Noncrossing Partitions

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Abstract. We characterize the reflection orderings of finite Coxeter groups that give EL-shelling orders of their noncrossing partition lattices.

Coxeter Groups

Let $S = \{s_1, s_2, \ldots, s_n\}$. A *Coxeter system* (W, S) is a group W with a presentation of the form

$$W = \langle S : (s_i s_j)^{m_{ij}} = e \rangle$$

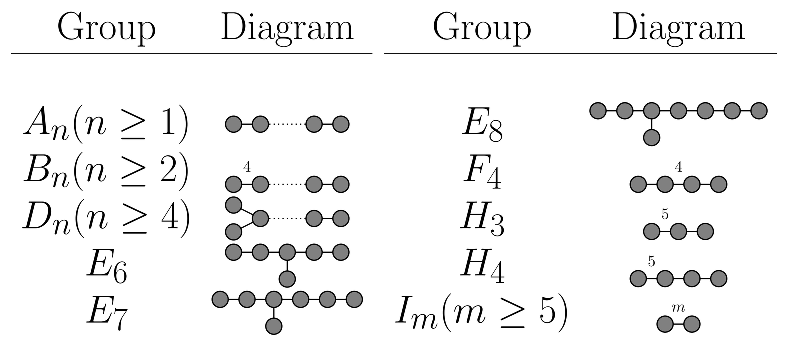
with $m_{ii} = 1$ and $m_{ij} \geq 2$ for $i \neq j$. We call the generators S the *simple reflections*, since any Coxeter group has a representation on $\mathbb{R}^n$ in which the simple reflections act as reflections in hyperplanes. The set of *reflections* of W

$$T = \{wsw^{-1} | s \in S, w \in W\}$$

is the set of all conjugates of the simple reflections.

Coxeter Diagrams

We represent the presentation of a Coxeter system using a *Coxeter diagram*, a graph with one vertex v_s for each generator $s \in S$. If $m_{ij} = 3$, we connect v_i and v_j with an unlabeled edge, and we label the edge m_{ij} if $m_{ij} \geq 4$. The finite irreducible Coxeter groups have been fully classified, and their Coxeter diagrams are illustrated below.



Partially Ordered Sets

A *poset* is a pair $(P, \prec)$ where P is a set and $\prec$ is a reflexive, antisymmetric, and transitive relation on P. An *m-chain* is a set of strictly increasing elements $x_1 < x_2 < \cdots < x_{m+1}$ in a poset P, or a totally ordered subset of P. A *lattice* is a poset in which any pair of elements have a (unique) greatest lower bound and a (unique) least upper bound.

Noncrossing Partition Lattices

A *standard Coxeter element* c is a product of the simple reflections in any order, each occurring once—in other words, an element of the form $c = s_{\sigma(1)} s_{\sigma(2)} \cdots s_{\sigma(n)}$, where σ is a permutation of $\{1, 2, \ldots, n\}$.

The *absolute length* $\ell_T : W \to \mathbb{Z}$ is the minimal length of a word for $w \in W$ as a product of reflections T. Absolute length induces a partial order $\leq_T$ on W by

$$\pi \leq_T \mu \quad \Leftrightarrow \quad \ell_T(\mu) = \ell_T(\pi) + \ell_T(\pi^{-1}\mu) \forall \pi, \mu \in W.$$

For c a Coxeter element, we define the *noncrossing partition lattice* to be the interval

