OBERWOLFACH PROBLEM SESSION: ENUMERATIVE COMBINATORICS 2022

PREPARED BY NATHAN WILLIAMS

1. Andrew Elvey Price: 3/4 plane walks from $(2,0)$ to $(-1,0)$

Definition 1.1. A *small step* in $\mathbb{Z}^2$ is a step of the form

$$\{(1,0),(-1,0),(0,1),(0,-1),(1,1),(1,-1),(-1,1),(-1,-1)\}.$$

Fix a multiset $\widehat{S}$ of small steps, and consider all walks $W(\widehat{S})$ from $(2,0)$ to $(-1,0)$ using the collection of steps in $\widehat{S}$ that avoid the negative x and y axes, except at the final point of the path (drawn in red in Figure 1). Let $X(\widehat{S})$ be the subset of walks in $W(\widehat{S})$ that do not touch the line segment $\{(1,y) : y \leq 0\}$(drawn in green in Figure 1), and let $Y(\widehat{S}) = W(\widehat{S}) \setminus X(\widehat{S})$. An example path is drawn in blue in Figure 1.

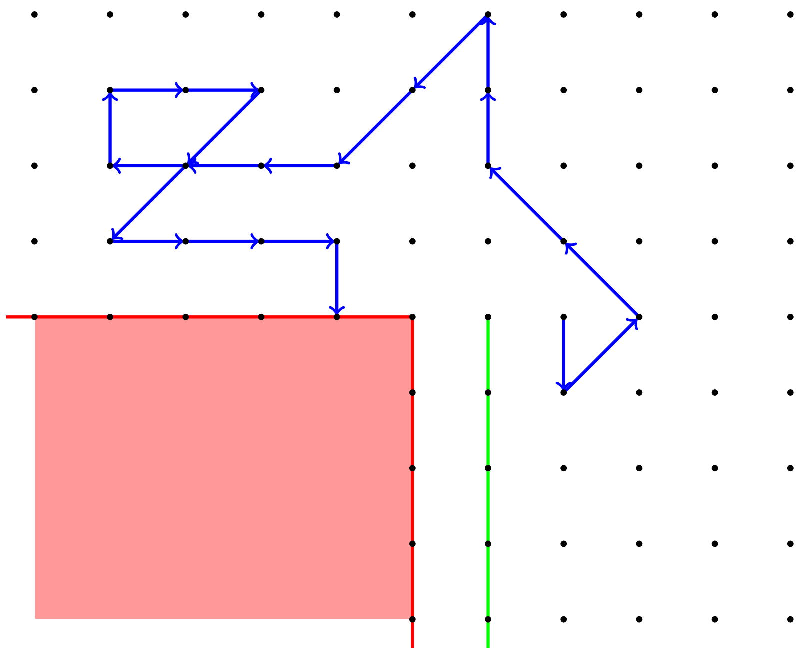


FIGURE 1. A walk in $X(\widehat{S})$ for $\widehat{S} = \{\nearrow, 2\times\downarrow, 2\times\nwarrow, 3\times\uparrow, 3\times\leftarrow, 4\times\swarrow, 5\times\rightarrow\}$.

Theorem 1.2 (A. Elvey Price)**.** For any multiset $\widehat{S}$ of small steps, $|X(\widehat{S})| = |Y(\widehat{S})|$.

The existing proof of this surprising fact uses elliptic functions.

Problem 1.3. Find a bijection between $X(\widehat{S})$ and $Y(\widehat{S})$.

2. Nathan Williams: Parking functions via Deodhar subwords

Taking all indices modulo $n+1$, the affine symmetric group has the familiar presentation

$$\widetilde{S}_{n+1} = \left\langle s_0, s_1, \ldots, s_n : s_i^2 = (s_i s_{i+1})^3 = (s_i s_j)^2 = e \right\rangle,$$

where $i \neq j$ and $i \neq j \pm 1$.

Theorem 2.1 (P. Galashin, T. Lam, N. Williams)**.** Let P_n be the set of subwords of $(s_0, s_1, \ldots, s_n)^n$ of length $n(n-1)$ whose product is the identity and whose consecutive products decrease in weak order whenever possible. Then $|P_n| = (n+1)^{n-1}$.

The interesting condition on consecutive products comes from the Deodhar decomposition of braid varieties. The 4^2 subwords in P_3 are given in Figure 2. The proof uses a trace formula for the affine Hecke algebra due to Opdam and a Tessler matrix identity due to Haglund. (See [GLTW22] for some related problems.)

