the Fuss–Catalan level of generality is provided by the *Artin monoid* $B^+(W)$. This setting allows us to not only give a uniform treatment of previous work on Fuss–generalizations of noncrossing partitions and clusters [Arm09, FR05], but also to find the missing Fuss–generalization of sortable elements.

Definition-Theorem 4 ([STW15]). *The Fuss c-sortable elements for a finite Coxeter group W are a certain subset of elements in the positive Artin monoid interval $[e, w_\circ^m]$, where $w_\circ$ is the Garside element. Their cardinality is $\mathsf{Cat}^{(m)}(W) := \prod_{i=1}^n \frac{mh+d_i}{d_i}$, where h is the Coxeter number of W, and $d_1 \leq d_2 \leq \cdots \leq d_n$ are the degrees of W.*

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