DUAL BRAID PRESENTATIONS AND CLUSTER ALGEBRAS

by

Priyojit Palit

APPROVED BY SUPERVISORY COMMITTEE:

Nathan Williams, Chair

Maxim Arnold

Carlos Arreche

Mieczyslaw K. Dabkowski

Viswanath Ramakrishna

This dissertation is dedicated to

my parents, Ashis Palit and Bandana Palit,

and my wife, Purbi Adhya.

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by

PRIYOJIT PALIT, BSc, MS

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DUAL BRAID PRESENTATIONS AND CLUSTER ALGEBRAS

Priyojit Palit, PhD
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Supervising Professor: Nathan Williams, Chair

Presentations for Coxeter groups and their braid groups are encoded by Dynkin diagrams. In their foundational work on cluster algebras, Fomin and Zelevinsky defined an operation on quivers (oriented Dynkin diagrams) called mutation. It is reasonable to ask if a quiver mutation-equivalent to (an orientation of) a Dynkin diagram also encodes a presentation of a Coxeter or braid group. By explicitly writing down a set of relations, Barot and Marsh constructed such presentations for Coxeter groups, which Grant and Marsh generalized to the corresponding braid groups. We explain and generalize these results for simply-laced types using presentations encoded by reduced factorizations (into reflections) of a Coxeter element—the results above are recovered by specializing to certain two-part factorizations (in bijection with vertices of the cluster exchange graph) and certain compositions of Hurwitz moves (paralleling quiver mutation).

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INTRODUCTION

1.1 Background

1.1.1 Braid and Symmetric Groups

Informally, a braid on n strands is an isotopy of diagrams which represents n 'braided' strings in a 3-dimensional Euclidean space, whose endpoints are fixed at their top and bottom at n distinct points, with the restriction that the strings may not pass through each other or double back. For example, Figure 1.1 is a schematic diagram of one such braid on 3 strands.

Figure 1.1. Example of a braid on 3 strands.

For any fixed integer n, there are infinitely many such braids on n strands and given any two such braids we can stack them on top of each other and obtain a new braid on n strands. These braids form a group, known as the braid group under concatenation (or stacking), see Figure 1.2, where the identity element is the braid whose strands are all untangled. Formally we can define a braid group as follows.

Definition 1.1.1 (Braid group on n strands)**.** Let $p_1, \ldots, p_n$ be n distinct points in $\mathbb{R}^2$. Let $(f_1, \ldots, f_n)$ be an n-tuple of functions

$$f_i : [0, 1] \longrightarrow \mathbb{R}^2$$

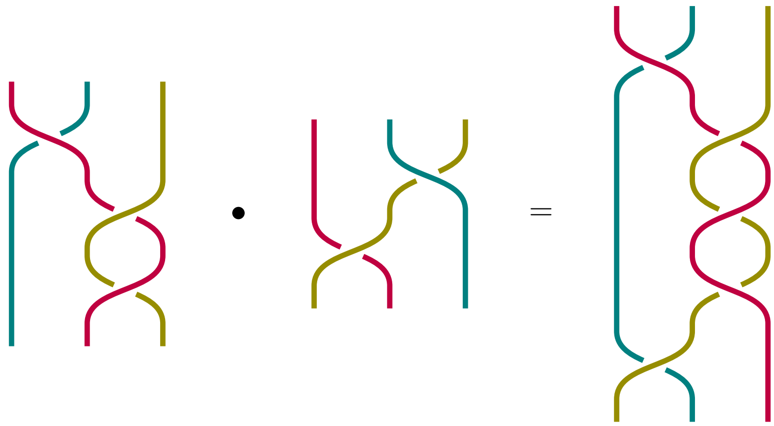


Figure 1.2. Concatenation of two braids on 3 strands.

such that

$$f_i(0) = p_i, f_i(1) = p_j \text{ for some } j = 1, \ldots, n,$$

and such that the n paths

$$\begin{aligned} [0,1] &\longrightarrow \mathbb{R}^2 \times [0,1] \\ t &\longrightarrow (f_i(t), t), \end{aligned}$$

called *strands*, have disjoint images. These n strands are called a *braid*. The *braid* group $\boldsymbol{B}_n$ on n strands is the group of isotopy classes of braids. The product of a braid $(f_1(t), \ldots, f_n(t))$ and a braid $(g_1(t), \ldots, g_n(t))$ is defined by

$$(f \bullet g)_i(t) = \begin{cases} f_i(2t), & 0 \le t \le \frac{1}{2} \\ g_j(2t-1), & \frac{1}{2} \le t \le 1 \end{cases} \tag{1.1}$$

where j is such that $f_i(1) = p_j$.

In (Artin, 1947), Artin gave the following presentation (now known as the Artin's presentation) of the *braid group* on n strands.

$$\boldsymbol{B}_n := \langle \boldsymbol{s}_1, \ldots, \boldsymbol{s}_{n-1} | \boldsymbol{s}_i \boldsymbol{s}_{i+1} \boldsymbol{s}_i = \boldsymbol{s}_{i+1} \boldsymbol{s}_i \boldsymbol{s}_{i+1}, \boldsymbol{s}_i \boldsymbol{s}_j = \boldsymbol{s}_j \boldsymbol{s}_i \rangle$$

where $1 \leq i \leq n-2$ and $i-j>2$. Here $\boldsymbol{s}_i$ and $\boldsymbol{s}_i^{-1}$ can be visualized as Figure 1.3 and Figure 1.4, respectively. It is easy to see how any braid on n strands can be produced by taking products of $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_{n-1}$ and their inverses.

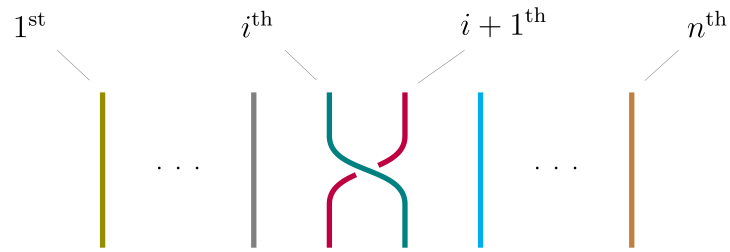


Figure 1.3. A visualization of the braid $\boldsymbol{s}_i$.

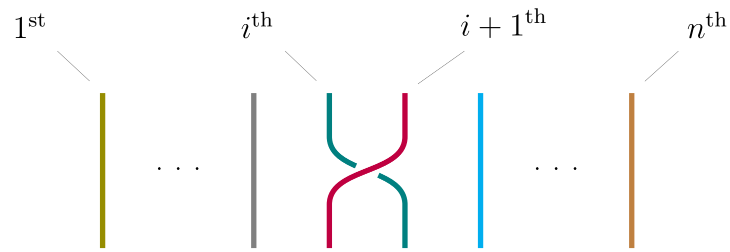


Figure 1.4. A visualization of the braid $\boldsymbol{s}_i^{-1}$.

By forgetting *how* the strands in a braid twist around each other, and only focusing on where the strands start and end, every braid on n strand determines a permutation on n elements. By assigning the braids $\boldsymbol{s}_i$ and $\boldsymbol{s}_i^{-1}$ to the transposition $(i, i+1)$, we obtain a surjective map from the braids on n strands to the set of bijective functions from a set with n elements to itself. This map is compatible with the composition defined in Equation (1.1), therefore we obtain a surjective group homomorphism from the braid group on n strands to the symmetric group

$$\mathfrak{S}_n := \langle s_1, \ldots, s_{n-1} : s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, s_i s_j = s_j s_i, s_i^2 = e \rangle$$

where $1 \leq i \leq n-2$ and $i-j>2$.

1.1.2 Artin and Coxeter groups

A *finite Coxeter group* is an abstract group generated by a set of *simple reflections*, denoted by S, with a presentation of the following form:

$$W := \left\langle S : \begin{array}{c} s_i^2 = s_j^2 = e, \\ \underbrace{s_i s_j s_i \cdots}_{m_{ij} \text{ terms}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij} \text{ terms}}, \end{array} \text{ for } s_i, s_j \in S \text{ with } s_i \neq s_j \right\rangle \tag{1.2}$$

where $2 \leq m_{ij} \leq \infty$. For example, symmetric group $\mathfrak{S}_n$ is a Coxeter group. The relations $s_i s_j s_i \cdots = s_j s_i s_j \cdots$ are called *braid relations*.

The spherical *Artin group* corresponding to a finite Coxeter group W is given by *Artin's presentation*:

$$\boldsymbol{B}(W) := \left\langle \boldsymbol{S} : \underbrace{\boldsymbol{s}_i \boldsymbol{s}_j \boldsymbol{s}_i \cdots}_{m_{ij} \text{ terms}} = \underbrace{\boldsymbol{s}_j \boldsymbol{s}_i \boldsymbol{s}_j \cdots}_{m_{ij} \text{ terms}}, \text{ for } s_i, s_j \in S \text{ with } s_i \neq s_j \right\rangle, \tag{1.3}$$

where $\boldsymbol{S}$ is a formal copy of the generators S, subject to only the braid relations. Coxeter groups are closely related to Artin groups—each Coxeter group is a quotient of the corresponding Artin group in a natural way. Braid group $\boldsymbol{B}_n$ is an example of an Artin group. For a given presentation of a Coxeter group W, the corresponding Artin group $\boldsymbol{B}(W)$ has a similar presentation, obtained by simply forgetting the involutions in the Coxeter group's presentation (Deligne, 1972; Brieskorn and Saito, 1972).

1.1.3 Presentations from Quivers

Artin's presentation for an Artin group $\boldsymbol{B}(W)$ and the Coxeter group W is encoded by Dynkin diagram—vertices correspond to generators, edges to braid relations, and missing edges to commutation relations (for example, see Figure 1.5).

Orienting the edges of a Dynkin diagram gives a directed graph called a *quiver*, for which Fomin and Zelevinsky (Fomin and Zelevinsky, 2002) have defined a notion of *quiver mutation*—input a quiver $\mathcal{Q}$ and any vertex v of $\mathcal{Q}$, and the output is a new quiver $\mu_v^{\text{quiv}}(\mathcal{Q})$

$$s_1 — s_2 — s_3 \qquad \Longleftrightarrow \qquad \left\langle \boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3 : \begin{array}{c} \boldsymbol{s}_1\boldsymbol{s}_2\boldsymbol{s}_1 = \boldsymbol{s}_2\boldsymbol{s}_1\boldsymbol{s}_2 \\ \boldsymbol{s}_2\boldsymbol{s}_3\boldsymbol{s}_2 = \boldsymbol{s}_3\boldsymbol{s}_2\boldsymbol{s}_3 \\ \boldsymbol{s}_1\boldsymbol{s}_3 = \boldsymbol{s}_3\boldsymbol{s}_1 \end{array} \right\rangle = \boldsymbol{B}_4$$

Figure 1.5. Artin's presentation for the braid group $\boldsymbol{B}_4$ encoded by a Dynkin diagram of type A_3.

with some local changes to edges near v (see Definition 8.1.2). A *cluster exchange graph* is a connected graph whose vertices are labelled by quivers and edges by quiver mutations. A quiver $\mathcal{Q}$ is said to be *mutation equivalent* to another quiver $\mathcal{Q}'$ if one can be obtained from the other by a finite number of quiver mutations, and the set of all quivers, mutation equivalent to the quiver $\mathcal{Q}$ is called the *mutation class* of $\mathcal{Q}$.

Since Dynkin diagrams encode a presentation of the corresponding Coxeter group W (and the Artin group $\boldsymbol{B}(W)$), it is reasonable to ask if a quiver mutation-equivalent to (an orientation of) a Dynkin diagram also encodes a presentation.