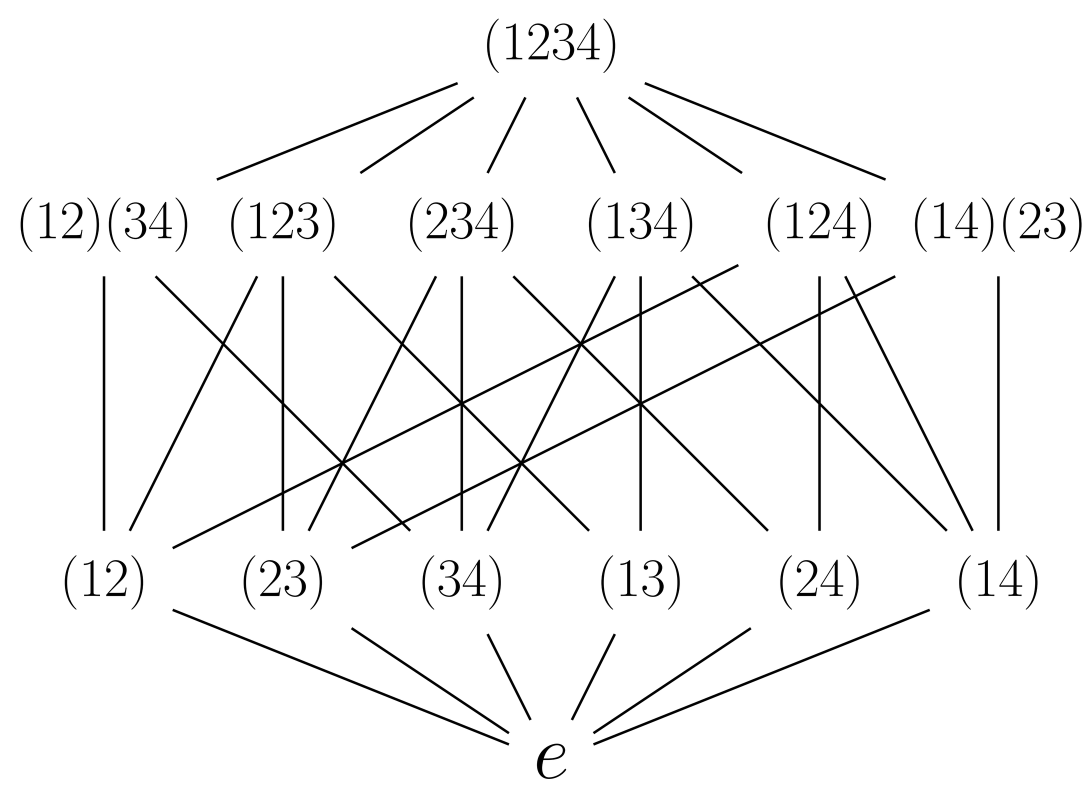
$$\mathrm{NC}(W, c) := \{w \in W : w \leq_T c\}.$$

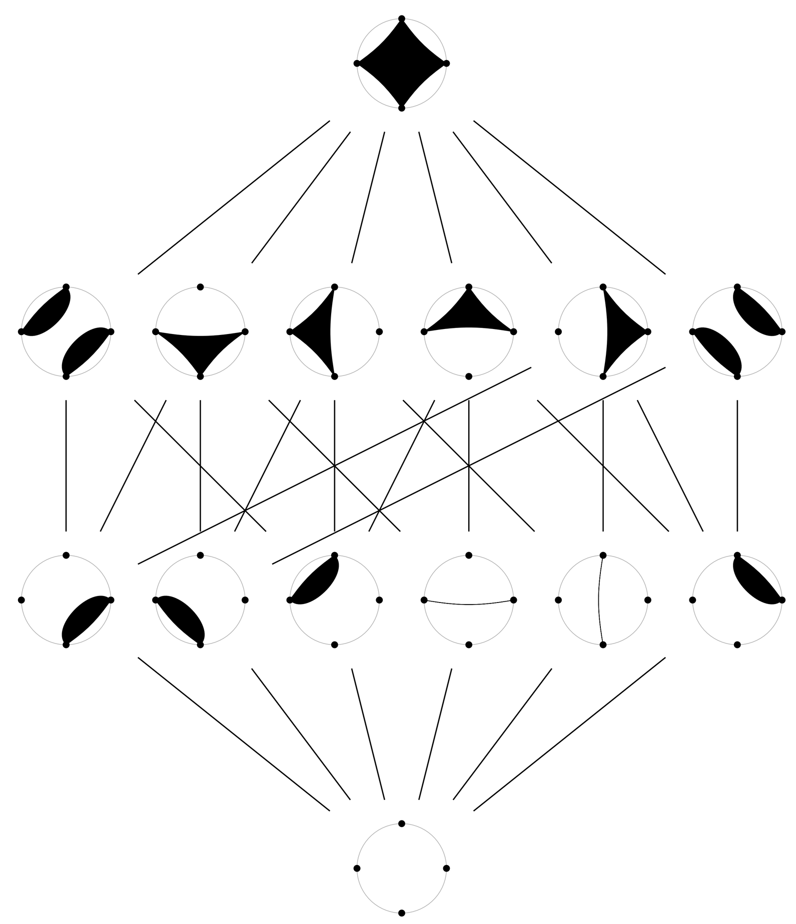
Since any two Coxeter elements c, c' are conjugate in W, $\mathrm{NC}(W, c) \simeq \mathrm{NC}(W, c')$.