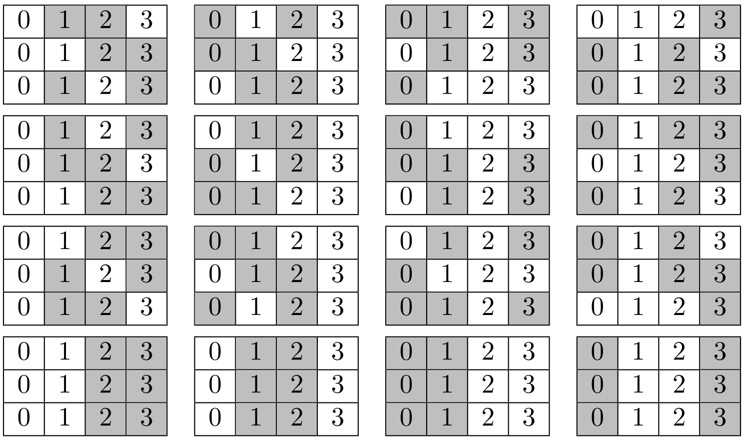


FIGURE 2. The letters in each of the 4^2 subwords in P_3 are indicated by a gray background. For example, the subword in the top left should be read as $(e, s_1, s_2, e, e, e, s_2, s_3, e, s_1, e, s_3)$.

Problem 2.2. Find a bijection between P_n and parking functions.

It might be useful to aim for noncrossing parking functions [ARR15]—there is some progress using a refinement coming from the "root configuration" of a subword.

3. Jay Pantone: 321-avoiding permutations obeying parity

We make the following two definitions regarding the one-line notation of a permutation.

Definition 3.1. A permutation π *contains the pattern* 321 if it has three (not-necessarily-consecutive in position or value) decreasing entries. Otherwise, π *avoids* 321.

Definition 3.2. A permutation π *obeys parity* if $\pi(i) = i \bmod 2$.

Problem 3.3 (P. Alexandersson [ha]). Let $P^{321}(n)$ be the number of permutations of length n that both avoid 321 and obey parity. Is the generating function $\sum_{n=1}^{n} P^{321}(n)q^n$ algebraic, D-finite, or D-algebraic? What are the asymptotics of $P^{321}(n)$?

Example 3.4. There are six permutations of length 6 that both avoid 321 and obey parity:

$$\{123456, 341256, 145236, 125634, 561234, 345612\}.$$

- This sequence appears as A354298 in the OEIS [SI20].
- From 65 terms, it appears the growth rate is approximately $(2.31)^n n$.
- The terms demonstrate an even-odd behavior.

4. Bruce Sagan: Log-concavity of q-Stirling numbers and their type B analogues

The *Stirling numbers of the second kind* $S(n, k)$ count set partitions of $[n] := \{1, 2, \ldots, n\}$ with k blocks $B_1, B_2, \ldots, B_k$. They satisfy the recursion

$$S(n, k) = S(n-1, k-1) + kS(n-1, k) \text{ with } S(0, k) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

For π a set partition of $[n]$ with k blocks, we put π in standard form $[B_1, B_2, \ldots, B_k]$ so that $1 = \min B_1 < \min B_2 < \cdots < \min B_k$. An *inversion* of π is a pair (b, B_j) where $b \in [n]$ and B_j is a block such that

- $b \in B_i$ for some $i < j$, and
- $b > \min B_j$.

The *inversion number* $\operatorname{inv}(\pi)$ of a set partition is its number of inversions. Carlitz's *q-Stirling numbers of the second kind* $S_q[n,k]$ count set partitions of $[n]$ with k blocks by inversion number [Car33, WW91]:

$$S_q[n,k] = \sum_{\pi \text{ a set partition of } [n] \text{ with } k \text{ blocks}} q^{\operatorname{inv}(\pi)}.$$

Writing $[k]_q = 1 + q + q^2 + \cdots + q^{k-1}$, these q-numbers satisfy the recursion

$$S_q[n,k] = S_q[n-1,k-1] + [k]_q S_q[n-1,k] \text{ with } S_q[0,k] = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Definition 4.1. A polynomial $\sum_{i\geq 0} a_i q^i$ is *log-concave* if $a_i^2 \geq a_{i-1}a_{i+1}$ for all $i \geq 1$.

Conjecture 4.2 ([SS22, Conjecture 7.4])**.** The polynomials $S_q[n,k]$ are log-concave.

- Have checked by computer all $n, k \leq 50$.
- For $k \geq 2$, have confirmed that the coefficients of $S_q[n,k]$ are asymptotically normal as $n \to \infty$ ([SS22, Theorem 7.7]).
- Have tried standard techniques, like Lorentzian polynomials [BH20]. Haven't tried the theory of atlases [CP21].
- Q: What about q-log concavity? A: This is a application of ideas in [Sag92].