Building on work of Barot and Marsh (Barot and Marsh, 2015), Grant and Marsh (Grant and Marsh, 2017) constructed such presentations (see Theorem 11.0.1) from Dynkin diagrams of simply-laced type, showing that each quiver $\mathcal{Q}$ in the same mutation class as a Dynkin quiver encode a presentation $\boldsymbol{B}(\mathcal{Q})$ of the Artin group of the same Dynkin type as $\mathcal{Q}$. The validity of these presentations was checked by giving an isomorphism between $\boldsymbol{B}(\mathcal{Q})$ and $\boldsymbol{B}(\mu_v^{\text{quiv}}(\mathcal{Q}))$. This is illustrated in Figure 1.6. Using different approaches this result has also been independently proved in (Qiu, 2016) for simply-laced type, and in (Haley et al., 2017) for finite type.

1.1.4 Coxeter Elements, Factorizations, and Hurwitz Moves

Let us write $t^s = sts^{-1}$ and ${}^st = s^{-1}ts$. The set of *reflections* of a finite Coxeter group W is defined to be the closure of simple reflections S under conjugation, and is generally denoted

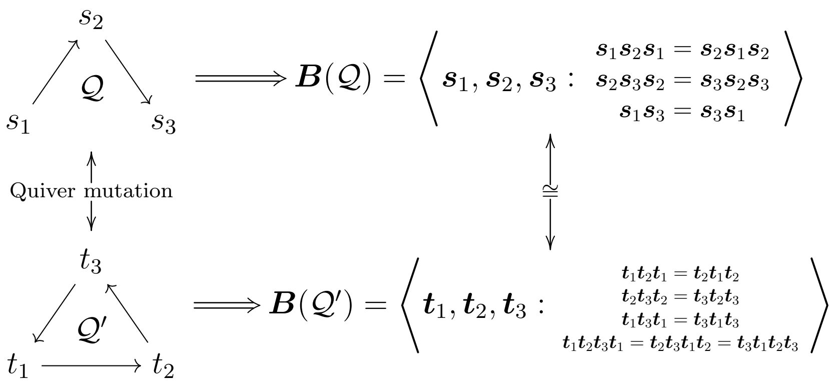


Figure 1.6. An example demonstrating the isomorphism between two groups—$\boldsymbol{B}(\mathcal{Q})$ and $\boldsymbol{B}(\mathcal{Q}')$—obtained using Grant and Marsh's presentation encoded by quivers $\mathcal{Q}$ and $\mathcal{Q}'$, which are mutation equivalent.

by T,

$$T := \{s^w : s \in S, w \in W\}.$$

Any element $w \in W$ can be written as an expression in the alphabet S or T. An *S-decomposition* of w is any expression $s_1 s_2 \cdots s_l$ such that $w = s_1 s_2 \cdots s_l$ with $s_1, s_2, \ldots, s_l \in S$. Similarly, a *$T$-decomposition* of w is any expression $t_1 t_2 \cdots t_l$ such that $w = t_1 t_2 \cdots t_l$ with $t_1, t_2, \ldots, t_l \in T$. An S-decomposition (resp. T-decomposition) of w is *reduced* if it is of minimal length among all S-decompositions (resp. T-decompositions) of w.

A *Coxeter element* in a Coxeter group W is a product of all the simple reflections (each appearing exactly once) in any order—associating these simple reflections to the vertices of a Dynkin diagram, the order (up to commutation) is equivalent to orienting the edges to obtain a quiver. For example, for a reduced S-decomposition $\mathsf{c} = s_1 \cdots s_n$ of a Coxeter element, if s_i appears before s_j in c and if their corresponding nodes in the associated Dynkin diagram share an edge, then we will orient it from the node corresponding to s_j to the node corresponding to s_i, (see Figure 1.7).

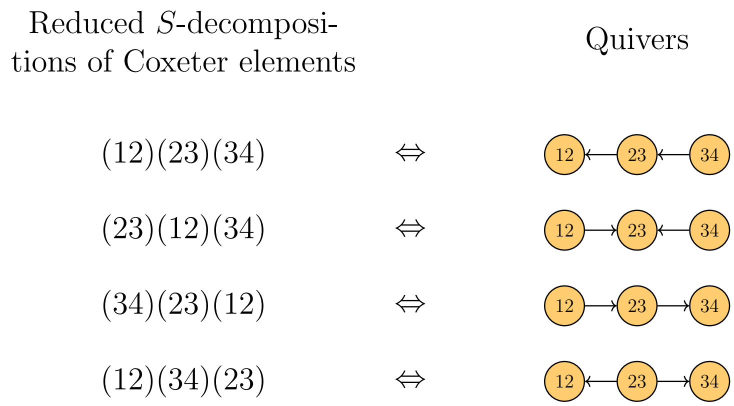


Figure 1.7. Reduced S-decompositions of Coxeter elements in $\mathfrak{S}_4$ encode quivers.

Given a T-decomposition $t_1 t_2 \cdots t_k t_{k+1} \cdots t_n$ of a Coxeter element c we can perform a *Hurwitz move* on $t_1 t_2 \cdots t_k t_{k+1} \cdots t_n$ at k to obtain a new T-decomposition

$$\mu_k(t_1 t_2 \cdots t_k t_{k+1} \cdots t_n) := t_1 t_2 \cdots t_{k+1}^{t_k} t_k \cdots t_n. \tag{1.4}$$

By proposition 1.6.1 in (Bessis, 2003), the set of reduced T-decompositions of a Coxeter element is connected under Hurwitz move, i.e. if we keep performing Hurwitz moves on a given T-factorization of a Coxeter element, then we will eventually get possible T-decomposition of c. We have already seen how the S-decompositions of Coxeter elements encode quivers which in turn encode presentations of Artin groups, it is natural to ask if the T-decompositions of a Coxeter element also encode 'meaningful' presentations of the corresponding Artin group. In this dissertation we answer this question for the simply-laced finite Coxeter groups—the groups of types A, D and E.

1.2 Presentations from Factorizations of Coxeter Elements

Our first result gives presentations of the Artin group $\boldsymbol{B}(W)$ encoded by reduced T- decompositions of a Coxeter element c in a Coxeter group W.

Theorem 1.2.1 (Theorem 5.1.2)**.** *Let c be a Coxeter element in W and let $t_1 t_2 \cdots t_n$ be a reduced T-decomposition of c. Define*

$$\boldsymbol{B}(t_1, t_2, \ldots, t_n) := \left\langle \boldsymbol{t}_1, \boldsymbol{t}_2, \ldots, \boldsymbol{t}_n \middle| \mathrm{Rel}(t_1, \ldots, t_n) \right\rangle \tag{1.5}$$

where

$$\mathrm{Rel}(t_1, \ldots, t_n) = \begin{cases} \boldsymbol{t}_i\boldsymbol{t}_j = \boldsymbol{t}_j\boldsymbol{t}_i & \textit{if } t_i \parallel t_j, \\ \boldsymbol{t}_i\boldsymbol{t}_j\boldsymbol{t}_i = \boldsymbol{t}_j\boldsymbol{t}_i\boldsymbol{t}_j & \textit{if } t_i \between t_j, \\ [\boldsymbol{t}_{i_1}, \boldsymbol{t}_{i_2} \cdots \boldsymbol{t}_{i_s} \cdots \boldsymbol{t}_{i_2}^{-1}] = e & \textit{if } t_{i_1} \between t_{i_2} \between \cdots \between t_{i_{s-1}} \between t_{i_s} \between t_{i_1}, \\ & t_{i_j} \parallel t_{i_k} \textit{ for } k \neq j-1, j+1. \end{cases}$$

then

$$\boldsymbol{B}(t_1, t_2, \ldots, t_n) \cong \boldsymbol{B}(W).$$

where $t_p \parallel t_q$ and $t_p \between t_q$ denote that t_p and t_q commute and don't commute, respectively.

By Grant and Marsh's result Theorem 11.0.1, quivers in the same mutation class encode presentations of an Artin group. By Theorem 1.2.1, reduced T-decompositions of a Coxeter element also encode presentations of an Artin group. Our second result relates these two presentations. We will refer to the presentations arising from Grant and Marsh's result as in Theorem 11.0.1 as *quiver presentations*, and presentations arising from Theorem 1.2.1, *factorization presentations*.

1.3 Quiver Presentations from Factorization Presentations

For our second result we use a combinatorial model of the cluster exchange graph to recover quiver presentations as a special case of factorization presentations. This model consists of two-part factorizations (in bijection with the vertices of the cluster exchange graph) and factorization mutations (paralleling quiver mutation).

For a given reduced S-decomposition c of a Coxeter element c we define a total order on the set of reflections, called the *Coxeter order* by Definition 9.2.7. We write $t_1 \leq_{\mathsf{c}} t_2$

for any two reflections t_1 and t_2, if t_1 precedes t_2 in the Coxeter order. For the symmetric group $\mathfrak{S}_n$ and the reduced S-decomposition $(12)(23)\cdots((n-1)n)$ of a Coxeter element in $\mathfrak{S}_n$, the Coxeter order on the reflections is just lexicographic order on the transpositions (ij) for $1 \leq i < j \leq n$. That is, $(ij) \leq_c (kl)$ if and only if $i < k$ or $i = k$ and $j \leq k$. For a given Coxeter element $c \in W$ and a reduced S-decomposition $\mathsf{c} = \mathsf{s}_1 \cdots \mathsf{s}_n$ of c, a *two-part factorization* is a reduced T-decomposition $\ell_1 \cdots \ell_i r_1 \cdots r_j$ of c, which can be written as

$$\ell_1 \cdots \ell_i | r_1 \cdots r_j,$$

such that

$$\ell_1 \leq_{\mathsf{c}} \cdots \leq_{\mathsf{c}} \ell_i, r_1 \leq \cdots \leq_{\mathsf{c}} r_j, \ell_1 \cdots \ell_i r_1 \cdots r_j = c, \text{ and } i + j = n.$$

Clearly, a Hurwitz move on a two-part factorization may result in a factorization that is not a two-part factorization, for example, consider the reduced S-decomposition $\mathsf{c} = (12)(23)(34)$ of a Coxeter element in $\mathfrak{S}_4$. The Hurwitz move, $(23)(34)|(14) \xrightarrow{\mu_1} (24)(23)|(14)$ gives a reduced T-decomposition which is not a two-part factorization (because $(23) \leq_{\mathsf{c}} (24)$). *Factorization mutation* is a particular sequence of Hurwitz moves on a two-part factorizations that preserves two-part factorizations (see Definition 10.0.14). To perform a factorization mutation we choose a reflection on the left-hand side of the two-part factorization and through a series of Hurwitz moves, 'move' it to a new position on the right-hand side of the two-part factorization such that the resulting decomposition is still a two-part factorization. For example, a factorization mutation on the reflection (23) is the following sequence of Hurwitz moves

$$(23)(34)|(14) \xrightarrow{\mu_1} (24)(23)|(14) \xrightarrow{\mu_2} (24)|(14)(23).$$

Since two-part factorizations of a Coxeter element are a special type of reduced T-decompositions, and factorization mutations are repeated Hurwitz moves, therefore by Theorem 1.2.1, the factorization presentations arising from all the two-part factorizations form a subset of the factorization presentations arising from all the reduced T-decompositions. By associating quivers to two-part factorizations—as in Definition 10.0.16—we show that if two two-part factorizations are connected by a factorization mutation then their associated quivers are connected by quiver mutation (see Theorem 10.0.17). Using these observations, we show that the factorization presentations arising from these two-part factorizations of a Coxeter element (using Theorem 1.2.1) recover Grant and Marsh's quiver presentations arising from the associated quivers (see Chapter 11).