Theorem 1. $\mathrm{NC}(W, c)$ *is a lattice.*

Example: the Symmetric Group

The *symmetric group* S_n is the set of all permutations of $\{1, 2, \ldots, n\}$ with group operation given by function composition. The simple reflections are the adjacent transpositions $S = \{(i, i+1) | 1 \leq i < n\}$, while the reflections are all transpositions $T = \{(ij) | 1 \leq i < j \leq n\}$. The poset $\mathrm{NC}(S_4, (1234))$ is drawn below; it is isomorphic to the lattice of noncrossing set partitions ordered by refinement.





Reflection Sequences

A word in simple reflections can be thought of as a path in the hyperplane arrangement corresponding to the reflections of W—at each step, exactly one hyperplane is crossed. A word is *reduced* if each hyperplane is crossed at most once. The *long element* $w_\circ$ is the unique element whose reduced words cross every hyperplane—such reduced words impose a total ordering on the hyperplanes, and hence reflections T.

Definition 2. The *c-sorting word* for $w_\circ$ is the lexicographically first subword of $c^\infty = (s_1 s_2 \cdots s_n)^\infty$ that is a reduced expression for $w_\circ$.

Simplicial Complexes

An *abstract simplicial complex* Δ on a set of vertices V is a finite collection of subsets such that

- $\{v\} \in \Delta \; \forall v \in V$
- $G \in \Delta$ and $F \subseteq G \Rightarrow F \in \Delta$

The elements of Δ are called *faces*, or simplices, of the simplicial complex and the maximal elements of Δ are called *facets*. We say that a face F has dimension d if $d = |F| - 1$. We refer to faces of dimension d as $d-$faces and write $\dim(F) = d$. A simplicial complex is considered *pure* of dimension d when all of its facets are d-dimensional.

Definition 3. A simplicial complex is *shellable* if its facets can be arranged in a linear order $F_1, F_2, ..., F_t$ such that the subcomplex $(\bigcup_{i=1}^{k-1} \langle F_i \rangle) \cap \langle F_k \rangle$ is pure and has dimension $\dim F_k - 1$ $\forall k = 2, ..., t$. (Such an ordering on the facets is called a *shelling*.)

Poset Topology

The *order complex* of a poset P is the simplicial complex $\Delta(P) := \{\text{chains in } P\}$. For each face in Δ, let $\langle F \rangle$ be the subcomplex $\langle F \rangle = \{G | G \subseteq F\}$. The facets of $\Delta(P)$ correspond to the maximal chains of P. A poset P is pure $\Leftrightarrow \Delta(P)$ is pure.
An *edge labelling* can be defined as a map $\lambda : E(P) \to \Lambda$, where $E(P)$ is the set of edges of the Hasse diagram of P and Λ is some poset.

Definition 4. A poset is *edge-lexicographic shellable* (EL-shellable) if

- it has an edge labelling by a totally ordered set
- every interval has a unique increasing maximal chain
- this maximal chain is lexicographically first among all other maximal chains

The lexicographic order of the maximal chains of an EL-shellable poset P is a shelling of $\Delta(P)$.

EL-Shellability of Noncrossing Partition Lattices

For $J \subseteq S$, $W_J := \langle J \rangle$ is the *standard parabolic subgroup* generated by J. A *parabolic subgroup* is any conjugate of a standard parabolic subgroup. A *rank 2 parabolic subgroup* is a subgroup generated by two reflections.

Definition 5. We say a total order $t_1 < t_2 < \cdots < t_n$ of T is *c-aligned* if, for any rank 2 parabolic subgroup, the restriction to the reflections of the rank 2 parabolic subgroup is a cyclic rotation of the ordering given by $w_\circ(c)$.

Theorem 6. *A total order on T is an EL-shelling order for* $\mathrm{NC}(W, c) \Leftrightarrow$ *it is c-aligned.*

The proof is by induction on rank.

References

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