There is a natural extension of q-Stirling numbers from set partitions to the signed set partitions of type B_n. Let

$$S_q^B[n,k] = S_q^B[n-1,k-1] + [2k+1]_q S_q^B[n-1,k] \text{ with } S_q^B[0,k] = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Note that the polynomials $S_q^B[n,k]$ are *not* log-concave.

Definition 4.3. A polynomial $\sum_{i\geq 0} a_i q^i$ is *parity log-concave* if $\sum_{i\geq 0} a_{2i} q^i$ and $\sum_{i\geq 0} a_{2i+1} q^i$ are log-concave.

Conjecture 4.4 ([SS22, Conjecture 7.5])**.** The polynomials $S_q^B[n,k]$ are parity log-concave.

5. Theo Douvropoulos: Deformations of braid arrangements

Figure 3 displays the braid arrangement, the Shi arrangement, and the Catalan arrangement for the symmetric group S_3, along with their defining set of hyperplanes and their characteristic polynomials.

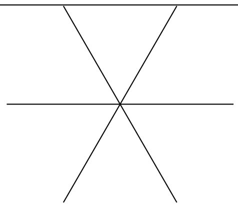
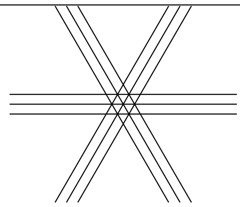
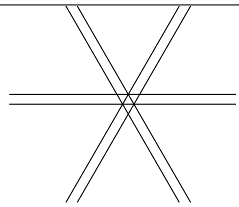
Braid	Catalan	Shi
		
$\mathscr{A}_{\text{Braid}} = \{x_i - x_j = 0\}$	$\mathscr{A}_{\text{Cat}} = \{x_i - x_j \in \{-1,0,1\}\}$	$\mathscr{A}_{\text{Shi}} = \{x_i - x_j \in \{0,1\}\}$
$\chi(\mathscr{A}_{\text{Braid}}, t) = t\prod_{i=1}^{n-1}(t-i)$	$\chi(\mathscr{A}_{\text{Cat}}, t) = t\prod_{i=1}^{n-1}(t-n-i)$	$\chi(\mathscr{A}_{\text{Shi}}, t) = t(t-n)^{n-1}$

Figure 3. The $n = 3$ braid arrangement (*left*), Catalan arrangement (*middle*), and Shi arrangement (*right*).

We generalize the last two kinds of arrangements as follows. Pick n positive integers $\mathbf{k} = (k_1, \ldots, k_n)$ and define

$$\mathscr{A}_{\text{Cat}}^{\mathbf{k}} = \big\{x_i - x_j \in \{-k_i - k_j, \ldots, k_i + k_j\}\big\} \text{ and}$$
$$\mathscr{A}_{\text{Shi}}^{\mathbf{k}} = \big\{x_i - x_j \in \{-k_i - k_j + 1, \ldots, k_i + k_j\}\big\}.$$

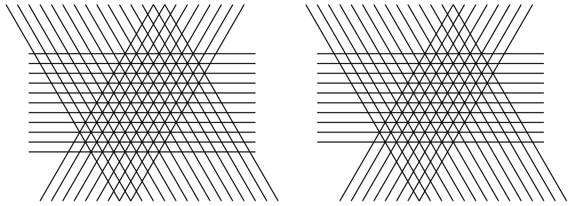


FIGURE 4. The arrangements $\mathscr{A}^{\mathbf{k}}_{\mathrm{Cat}}$ and $\mathscr{A}^{\mathbf{k}}_{\mathrm{Shi}}$ for $\mathbf{k} = (1, 3, 4)$.

Theorem 5.1. For any n positive integers $\mathbf{k} = (k_1, \ldots, k_n)$, we have

$$\chi(\mathscr{A}^{\mathbf{k}}_{\mathrm{Cat}}, t) = t \prod_{i=1}^{n-1} (t - 2(k_1 + \cdots + k_n) - i) \text{ and}$$
$$\chi(\mathscr{A}^{\mathbf{k}}_{\mathrm{Shi}}, t) = t(t - 2(k_1 + \cdots + k_n))^{n-1}.$$

There are other versions, but they are not quite as clean.

Problem 5.2. Find a combinatorial interpretation for the regions of $\mathscr{A}^{\mathbf{k}}_{\mathrm{Cat}}$ and $\mathscr{A}^{\mathbf{k}}_{\mathrm{Shi}}$, à la Athanasiadis–Linusson [AL99], Stanley–Pak [Sta96, Section 5], or Bernardi [Ber18].