Theorem 1.3.1 (Theorem 11.0.2)**.** *Let $s_1 \cdots s_n$ be a reduced S-decomposition of a Coxeter element c. Let $Fact_2(c)$ denote the set of all two-part factorizations of c and $\mathcal{Q}$ denote the quiver associated to the two-part factorization $s_1 \cdots s_n|\cdot$. Then the factorization presentations arising from the reduced T-decompositions in $Fact_2(c)$ using Theorem 1.2.1 are precisely the quiver presentations arising from the quivers in the mutation class of $\mathcal{Q}$ using Theorem 2.12. in (Grant and Marsh, 2017).*

1.4 Dual Braid Presentation from Factorization Presentations

Our third result draws a parallel between our presentations and Bessis' dual braid presentation. Building on work of Birman-Ko-Lee (Birman et al., 1998), Bessis (Bessis, 2003) gave a second, different presentation for Artin groups associated with finite Coxeter groups, by replacing the set of simple reflections S by the set of all the reflections T, leading to a 'dual' presentation for the Artin group $\boldsymbol{B}(W)$ called the *dual braid presentation*.

$$\boldsymbol{B}(W) \cong \left\langle \boldsymbol{T} \middle| \boldsymbol{t}_i \boldsymbol{t}_j = \boldsymbol{t}_k \boldsymbol{t}_i, \text{ for } t_i, t_j, t_k \in T \text{ with } t_i t_j = t_k t_i \text{ and } t_i t_j \leq_T c \right\rangle \tag{1.6}$$

where $\boldsymbol{T}$ is a formal copy of the set of all the reflections T and c is a Coxeter element. In Chapter 7 we prove that Bessis's dual braid presentation is the 'union' of all the presentations given by Theorem 1.2.1.

Theorem 1.4.1 (Theorem 7.0.6)**.** *For a Coxeter element c in W, the dual braid presentation of the Artin group $\boldsymbol{B}(W)$ is generated by $\boldsymbol{T}$ (a formal copy of the set of reflections), subject to the relations $\{\mathrm{Rel}(t_1, \ldots, t_n) : t_1 \cdots t_n$ is a reduced T-decomposition of $c\}$.*

CHAPTER 2

BACKGROUND ON ARTIN AND COXETER GROUPS

This chapter is a review of the theory of Artin and Coxeter groups. Most of this material has been well studied and a much detailed exploration of this material can be found in (Humphreys, 1990; Bourbaki, 2002). We will first establish the standard conventions and notations for Coxeter and Artin groups and their connection with abstract reflection groups, followed by a review of how they are encoded and classified by Coxeter-Dynkin diagrams. We will conclude by defining two different types of orders on the set of elements of a Coxeter group.

2.1 Finite Real Reflection Groups and Finite Coxeter Groups

Definition 2.1.1 (Reflection)**.** Let E be an n-dimensional Euclidean space. A mapping $t : E \mapsto E$ is a *reflection* in E if t

1. is an isometry

2. is an involution

3. fixes a hyperplane pointwise

4. swaps the half-spaces defined by the hyperplane.

Definition 2.1.2 (Finite reflection group)**.** Let E be an n-dimensional Euclidean space. A *finite reflection group* is a group generated by reflections in E such that the hyperplanes fixed by these reflections pass through the origin.

If the requirement that the hyperplanes pass through the origin is relaxed then we have *affine reflection groups*. If we allow the underlying space to be a finite complex vector space then we will have *complex reflection group*. In this dissertation, all reflection groups are *real* and *finite*.

Definition 2.1.3. A *Coxeter group* is a group with the following presentation:

$$W := \langle s_1, s_2, \ldots, s_n : (s_i s_j)^{m_{ij}} = e \rangle_{\text{group}},$$

where $m_{ij} \in \mathbb{N} \cup \{\infty\}$, $m_{ij} = 1$ when $i = j$ and $m_{ij} = \infty$ when $s_i s_j$ is of infinite order.

For $m_{ij} < \infty$ we have *finite Coxeter groups*. When $i \neq j$ and $m_{ij} > 2$, the relations $(s_i s_j)^{m_{ij}} = e$ (also written as $\underbrace{s_i s_j s_i \cdots}_{m_{ij} \text{ terms}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij} \text{ terms}}$) are *braid relations*. When $i \neq j$ and $m_{ij} = 2$, the relations $(s_i s_j)^{m_{ij}}$ (also written as $s_i s_j = s_j s_i$) are *commutation relations*. The collection (W, S), where $S := \{s_1, \ldots, s_n\}$ is called a *Coxeter system* of *rank* $n := |S|$. Denote the closure of S under conjugation by $T := \{wsw^{-1} | s \in S, w \in W\}$.

Every reflection group in an n-dimensional Euclidean space is isomorphic to some Coxeter group (Coxeter, 1934) and every *finite* Coxeter group is isomorphic to some reflection group in an n-dimensional Euclidean space (Coxeter, 1935). In these isomorphisms the set T in the Coxeter group always maps to the set of reflections in the reflection group, therefore T is called the *set of reflections* and S—the *set of simple reflections* or *simple generators*. In this dissertation all Coxeter groups are of finite type.

Definition 2.1.4. A pair (A, T_A) is called an *abstract finite real reflection group*, if there exists a faithful representation $\rho : A \hookrightarrow \mathrm{GL}(V_{\mathbb{R}})$ satisfying

$$\forall a \in A, \mathrm{codim}(\ker(\rho(a) - \mathrm{Id}_{\mathrm{GL}(V_{\mathbb{R}})})) = 1 \Leftrightarrow a \in T_A$$

where A is a finite group, T_A a generating subset of A and $V_{\mathbb{R}}$ a finite dimensional $\mathbb{R}$-vector space.

In this dissertation all abstract reflection groups will be finite and real, therefore we will simply call them abstract reflection groups. From this definition, it is clear that every abstract reflection group is also a finite Coxeter group (since every abstract reflection group is isomorphic to a finite reflection group). Thus, given any abstract reflection group (A, T_A)

we can choose a set $S_A \subset T_A$ such that (A, S_A) is a Coxeter system (the choice of S_A is not unique), and given any Coxeter system (W, S), the pair (W, T) where $T = \{wsw^{-1} : w \in W, s \in S\}$ is an abstract reflection group. In this dissertation we will denote an abstract reflection group by (W, T) and its corresponding Coxeter system by (W, S).

Definition 2.1.5. Given a finite Coxeter system (W, S) there is a corresponding spherical *Artin system* $(\boldsymbol{B}(W), \boldsymbol{S})$—an *Artin group* $\boldsymbol{B}(W)$, generated by a formal copy of the simple reflections S denoted by $\boldsymbol{S}$, subject to only the braid relations and commutations (Deligne, 1972)(Brieskorn and Saito, 1972).

$$\boldsymbol{B}(W) := \langle \boldsymbol{S} : \underbrace{\boldsymbol{s}_i\boldsymbol{s}_j\boldsymbol{s}_i\cdots}_{m_{ij}} = \underbrace{\boldsymbol{s}_j\boldsymbol{s}_i\boldsymbol{s}_j\cdots}_{m_{ij}}, s_i, s_j \in S, s_i \neq s_j\rangle_{\text{group}}$$

Similarly, an Artin group presentation becomes a Coxeter group presentation upon adding the relations $s^2 = e$ for each s in the generating set. Since the relations in the presentation are between positive words therefore this same presentation can also be seen as a monoid presentation, namely *positive Artin monoid*

$$\mathbf{B}^+ := \langle \boldsymbol{S} : \underbrace{\boldsymbol{s}_i\boldsymbol{s}_j\boldsymbol{s}_i\cdots}_{m_{ij}} = \underbrace{\boldsymbol{s}_j\boldsymbol{s}_i\boldsymbol{s}_j\cdots}_{m_{ij}}, s_i, s_j \in S, s_i \neq s_j\rangle_{\text{monoid}}$$

Example 2.1.6. The symmetric group $\mathfrak{S}_n$ (the group of bijective functions from a set with n elements to itself) is a Coxeter group. The corresponding Artin group is the braid group on n strands, $\boldsymbol{B}_n$. In particular, $\mathfrak{S}_3$ and $\boldsymbol{B}_3$ are the Coxeter group and braid group with the following presentations, respectively:

$$\mathfrak{S}_3 = \langle s_1, s_2 : s_1s_2s_1 = s_2s_1s_2, s_1^2 = s_2^2 = e\rangle_{\text{group}}, \qquad \boldsymbol{B}_3 = \langle \boldsymbol{s}_1, \boldsymbol{s}_2 : \boldsymbol{s}_1\boldsymbol{s}_2\boldsymbol{s}_1 = \boldsymbol{s}_2\boldsymbol{s}_1\boldsymbol{s}_2\rangle_{\text{group}}$$

The reflections s_1 and s_2 correspond to the transpositions (12) and (23) respectively. The set $S = \{s_1, s_2\}$ is the set of simple reflections, whereas the set of reflections is $T =$

$\{s_1, s_2, s_1s_2s_1\}$ where $s_1s_2s_1$ corresponds to the transposition (13). The elements $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ in the braid group $\boldsymbol{B}_3$ may be visualized as 3 strands with two of them crossing each other in a particular way as illustrated in Figure 2.1.

Figure 2.1. A visualization of the elements $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ in the braid group $\boldsymbol{B}_3$. $\boldsymbol{s}_1$ twists the first two strands whereas $\boldsymbol{s}_2$ twists the second and the third strands.

The braid relation in $\boldsymbol{B}_3$ is illustrated in Figure 2.2.

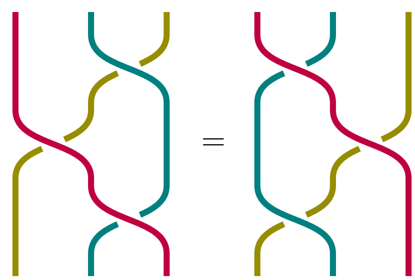


Figure 2.2. Braid relation: $\boldsymbol{s}_2\boldsymbol{s}_1\boldsymbol{s}_2 = \boldsymbol{s}_1\boldsymbol{s}_2\boldsymbol{s}_1$.

The corresponding positive braid monoid $\boldsymbol{B}_3^+$ has the exact same presentation as that of the group $\boldsymbol{B}_3$.

2.2 Coxeter-Dynkin Diagrams

Coxeter groups are encoded by Coxeter-Dynkin diagrams—vertices correspond to simple reflections of the Coxeter group, while edges encode the relations as follows: if s_1 and s_2 are two simple reflections such that $(s_1s_2)^3 = e$ then their corresponding vertices are connected by an unlabeled edge (• ——— •), whereas if $(s_1s_2)^2 = e$ then the corresponding vertices don't share an edge (•   •). If $(s_1s_2)^{m_{ij}} = e$ where $m_{ij} \neq 2$ or 3 then the corresponding vertices share an edge with an assigned weight of m_{ij} ($\bullet \overset{m_{ij}}{\longrightarrow} \bullet$). By this convention the lowest assigned weight to an edge in a Coxeter-Dynkin diagram is 4.

In Figure 2.3 we show the Coxeter-Dynkin diagram and the corresponding hyperplane arrangement for the Coxeter group $\mathfrak{S}_2$. In (Coxeter, 1935) H. S. M. Coxeter classified all the finite Coxeter groups in terms of Coxeter-Dynkin diagrams thus classifying the corresponding Artin groups as well, see Figure 2.4.

A Coxeter-Dynkin diagram with no assigned weights to its edges is called a *simply-laced Coxeter-Dynkin diagram*. Thus Coxeter-Dynkin diagrams of type A_n, D_n and E_n are simply-laced Dynkin diagram and the corresponding Coxeter groups are called *simply-laced Coxeter reflection group*.

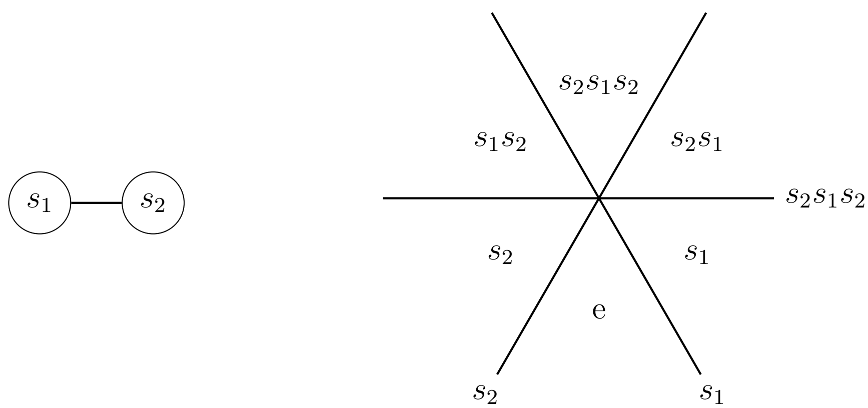


Figure 2.3. The Coxeter group $\mathfrak{S}_3$ encoded by a Dynkin diagram of type A_2 with its corresponding hyperplane arrangement.

2.3 Weak Order

For any group $(G, \bullet)$ and any subset $H \subset G$, an expression $\mathsf{h}_1^{a_1}\mathsf{h}_2^{a_2}\cdots\mathsf{h}_k^{a_k}$ (or the sequence $(h_1^{a_1}, h_2^{a_2}, \ldots, h_k^{a_k})$) is an *$H$-decomposition* of an element $g \in G$ if $h_1^{a_1} \bullet h_2^{a_2} \bullet \cdots \bullet h_k^{a_k} = g$, where $h_1, h_2, \ldots, h_k$ (not necessarily all distinct) are in H and $a_i = \pm 1$. Sometimes, we also use the phrases—"a word in H" or "an H-word" for an element g to denote the H-decomposition

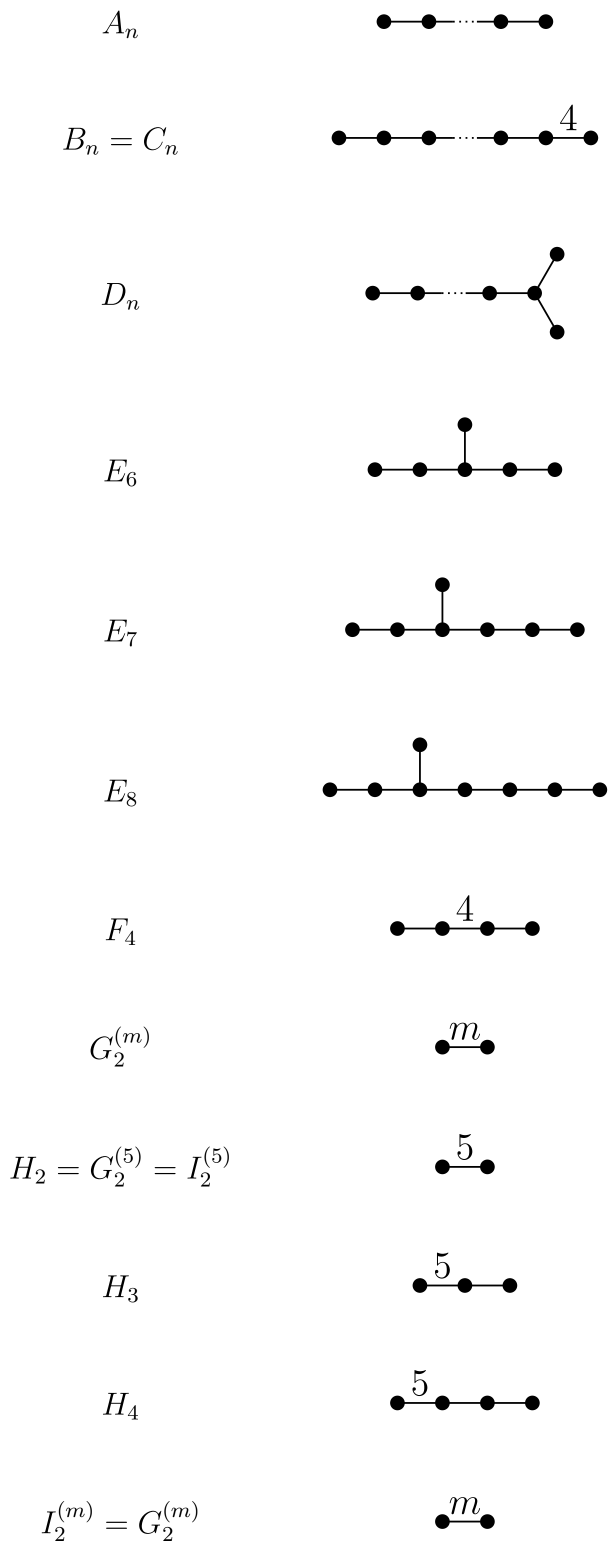


Figure 2.4. Types of finite Coxeter groups and their Coxeter-Dynkin diagrams.

of g. H is called a *generating set* if every element in G can be expressed as a word in H. In this context elements in the subset H are sometimes referred to as *letters*. Note that a generating set for a given group may not be unique. We will use $\mathsf{sans\text{-}serif}$ font to distinguish words from *elements*. For example, if w denotes an element then we may use w to denote a particular H-decomposition of w. The *length of an H-decomposition* is the length of the H-decomposition as a sequence. An H-decomposition of $g \in G$ will be called *reduced H-decomposition* if it is of minimal length among reduced H-decompositions of g, and the set of all such reduced H-decompositions of g will be denoted by $Red_H(g)$. The *length* of an element $g \in G$ is the length of any reduced H-decomposition of g, denoted by $l_H(g)$.

In this way we can define S-decomposition, T-decomposition and all the other related concepts for a Coxeter system (W, S). Two words w and w' in S (or in T) are called *commutation equivalent* if one can be written as the other by a sequence of commutations of consecutive commuting letters, and we will write $\mathsf{w} \equiv \mathsf{w}'$. A word u is *initial* in a word w if u appears as a prefix of a word w' such that $\mathsf{w}' \equiv \mathsf{w}$ where u, w and w' are S-words (or T-words). Similarly a word v is *final* in a word w if v appears as a suffix of a word w' such that $\mathsf{w}' \equiv \mathsf{w}$ where v, w and w' are S-words (or T-words).

In a Coxeter system (W, S) a factorization $w = u.v$ is a *reduced S-factorization* (where u, v, w are elements in W) if $l_S(w) = l_S(u) + l_S(v)$. Here we must make a clear distinction between the words *S-factorization* and *S-decomposition*. S-decomposition denotes a factorization into elements in S, whereas S-factorization denotes just any factorization with the condition that the sum of the length of each of the factors add up to the length of the original element.

$u \in W$ is *S-initial* in $w \in W$ if there exists a $v \in W$ such that the factorization $w = u.v$ is a reduced S-factorization. Similarly $u \in W$ is *S-final* in $w \in W$ if there exists a $v \in W$ such that the factorization $w = v.u$ is a reduced S-factorization. The *weak order* $\mathbf{Weak}(W) = (W, \leq_S)$ is defined by $u \leq_S w$ if and only if u is S-initial in w as elements. Weak order is analogously defined for $\boldsymbol{B}^+$.

2.4 Absolute Order

The absolute order $\mathbf{Abs}(W) = (W, \leq_T)$ is defined by $u \leq_T w$ if and only if there exists a v in W such that $u.v = w$ and $l_T(u)+l_T(v) = l_T(w)$. Observe that if $u \in W$ is T-initial in $v \in V$ then u is not necessarily S-initial in v—because $l_T(v)$ may not be equal to $l_S(v)$. If u is T-initial in v then u is also T-final in v, because if $uv' = v$ then there exist a $v'' = uv'u^{-1}$ such that $v''u = v$ where $l_T(v'') = l_T(v')$. This is not true for S-factorizations i.e. if u is S-initial in v then u is not necessarily S-final in v.

CHAPTER 3

ROOT SYSTEMS

In this chapter we establish the notations and conventions for root systems. Roots are vectors in a vector space equipped with some additional properties which make them the generators of finite Coxeter groups. Therefore the study of finite root systems is closely related to the study of finite Coxeter groups. A detailed treatment can be found in (Humphreys, 1990).

3.1 Root Systems and Weyl Groups

Definition 3.1.1 (Root System)**.** For any standard Euclidean space E (with a positive definite inner product $\langle \cdot, \cdot \rangle$), a finite subset R of E will be called a *root system*, if it satisfies the following axioms

1. $0 \notin R$ and R spans E.

2. If $v \in R$ and $kv \in R$, where $k \in \mathbb{R}$ then either $k = 1$ or $k = -1$.

3. For $u, v \in R$, $\frac{2\langle u,v \rangle}{\langle v,v \rangle} \in \mathbb{Z}$.

4. For $u, v \in R$, $\left(u - 2\frac{\langle u,v \rangle}{\langle v,v \rangle} v \right) \in R$.

Using axiom 4 we can define a linear mapping $t_v : E \mapsto E$ (see Figure 3.1) by

$$t_v(u) := u - 2\frac{\langle u, v \rangle}{\langle v, v \rangle} v, \forall u \in E.$$

For any vector $v \in E$ if we denote the hyperplane orthogonal to v by H_v then one can see that $\frac{\langle u,v \rangle}{\langle v,v \rangle} v$ is the projection of the vector u onto the line through v, consequently the map $t_v(u)$ produces the reflection of the vector u on the hyperplane H_v. In particular, $t_v(v) = -v$ and $t_v(u) = u$ for $u \in H_v$, therefore the mappings t_v's are reflections in E. We can now restate axiom 4 as—for $u, v \in R$, $t_v(u) \in R$.

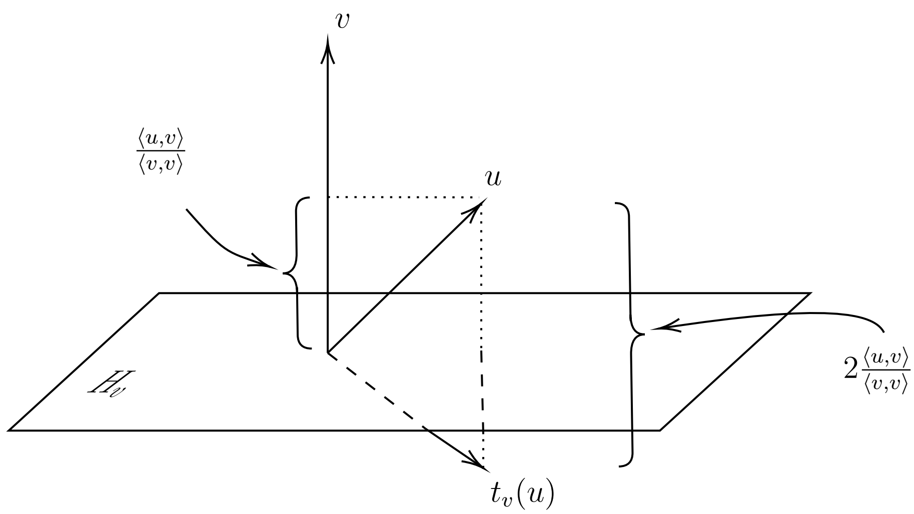


Figure 3.1. Reflection of u onto the hyperplane perpendicular to v.

Definition 3.1.2 (Weyl Group)**.** *Weyl group $W(R)$* is the reflection group generated by the reflections t_v, for all $v \in R$. Sometimes we may denote it by just W when the corresponding root system is understood.

Since all reflection groups are finite Coxeter groups therefore Weyl groups are also finite Coxeter groups consequently, Weyl groups can be classified by finite Coxeter groups—the *type of a root system* is determined by the type of finite Coxeter group it is isomorphic to. However, all finite Coxeter groups are not isomorphic to a Weyl group—there are no root systems of the type H_2, H_3, H_4 and $I_2(m)$ for $m \geq 7$. The root system of B_n and C_n have the same underlying Weyl group.