6. Valentin Féray: Catalan numbers and polynomials in $1/\pi$.

Here are two interesting sums that arise from work that V. Féray will present later in the week.

Theorem 6.1. Write $\mathrm{Cat}_\ell = \frac{1}{\ell+1}\binom{2\ell}{\ell}$. Then

$$\sum_{\ell=0}^{\infty} \mathrm{Cat}_\ell^2 \cdot 16^{-\ell} = \frac{16}{\pi} - 4$$
$$\sum_{\ell_1,\ell_2 \geq 0}^{\infty} \mathrm{Cat}_{\ell_1} \cdot \mathrm{Cat}_{\ell_2} \cdot \mathrm{Cat}_{\ell_1+\ell_2} \cdot 16^{-\ell_1-\ell_2} = 8 - \frac{64}{3\pi}.$$

The general form of such sums is as follows. Let τ_1, τ_2 be two set partitions of $[k]$ such that $([k], \tau_1 \uplus \tau_2)$ is a connected hypertree with vertex degrees exactly 2—that is, the join $\tau_1 \vee \tau_2$ is the maximal partition $\{[k]\}$ into one part and $\#(\tau_1) + \#(\tau_2) = k + 1$, where $\#(\tau_i)$ is the number of parts of τ_i.

Problem 6.2. Is

$$\sum_{\ell_1,\ldots,\ell_k \geq 0} \left(\prod_{B \in \tau_1 \uplus \tau_2} \mathrm{Cat}_{\sum_{i \in B} \ell_i} \right) 16^{-\sum_{i=1}^{k} \ell_i} \in \mathbb{Q}\left[\frac{1}{\pi}\right]?$$

- There are examples where the sum gives a degree 2 polynomial in $1/\pi$.
- Q: Is there a connection to restricted meanders? A: Perhaps—the problem arises from a meander question.
- Q: Is there a relation to the Green function for $\mathbb{Z}^2$? A: Perhaps (not always affine).

7. Philippe Di Francesco: Enumeration of planar bicubic maps

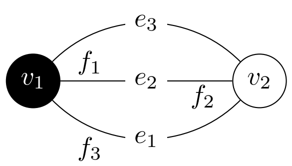
This is a problem with an apparently similar complexity to problems on meanders—the enumeration of edge-rooted Hamiltonian cycles in genus 0 planar bicubic (that is, bipartite and trivalent) maps. An example is given on the left of Figure 5.

We now cut the rooted edge and use the given Hamiltonian cycle to redraw the map as a path of the vertices in the order visited by the Hamiltonian cycle, with some additional noncrossing arcs connecting vertices. An example is given on the right of Figure 5.

Let H_{2n} be the number of such maps for a fixed number n of vertices. Then it is predicted by physics, and verified to 3 significant digits (using, for example, the transfer matrix method), that

$$H_{2n} \sim c\frac{\mu^{2n}}{n^\gamma}, \text{ where } \gamma = \frac{13 + \sqrt{13}}{6} \text{ and } \log(\mu^2) = 2.313.$$

Problem 7.1. Is there some mathematical (probabilistic or even combinatorial) approach to proving this prediction?



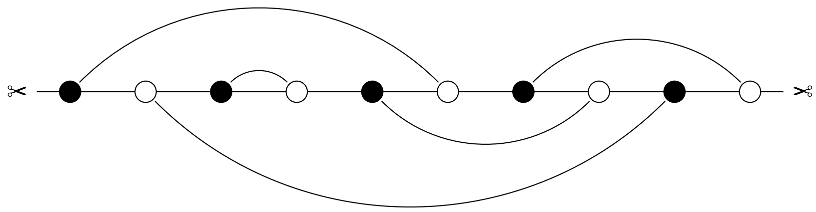


FIGURE 5. *Left:* an example of a vertex bicolored trivalent map of genus zero—here $F - E + V = 3 - 3 + 2 = 2 = 2 - 2g$, so that $g = 0$. *Right:* A redrawn map, where the path is the given Hamiltonian cycle and the rooted edge is indicated by scissors.

As a case study, if we remove the bicoloring of the vertices, then the quantity in question is easily computed to be

$$\sum_{m=0}^{n} \binom{2n}{2m} \mathrm{Cat}_m \mathrm{Cat}_{n-m} = \mathrm{Cat}_n \mathrm{Cat}_{n+1} \sim \frac{4}{\pi} \frac{4^{2n}}{n^3},$$

so that in this case we have $c = 4/\pi$, $\gamma = 3$, and $\mu^2 = 16$—and the same machinery from physics predicts these constants.

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