Theorem 3.1.3. *If u and v are two roots in R such that u is not a scalar multiple of v and $||u|| \leq ||v||$ then one of the following is true*

1. *The angle between u and v is $\frac{\pi}{2}$ and the ratio between $||u||$ and $||v||$ is unrestricted.*

2. *The angle between u and v is $\frac{\pi}{3}$ or $\frac{2\pi}{3}$ and $||u|| = ||v||$*

3. *The angle between u and v is $\frac{\pi}{4}$ or $\frac{3\pi}{4}$ and $||v|| = \sqrt{2}||u||$*

4. The angle between u and v is $\frac{\pi}{6}$ or $\frac{5\pi}{6}$ and $||v|| = \sqrt{3}||u||$

Proof. Using Schwartz inequality we get

$$\begin{aligned}\frac{2\langle u,v\rangle}{\langle v,v\rangle}\frac{2\langle v,u\rangle}{\langle u,u\rangle} &= \frac{4\langle u,v\rangle\langle v,u\rangle}{\langle v,v\rangle\langle u,u\rangle} \\ &= \frac{4\langle u,v\rangle^2}{||v||^2||u||^2} \leq 4.\end{aligned}$$

The equality implies that u and v are collinear, making them scalar multiples of each other (excluded by assumption). Since by axiom 4, $\frac{2\langle u,v\rangle}{\langle v,v\rangle}$ and $\frac{2\langle v,u\rangle}{\langle u,u\rangle}$ must be integers therefore $\frac{2\langle u,v\rangle}{\langle v,v\rangle}\frac{2\langle v,u\rangle}{\langle u,u\rangle} < 4$ implies $\frac{2\langle u,v\rangle}{\langle v,v\rangle}\frac{2\langle v,u\rangle}{\langle u,u\rangle} = 0, 1, 2$ or 3, consequently $4\cos^2\theta = 0, 1, 2$ or 3, where θ is the angle between u and v. Thus

$$\begin{array}{rclllll}\cos^2\theta &=& 0 & \frac{1}{4} & \frac{2}{4} & \frac{3}{4} \\ \cos\theta &=& 0 & \pm\frac{1}{2} & \pm\frac{1}{\sqrt{2}} & \pm\frac{\sqrt{3}}{2} \\ \theta &=& \frac{\pi}{2} & \frac{\pi}{3} \text{ or } \frac{2\pi}{3} & \frac{\pi}{4} \text{ or } \frac{3\pi}{4} & \frac{\pi}{6} \text{ or } \frac{5\pi}{6}\end{array}$$

Now we will look into the lengths $||u||$ and $||v||$, in particular, the relation between them. There are two possibilities, either $\frac{2\langle u,v\rangle}{\langle v,v\rangle}\frac{2\langle v,u\rangle}{\langle u,u\rangle} = 0$ or $\neq 0$. Clearly, if $\frac{2\langle u,v\rangle}{\langle v,v\rangle}\frac{2\langle v,u\rangle}{\langle u,u\rangle} = 0$, then the angle between them must be $\frac{\pi}{2}$ and the vectors u and v can be of any length. If however $\frac{2\langle u,v\rangle}{\langle v,v\rangle}\frac{2\langle v,u\rangle}{\langle u,u\rangle} \neq 0$ then by symmetry we may assume that $\frac{2\langle u,v\rangle}{\langle v,v\rangle} = \pm 1$ and $\frac{2\langle v,u\rangle}{\langle u,u\rangle} = \pm 1$, ± 2 or ± 3. Since $\frac{2\langle v,u\rangle}{\langle u,u\rangle}$ and $\frac{2\langle u,v\rangle}{\langle v,v\rangle}$ are either both positive or both negative therefore

$$\begin{aligned}&\frac{\frac{2\langle v,u\rangle}{\langle u,u\rangle}}{\frac{2\langle u,v\rangle}{\langle v,v\rangle}} = 1, 2, \text{ or } 3 \\ \Longrightarrow\ &\frac{\frac{2\langle v,u\rangle}{||u||^2}}{\frac{2\langle u,v\rangle}{||v||^2}} = \frac{||v||^2}{||u||^2} = 1, 2, \text{ or } 3 \\ \Longrightarrow\ &\underbrace{||v|| = ||u||}_{\theta=\frac{\pi}{3} \text{ or } \frac{2\pi}{3}}, \underbrace{||v|| = \sqrt{2}||u||}_{\theta=\frac{\pi}{4} \text{ or } \frac{3\pi}{4}} \text{ or } \underbrace{||v|| = \sqrt{3}||u||}_{\theta=\frac{\pi}{6} \text{ or } \frac{5\pi}{6}}.\end{aligned}$$

Thus we have proved all our claims. □

3.2 Bases

Definition 3.2.1 (Base)**.** Let E be the standard Euclidean space and R be a root system of E, a subset $\Delta \subset R$ is called a base of R if

1. Δ is a basis for E, as a vector space.

2. Any $v \in R$ can be expressed as $v = \sum_i k_i u_i$, where $u_i \in \Delta$ and k_i's are all of the same sign (or zero).

It is not obvious that such a base exists. However with the help of Theorem 3.2.5 we can show that a root system will always have a base. Let x be any vector in E such that x is not perpendicular to any of the roots in R. Such an x exists, because the total measure of all the hyperplanes corresponding to all the roots in R is 0. A root v will be called *x-positive* if $\langle v, x\rangle > 0$. In other words v is x-positive if v lies on the same side of the half-space as x, defined by the hyperplane perpendicular to x see Figure 3.2. For a given vector x in E which is not perpendicular to any r in R, we say, $v \in R$ is *x-indecomposable* if v is x-positive and v can't be written as a sum of other x-positive roots i.e. v can't be written as $\sum u_i$ where u_i's are x-positive roots.

Lemma 3.2.2. *If $u, v \in R$ for a root system R then*

1. $\frac{2\langle u,v\rangle}{\langle v,v\rangle} < 0$ *then $u + v$ is a root.*

2. $\frac{2\langle u,v\rangle}{\langle v,v\rangle} > 0$ *then $u - v$ is a root.*

Proof. As stated in the proof of Theorem 3.1.3, by symmetry we may assume that $\frac{2\langle u,v\rangle}{\langle v,v\rangle} = \pm 1$. Since $\langle v, v\rangle$ is positive therefore $\frac{2\langle u,v\rangle}{\langle v,v\rangle}$ has the same sign as $\langle u, v\rangle$. Additionally

$$t_v(u) = u - \underbrace{\frac{2\langle u, v\rangle}{\langle v, v\rangle}}_{=\pm 1} v = \begin{cases} u + v, & \text{if } \langle u, v\rangle < 0 \\ u - v, & \text{if } \langle u, v\rangle > 0 \end{cases}$$

$u + v$ and $u - v$ are in R by axiom 4. □

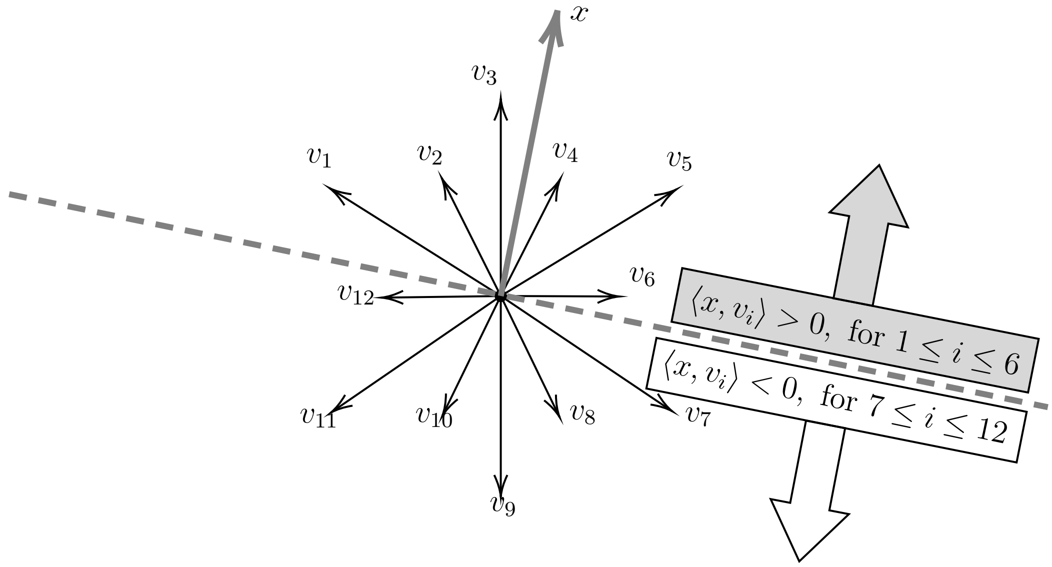


Figure 3.2. A root system and its x-positive roots.

Lemma 3.2.3. *Let v be an x-positive root in R, for a vector $x \in E$ not perpendicular to any root in R, v can be written as*

$$v = \sum u_i,$$

where u_i's are x-indecomposable. As a result, the set of x-positive roots span E.

Proof. Let S_x be the set of all those $r \in R$ such that r is x-positive. Let S_x^v be the subset of S_x corresponding to a root v in S_x such that if $u \in S_x^v$ then $\langle u, x \rangle < \langle v, x \rangle$. We will prove this through induction on $|S_x^v|$. For any $v \in S_x$, if $|S_x^v| = 0$, then v is x-indecomposable, thus the base case is true. Let us assume that our claim is true for any $v \in S_x$ if $|S_x^v| \leq k$. Now, if for some $v \in S_x$, $S_x^v = k + 1$, $v = u + w$ for some $u, v \in S_x$ then since $\langle u, x \rangle < \langle v, x \rangle$ and $\langle w, x \rangle < \langle v, x \rangle$ we have $|S_x^u| < |S_x^v|$ and $|S_x^w| < |S_x^v|$, consequently $S_x^u \leq k$ and $S_x^w \leq k$, thus using induction hypothesis we can say that u and w can be written as sum of x-indecomposable roots, as a result v can be written as a sum of x-indecomposable roots. Since any root in the half-space defined by the hyperplane perpendicular to x can be written as a sum of x-indecomposable roots therefore any vector v in the same half-space can also be written as a sum of these x-positive roots, and any vector in the other half-space can be written as a sum of negative x-positive roots. Therefore x-positive roots span E. $\square$

Lemma 3.2.4. *If u and v are x-indecomposable then* $\langle u, v\rangle \le 0$

Proof. Let x be any vector not perpendicular to any root in R. By Lemma 3.2.2 if $\langle u, v\rangle > 0$ then $r = u - v$ is a root, and thus either r or $-r$ is in the same half-space as x, i.e. either r or $-r$ is in x-positive. If r is x-positive then $u = v + r$, else if $-r$ is x-positive then $v = u + (-r)$, a contradiction, thus $\langle u, v\rangle \le 0$. $\square$

Theorem 3.2.5. *Every root system R has a base Δ.*

Proof. We will show that for any vector $x \in E$ not perpendicular to any of the roots in R, the set of x-indecomposable roots *is* the base for R. Let us denote the set of x-indecomposable roots by ∇.

We will first show that the set ∇ is linearly independent. If not then there exist non-zero k_i's such that $\sum k_i u_i = 0,\ u_i \in \nabla$. Since k_i's are non-zero therefore we can rewrite summation as

$$\sum a_i u_i = \sum b_j u_j,$$

where a_i's and b_j's are positive real numbers. Since u_i's are in ∇, therefore by Lemma 3.2.4

$$\left\langle \left(\sum a_i u_i\right), \left(\sum b_j u_j\right)\right\rangle = \sum a_i b_j \langle u_i, u_j\rangle \le 0.$$

Since $\sum a_i u_i = \sum b_j u_j$,

$$\left\langle \left(\sum a_i u_i\right), \left(\sum b_j u_j\right)\right\rangle = \left(\sum a_i u_i\right)^2 \ge 0$$

therefore $\sum a_i u_i = 0$, a contradiction. Thus ∇ is linearly independent. Additionally, by Lemma 3.2.3, ∇ spans E.

Now for the second condition in the definition of a base, observe that for a given vector x (not perpendicular to any of the roots in R) every x-positive root can be written as a sum of x-indecomposable roots, i.e. all the roots in the half-space defined by the hyperplane perpendicular to x can be expressed as $\sum u_i$, where u_i's are roots in the same half-space. Therefore any vector v in the same half-space can be expressed as $\sum k_i u_i$ where k_i's are all positive or zero, and any vector u in the other half-space can be expressed as $\sum \ell_i u_i$ where ℓ_i's are all negative or zero. $\square$

Definition 3.2.6 (Positive roots $R^+ \subset R$)**.** Fix a vector x in E such that x is not perpendicular to any root r in R. This fixes a base in R. A root $r \in R$ is positive if $\langle r, x \rangle > 0$.

Positive roots in a root system are not unique, it is dependent on the choice of a base in the root system. An equivalent but alternative definition avoiding the choice of a vector x is as follows:

Definition 3.2.7 (Positive roots $R^+ \subset R$ (Alternative definition))**.** For a root system R choose a base Δ of R. A root $r \in R$ is positive if r can be written as $r = \sum k_i u_i$ where k_i's are greater than or equal to 0 and u_i's are in Δ.

Definition 3.2.8 (Negative roots $R^- \subset R$)**.** For a root system R and a base $\Delta \subset R$, a root $r \in R$ is a negative root if it is not a positive root.

Definition 3.2.9 (Simple roots S)**.** The roots in the base of a root system are called simple roots.

The existence of such indecomposable roots in Δ is ascertained by Lemma 3.2.3. The set of simple roots correspond to the simple reflections in the corresponding Coxeter group. Instead of first developing a base for the root system one can first define the set of simple roots as follows.

Definition 3.2.10 (Simple roots (alternate definition))**.** For any root system R in the standard Euclidean space E, the set of hyperplanes H_v perpendicular to v for each v in R decompose E into finitely many simplical cones. Choose one of these simplical cones, the set of all outward normal roots to the facets of the chosen simplical cone are the simple roots.

The sum $\sum v_{s_i}$ where v_{s_i} are the simple roots, give us all the positive roots. Taking the negative of the positive roots give us the negative roots.

Definition 3.2.11 (Rank of a root system)**.** The rank of a root system is the number of simple roots in it.

Example 3.2.12. In the root system A_2, (see Item 2) for the choice of base $\Delta = \{u, v\}$, the simple roots are $S = \{u, v\}$, the positive roots are $R^+ = \{u, v, u+v\}$, the negative roots are $R^- = \{-u, -v, -(u+v)\}$ and the set of roots is $R = R^+ \cup R^-$.

3.3 Drawing all Rank-2 Root Systems.

Starting with any two non-collinear vectors in a root system R we can generate all the other roots in R simply by taking all the linear combinations with only the scalar multiples 1 and -1. We also know from Theorem 3.1.3 that only certain angles and length ratios can occur. Additionally since $\frac{2\langle u,v\rangle}{\langle v,v\rangle}\frac{2\langle v,u\rangle}{\langle u,u\rangle}$ is positive therefore $\frac{2\langle u,v\rangle}{\langle v,v\rangle}$ and $\frac{2\langle v,u\rangle}{\langle u,u\rangle}$ are either both positive or both negative therefore there are 6 cases to consider.

$$\begin{aligned} \tfrac{2\langle u,v\rangle}{\langle v,v\rangle} = 1 \text{ and } \tfrac{2\langle v,u\rangle}{\langle u,u\rangle} &= \quad 1, \quad 2 \quad \text{or} \quad 3 \\ \tfrac{2\langle u,v\rangle}{\langle v,v\rangle} = 1 \text{ and } \tfrac{2\langle v,u\rangle}{\langle u,u\rangle} &= -1, \ -2 \quad \text{or} \ -3 \end{aligned}$$

We can further reduce this to three cases with the following observation. If $w = t_v(u)$, i.e. $w = u - \frac{2\langle u,v\rangle}{\langle v,v\rangle}v = u - \frac{2\langle u,v\rangle}{\langle v,v\rangle}v$ then w is in the same R as u and v

$$\begin{aligned} \frac{2\langle w,v\rangle}{\langle v,v\rangle} &= \frac{2\left\langle u - \frac{2\langle u,v\rangle}{\langle v,v\rangle}v, v\right\rangle}{\langle v,v\rangle} \\ &= 2\frac{\langle u,v\rangle - \left\langle \frac{2\langle u,v\rangle}{\langle v,v\rangle}v, v\right\rangle}{\langle v,v\rangle} \\ &= 2\frac{\langle u,v\rangle - \frac{2\langle u,v\rangle}{\langle v,v\rangle}\langle v,v\rangle}{\langle v,v\rangle} \\ &= \frac{2\langle u,v\rangle - 4\langle u,v\rangle}{\langle v,v\rangle} \\ &= -\frac{2\langle u,v\rangle}{\langle v,v\rangle}. \end{aligned}$$

Therefore, whenever necessary, we can replace u by w and thus the only cases to consider are $\frac{2\langle v,u\rangle}{\langle u,u\rangle} = -1, -2$ or -3.

1. If $\frac{2\langle u,v\rangle}{\langle u,u\rangle} = 0$ then u and v are perpendicular and could be of any length. This gives us a $A_1 \times A_1$ type root system with a Dynkin diagram.

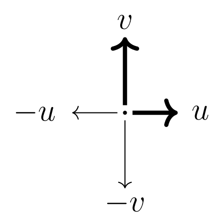


Figure 3.3. Root system $A_1 \times A_1$.

2. If $\frac{2\langle u,v\rangle}{\langle u,u\rangle} = -1$, then the angle between u and v is $2\pi/3$ and $||u|| = ||v||$. Additionally

$$t_u(v) = v - \frac{2\langle u, v\rangle}{\langle u, u\rangle}u = v + u.$$

This gives us a A_2 type root system.

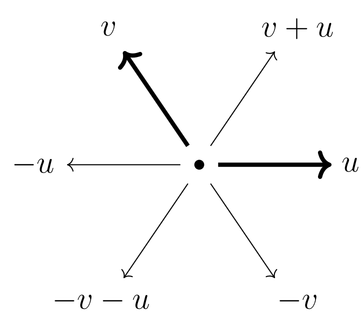
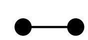


Figure 3.4. Root system A_2,  .

3. If $\frac{2\langle u,v\rangle}{\langle u,u\rangle} = -2$, then the angle between them is $3\pi/4$ and $||v|| = \sqrt{2}||u||$. And

$$t_u(v) = v - \frac{2\langle u, v\rangle}{\langle u, u\rangle}u = v + 2u.$$

This gives us a root system of type B_2 or C_2.

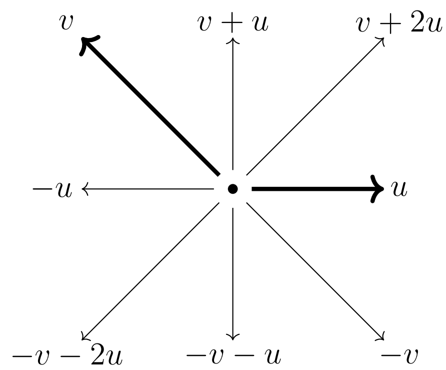
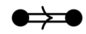


Figure 3.5. Root system B_2,  .

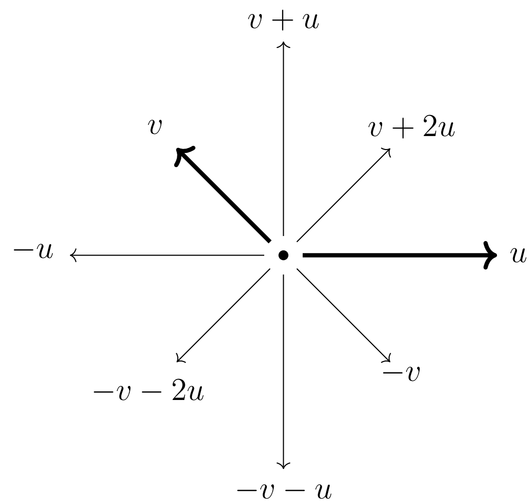
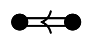


Figure 3.6. Root system C_2,  .

4. If $\frac{2\langle v,u\rangle}{\langle u,u\rangle} = -3$, then the angle between them is $5\pi/6$ and $||v|| = \sqrt{3}||u||$. Also

$$t_u(v) = v - \frac{2\langle v, u\rangle}{\langle u, u\rangle}u = v + 3u.$$

This gives us a root system of type G_2.

With the help of the background we reviewed so far we now prove an important result which we will later use in Chapter 6 in order to prove one of our main results. We will use the notation $s \between t$ to denote $sts = tst$ and the notation $s \parallel t$ to denote $st = ts$, for reflections s and t.

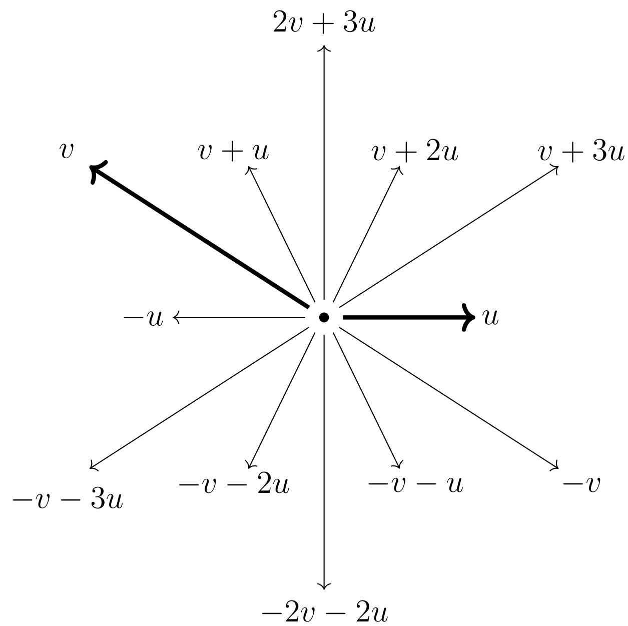


Figure 3.7. Root system G_2, [Dynkin diagram] .

Lemma 3.3.1. *Fix a finite, simply-laced Coxeter reflection group W, and let $t_1, \ldots, t_s$ be reflections occurring in a reduced T-decomposition of a Coxeter element such that $t_i \parallel t_j$ for $i \neq j \pm 1$ and $t_1 \between t_2 \between \cdots \between t_{s-1} \between t_s \between t_1$. Then $[t_1, t_2 \cdots t_s \cdots t_2] = e$.*

Proof. Pick roots $v_1, v_2, \ldots, v_s$ orthogonal to the hyperplanes corresponding to the reflections $t_1, \ldots, t_s$. These are linearly independent, since Coxeter elements have trivial fixed spaces (see, for example, (Athanasiadis et al., 2007, Lemma 3.11)). By replacing v_i with $-v_i$, we can assure that $t_i(v_{i+1}) = v_i + v_{i+1}$ for $1 \leq i < s-1$. On the other hand, we only know that $t_1(v_s)$ is either $v_s + v_1$ or $v_s - v_1$.

But we are assuming that $[t_1, t_2 \cdots t_s \cdots t_2] \neq e$, which may be written equivalently as $t_1 \cdots t_{s-1}(v_s) \neq t_2 \cdots t_{s-1}(v_s)$. We compute

$$t_2 \cdots t_{s-1}(v_s) = v_2 + \cdots v_s \neq$$

$$t_1(t_2 \cdots t_{s-1}(v_s)) = t_1(v_2 + \cdots + v_s) = v_1 + v_2 + \cdots + t_1(v_s),$$

so that $t_1(v_s) = v_s + v_1$.

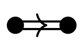
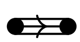
By comparing coefficients by linear independence, we observe that the powers

$$(t_1 t_2 \cdots t_s)^k (v_s)$$

produce an infinite number of distinct roots of W, contradicting the assumption that W was finite. $\square$

3.4 Dynkin Diagrams

Root systems are encoded by Dynkin diagrams which can be constructed as follows. If $R^+ \cap S$ denotes the set of positive roots in a given root system, then we construct a graph whose vertices are in bijection with the roots in $R^+ \cap S$ and the edges are decided as follows. If two roots in $R^+ \cap S$ are at an angle of

- $\pi/2$, then the corresponding vertices are non-adjacent (don't share an edge).
- $2\pi/3$, then there is an edge between the corresponding vertices ().
- $3\pi/4$, then there are 2 arrows between the corresponding vertices directed from the vertex corresponding to the long root to the vertex corresponding to the short root ().
- $5\pi/6$, then there are 3 arrows between the corresponding vertices directed from the vertex corresponding to the long root to the vertex corresponding to the short root ().

We don't have to consider any other angles between the roots because we know that these are the only angles that appear between roots in a root system. Note that though we can draw hyperplane arrangements corresponding to a Coxeter group of H_2 (with the angle between two hyperplanes $\pi/5$ or $4\pi/5$), but there is no root system with an angle of $4\pi/5$ between the simple roots.

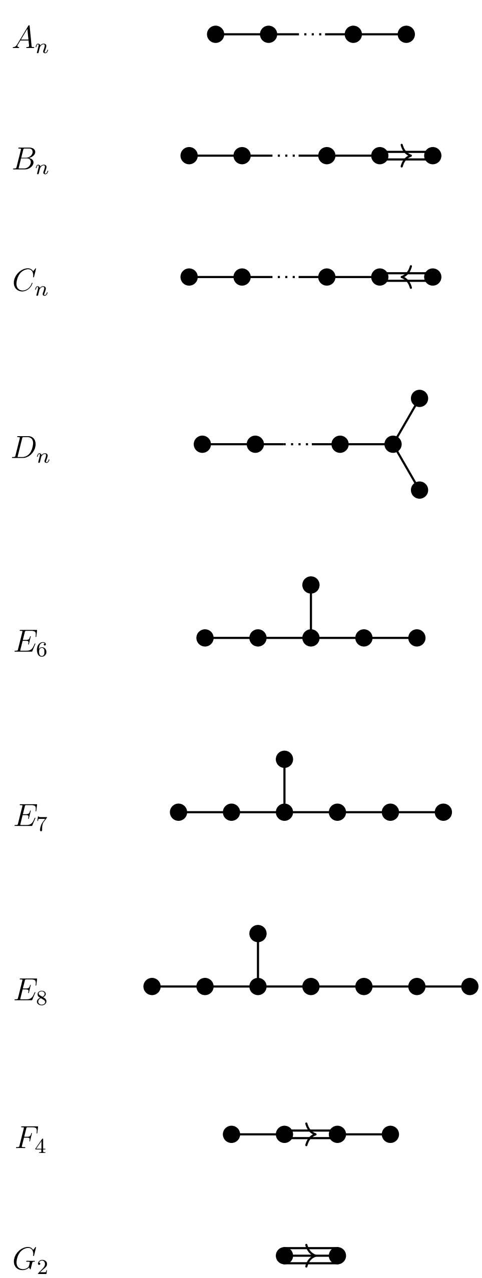


Figure 3.8. Dynkin diagrams illustrating the types of reduced root systems.

CHAPTER 4

DUAL COXETER SYSTEM AND HURWITZ MOVES

In this chapter we define Coxeter elements, and Hurwitz moves on reduced T-decompositions of Coxeter elements. Most of this material can be found in (Bessis, 2003). The main focus of this chapter will be to prove that reduced T-decompositions of a Coxeter element are connected under Hurwitz move. This will form the foundation of Chapter 5, where we will develop presentations for Artin groups encoded by reduced T-factorizations of a Coxeter element.

4.1 Coxeter Element and Dual Coxeter System

Definition 4.1.1. For an (abstract) reflection group (W, T) a *chromatic pair* is an ordered pair (L, R), $L, R \subset T$, such that $L \cap R = \phi$, the subgroups $\langle L \rangle$ and $\langle R \rangle$ are abelian, and $(W, L \cup R)$ is a Coxeter system.

Definition 4.1.2. For a given Coxeter system (W, S), a *Coxeter element c* is the product

$$c = \prod_{s \in S} s$$

where each simple reflection appears exactly once in the product.

Definition 4.1.3. Let $s_L := \prod_{s \in L} s$ and $s_R := \prod_{s \in R} s$ where (L, R) is a chromatic pair for a reflection group (W, T), then the elements of the form $c_{L,R} := s_L s_R$ are called *bipartite Coxeter elements*.

Definition 4.1.4. If (W, T) is a reflection group and c is a Coxeter element in (W, T) then we call the triple (W, T, c) a *dual Coxeter system*.

For a given abstract reflection group (W, T) one can choose a set $S \subset T$ such that S generates W and the conjugacy closure of S is T, then (W, S) forms a Coxeter system. In the "dual" approach (as discussed in (Bessis, 2003)) we instead choose a Coxeter element c (which is equivalent to choosing the generating set S in the classical approach) in order to obtain a dual Coxeter system (W, T, c) (which is equivalent to the Coxeter system (W, S)). Since all the Coxeter elements form a single conjugacy class (Lemma 1.7, (Reading, 2007)) we can define the following:

Definition 4.1.5. In a dual Coxeter system (W, T, c) the *Coxeter number* (denoted by h) is the order of any Coxeter element.

Lemma 4.1.6 (Steinberg)**.** *For an irreducible abstract reflection group (W, T), and a chromatic pair (L, R), the closure of $S := L \cap R$ under the conjugacy action of the Coxeter element $c_{L,R}$ is T. Additionally, if $\theta \subset T$ is an orbit of any reflection under the conjugation action of $c_{L,R}$ then either $|\theta| = h$ and $|\theta \cap S| = 2$ or $|\theta| = \frac{h}{2}$ and $|\theta \cap S| = 1$, where h is the Coxeter number.*

Proof. Let $L = \{s_1, \ldots, s_k\}$, $R = \{s_{k+1}, \ldots, s_n\}$ then an expression for $c_{L,R} = s_1 \cdots s_n$. $|\theta| \leq h$, since $c_{L,R}^h = e$. If s_i and s_j belong to the same orbit under conjugacy of $c_{L,R}$ then there exists an m such that $s_i c_{L,R}^m = c_{L,R}^m s_j$. Since $c_{L,R}^h t c_{L,R}^{-h} = t$ for any $t \in T$, therefore for any orbit θ, $|\theta| \leq h$, which in turn implies that $m \leq h$. We will show (through contradiction) that $m \geq \lfloor h/2 \rfloor$. If $m < \lfloor h/2 \rfloor$ then $(s_1 \cdots s_n)$ is a reduced S-word for the element $c_{L,R}^m$.

Let us first assume that $s_i \in L$ for some $i \leq k$ then $s_1 \cdots \hat{s}_i \cdots s_n (s_1 \cdots s_n)^{m-1}$ is a reduced S-word for $s_i c_{L,R}^m$, (where $\hat{s}_i$ stands for deleted s_i). Since $s_i c_{L,R}^m = c_{L,R}^m s_j$ therefore a reduced S-word for $c_{L,R}^m$ would be $(s_1 \cdots s_n)^{m-1} s_1 \cdots \hat{s}_j \cdots s_n$, (where $\hat{s}_j$ stands for deleted s_j) i.e. $s_j \in R$, (because if $s_j \notin R$ then $(s_1 \cdots s_n)^{m-1} s_1 \cdots s_n s_j$ would be a reduced S-word for $c_{L,R}^m s_j$ and $l_S(s_i c_{L,R}^m) \neq l_S(c_{L,R}^m s_j)$ which is a contradiction). Even though $s_1 \cdots \hat{s}_i \cdots s_n (s_1 \cdots s_n)^{m-1}$ and $(s_1 \cdots s_n)^{m-1} s_1 \cdots \hat{s}_j \cdots s_n$ are reduced S-words for the same element still s_i is initial in later expression but not in the former expression, which is a contradiction.

If we instead assume that $s_i \in R$ then $s_i(s_1 \cdots s_n)^m$ is a reduced S-word for $s_i c_{L,R}^m$. Since $s_i c_{L,R} = c_{L,R} s_j$ therefore the word $(s_1 \cdots s_n)^m s_j$ is a reduced S-word for $c_{L,R}$, which in turn implies that $s_j \in L$. Since we have assumed that $m \leq \lfloor h/2 \rfloor$ therefore

$$s_{k+1} \cdots s_n (s_1 \cdots s_n)^m s_1 \cdots s_k$$

is a reduced S-word for the element $c_{[R,L]}^{m+1}$. Since $s_i \in R$ and $s_j \in L$ therefore another reduced S-word for $c_{[R,L]}^{m+1}$ would be

$$s_{k+1} \cdots \hat{s}_i \cdots s_n s_i (s_1 \cdots s_n)^m s_j s_1 \cdots \hat{s}_j \cdots s_k.$$

Therefore the subword $s_i(s_1 \cdots s_n)^m s_j$ is a reduced expression. Now, since $s_i(s_1 \cdots s_n)^m = (s_1 \cdots s_n)^m s_j$ therefore $s_i(s_1 \cdots s_n)^m s_j = (s_1 \cdots s_n)^m$ which is a contradiction. Thus we have shown $m \geq \lfloor h/2 \rfloor$.

The reflections in the orbit of t can be written in an infinite sequence by conjugating t infinitely as

$$\theta_t = \left(t, c_{L,R} t c_{L,R}^{-1}, \ldots, c_{L,R}^{\{h-1\}} t c_{L,R}^{-\{h-1\}}, t, c_{L,R} t c_{L,R}^{-1}, \ldots, c_{L,R}^{\{h-1\}} t c_{L,R}^{-\{h-1\}}, \cdots \right).$$

where all the reflections in the orbit of t appear at least once in the first h terms of the sequence. Since $m \geq \lfloor h/2 \rfloor$ therefore at most 2 simple reflections appear in the first h terms of the sequence which gives us the inequality $\frac{|\theta|}{|\theta \cap S|} \geq h/2$ and $|\theta \cap S| = 1$ or 2 for any orbit θ. If for a particular orbit θ, $|\theta \cap S| = 2$ since $|\theta| \leq h$ we must have $|\theta| = h$. If $|\theta \cap S| = 1$ then it follows that $h/2 \leq |\theta| \leq h$. Since $\frac{|T|}{|S|} = \frac{h}{2}$ and each of the orbits maintain this orbit the same ratio therefore the closure of S is T.

We can conclude that the orbit containing only 1 simple reflection, contains $h/2$ reflection in them. □

The next lemma tells us how we can obtain a Coxeter system from a given dual Coxeter system.

Lemma 4.1.7. *For a given dual Coxeter system* (W, T, c) *of rank* n *and* $t \in T$ *there exists a chromatic pair* (L, R) *such that* $c = c_{L,R}$ *and* $t \in L$*. In particular there exists a reduced* T*-word* $\mathsf{c} = \mathsf{t}_1 \cdots \mathsf{t}_n$ *for the Coxeter element* c *such that* $t_1 = t$ *and* $(W, \{t_1, \ldots, t_n\})$ *is a Coxeter system.*

Proof. Let (L, R) with $L = \{l_1, \ldots, l_i\}$ and $R = \{r_1, \ldots, r_j\}$ be a chromatic pair such that $c = c_{L,R}$. For any $t \in T$, by Lemma 4.1.6, $t = c^k t' c^{-k}$ with $t' \in L \cup R$. If $t' \in L$ then the chromatic pair $\left(c^k L c^{-k}, c^k R c^{-k}\right)$ is such that $t \in c^k L c^{-k}$ and $c = c_{c^k L c^{-k}, c^k R c^{-k}}$. Now assume $t' \in R$. An expression $l_1 \cdots l_i r_1 \cdots r_j$ for the Coxeter element $c_{L,R}$ can be rewritten as:

$$\begin{aligned}
c_{L,R} &= l_1 \cdots l_i r_1 \cdots r_j \\
&= r_1 r_1^{-1} l_1 r_1 \cdots r_1^{-1} l_i r_1 \cdots r_j \\
&= r_1 r_2 r_2^{-1} r_1^{-1} l_1 r_1 r_2 \cdots r_2^{-1} r_1^{-1} l_i r_1 \cdots r_j \\
&\qquad \vdots \qquad\quad \vdots \qquad\quad \vdots \\
&= r_1 \cdots r_j (r_j^{-1} \cdots r_1^{-1} l_1 r_1 \cdots r_j) \cdots (r_j^{-1} \cdots r_1^{-1} l_i r_1 \cdots r_j) \\
&= s_R s_R^{-1} l_1 s_R \cdots s_R^{-1} l_i s_R \\
&= c_{R, s_R^{-1} L s_R}
\end{aligned}$$

Thus, with the new chromatic pair $(R, s_R^{-1} L s_R)$, $t \in R$ and $c = c_{R, s_R^{-1} L s_R}$ □

4.2 Parabolic Subgroup and Parabolic Coxeter Elements

Definition 4.2.1. Let (W, T) be an abstract reflection group, and (W, S) be the corresponding Coxeter group with S as the set of simple reflections that generate W. For any $I \subset S$ the subgroup (W_I, T_I) is a *parabolic subgroup* of the abstract reflection group (W, T) where $W_I := \langle I \rangle$ and $T_I := T \cap I$. An element $w \in W$ will be called a *parabolic Coxeter element* if w is a Coxeter element in some parabolic subgroup (W_I, T_I).

Lemma 4.2.2. *For any dual Coxeter system* (W, T, c) *the element* tc *where* $t \in T$ *is a parabolic Coxeter element.*

Proof. By Lemma 4.1.7, there exists a chromatic pair (L, R) such that $L = \{l_1, \ldots, l_i\}$, $R = \{r_1, \ldots, r_j\}$ and c has a $T-$decomposition $l_1 \cdots l_i r_1 \cdots r_j$ with $l_1 = t$. Therefore $\hat{t} l_2 \cdots l_i r_1 \cdots r_j$ is a reduced T-decomposition of tc. Therefore the ordered pair $(L \setminus t, R)$ is a chromatic pair in the Coxeter system $(W, L \setminus t \cup R)$ which implies that tc is a Coxeter element in the subgroup generated by $L \setminus t \cup R$, consequently, tc is a parabolic Coxeter element. ☐

Lemma 4.2.3. *Let* (W, S) *be a Coxeter system, there exists a Coxeter element* c *such that* $w \leq_T c$ *if and only if* w *is a parabolic Coxeter element.*

Proof. If w is a parabolic Coxeter element then clearly $w \leq_T c$ because there exists a reduced T-decomposition of $w = (w_1 \cdots w_k)$ and a reduced T-decomposition of $c = (c_1 \cdots c_l)$ such that $\{w_1, \ldots, w_k\} \subset \{c_1, \ldots, c_l\}$

Now if $w \leq_T c$ then w can be written as $wv = c$ for some $v \in W$ with $l_T(w) + l_T(v) = l_T(c)$. This also implies that there exists some $v' \in W$ such that $v'w = c$ with $l_T(v') + l_T(w) = l_T(c)$. Using induction on $l_T(v')$ we will show that $v'c$ is a parabolic Coxeter element where $l_T(v'c) = l_T(c) - l_T(v')$. If $l_T(v') = 1$ then $v'c$ is a parabolic Coxeter element by Lemma 4.2.2. For the inductive hypothesis we assume that $v'c$ is a parabolic Coxeter element if $l_T(v') = n$. If $l_T(v') = n + 1$ then $v = tu$ where $l_T(t) = 1$ and $l_T(u) = n$. Since $l_T(u) = n$ therefore uc is a parabolic Coxeter element. Since $tu \leq_T c$ and $t \not\leq_T u$ thus $t \leq_T uc$, which implies that tuc is a parabolic Coxeter element. ☐

4.3 Hurwitz Moves

Definition 4.3.1. Let (W, S) be a Coxeter system and let T be the set of reflections. For a positive integer n consider $\boldsymbol{B}_n$—the braid group on n strands with generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$.

Hurwitz action is a group action of $\boldsymbol{B}_n$ on T^n defined as,

$$\sigma_i(t_1, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots t_n) \mapsto (t_1, \ldots, t_{i-1}, t_{i+1}^{t_i}, t_i, \ldots, t_n)$$

In a similar way we define *Hurwitz move* on a word

$$\mu_i(\mathsf{t}_1 \cdots \mathsf{t}_{i-1}\mathsf{t}_i\mathsf{t}_{i+1} \cdots \mathsf{t}_n) \mapsto (\mathsf{t}_1 \cdots \mathsf{t}_{i-1}\mathsf{t}_{i+1}^{\mathsf{t}_i}\mathsf{t}_i \cdots \mathsf{t}_n)$$

We can use Hurwitz move to 'alter' a given reduced T-decomposition of a Coxeter element. Let c be a Coxeter element of W and let $\mathsf{t}_1 \cdots \mathsf{t}_i\mathsf{t}_{i+1} \cdots \mathsf{t}_n$ be a reduced T-decomposition of c. Performing a *Hurwitz move* at t_i on $\mathsf{c} = \mathsf{t}_1 \cdots \mathsf{t}_i\mathsf{t}_{i+1} \cdots \mathsf{t}_n$ gives a new T-decomposition $\mathsf{c}' = \mathsf{t}_1 \cdots \mathsf{t}_{i+1}^{\mathsf{t}_i}\mathsf{t}_i \cdots \mathsf{t}_n$, which corresponds to the Hurwitz action $\sigma_i(t_1, \ldots, t_i, t_{i+1}, \ldots, t_n)$. Since the product map $T^n \to W$ defined as $(t_1, \ldots, t_n) \mapsto \prod_{i=1}^n t_i$ is invariant under the Hurwitz action therefore Hurwitz moves only produce different reduced T-factorizations of a given Coxeter element without affecting the Coxeter element.

Example 4.3.2. Let us perform a Hurwitz move on the reduced T-decomposition (12)(23)(34) of a Coxeter element in $\mathfrak{S}_3$, at (23).

$$\begin{aligned} c &= (12)(23)(34) \\ &= (12)(23)(34)(23)(23) \\ &= (12)(24)(23) \end{aligned}$$

Lemma 4.3.3. *If* $\mathsf{t}_1 \cdots \mathsf{t}_n$ *is a reduced* T*-decomposition of a Coxeter element* c*, then the decomposition*

$$\mathsf{t}_1^{c^k} \cdots \mathsf{t}_n^{c^k}$$

is also a reduced T*-decomposition of* c *and is in the orbit of* $\mathsf{t}_1 \cdots \mathsf{t}_n$ *under Hurwitz move.*

Proof. First we will show (using induction) that in a Coxeter system of rank m performing m successive rotation on the leftmost reflection of a reduced T-decomposition $\mathsf{t}_1 \cdots \mathsf{t}_n$ of a Coxeter element c gives a new reduced T-decomposition of c in which ct_nc^{-1} is the right most reflection.

For the base case consider the Coxeter element $c = t_1t_2$ and the reduced T-decomposition $\mathsf{c} = \mathsf{t}_1\mathsf{t}_2$. On rotating t_1 we get $\mathsf{t}_2^{\mathsf{t}_1}\mathsf{t}_1$. On rotating $t_2^{t_1}$ we get $\mathsf{t}_1^{\left(\mathsf{t}_2^{\mathsf{t}_1}\right)}\mathsf{t}_2^{\mathsf{t}_1}$. Here, the right most reflection in the factorization $\mathsf{t}_1^{\left(\mathsf{t}_2^{\mathsf{t}_1}\right)}\mathsf{t}_2^{\mathsf{t}_1}$ is:

$$t_2^{t_1} = t_1t_2t_1^{-1} = t_1t_2t_2t_2^{-1}t_1^{-1} = ct_2c^{-1}$$

Now consider a Coxeter element $c_{n-1} = t_1t_2 \cdots t_{n-1}$ in a rank $n-1$ Coxeter system. On performing $n-1$ rotations on the leftmost reflection (successively) of the T-decomposition $\mathsf{t}_1\mathsf{t}_2 \cdots \mathsf{t}_{n-1}$ we obtain a new T-decomposition of c_{n-1} which can be written as:

$$\mathsf{t}_1^{P_1}\mathsf{t}_2^{P_2} \cdots {\mathsf{t}_{n-1}}^{P_{n-1}}$$

For our induction hypothesis we assume that $P_{n-1}t_{n-1} = c_{n-1}$, which would imply $t_{n-1}^{P_{n-1}} = t_{n-1}^{c_{n-1}}$. Observe that this is true in the base case where $P_{n-1} \equiv t_1$ and $t_{n-1} \equiv t_2$

Now for the inductive step consider a Coxeter element $c_n = t_1t_2 \cdots t_{n-1}t_n$ obtained by concatenating t_n to the right of c_{n-1}. Performing n successive rotations on the leftmost reflection of the reduced T-decomposition $\left({\mathsf{t}_n}^{P_{n-1}}\right)^{\left({\mathsf{t}_{n-1}}^{P_{n-1}}\right)}{\mathsf{t}_1}^{P_1} \cdots {\mathsf{t}_{n-1}}^{P_{n-1}}$ we obtain (using the same notation as in the $n-1$ case):

$$\left({\mathsf{t}_n}^{P_{n-1}}\right)^{\left({\mathsf{t}_{n-1}}^{P_{n-1}}\right)}{\mathsf{t}_1}^{P_1} \cdots {\mathsf{t}_{n-1}}^{P_{n-1}}$$

where

$$\begin{aligned}
\left(t_n{}^{P_{n-1}}\right)^{\left(t_{n-1}{}^{P_{n-1}}\right)} &= \left(P_{n-1}t_{n-1}P_{n-1}^{-1}\right)\left(P_{n-1}t_nP_{n-1}^{-1}\right)\left(P_{n-1}t_{n-1}P_{n-1}^{-1}\right)^{-1} \\
&= P_{n-1}t_{n-1}t_nt_{n-1}^{-1}P_{n-1}^{-1} \\
&= P_{n-1}t_{n-1}t_n\underbrace{\left(t_nt_n^{-1}\right)}_{=e}t_{n-1}^{-1}P_{n-1}^{-1} \\
&= \underbrace{P_{n-1}t_{n-1}}_{=c_{n-1}}t_nt_nt_n^{-1}\,\underbrace{t_{n-1}^{-1}P_{n-1}^{-1}}_{=c_{n-1}^{-1}} \text{ (using induction hypothesis)} \\
&= \underbrace{c_{n-1}t_n}_{=c_n}t_n\,\underbrace{t_n^{-1}c_{n-1}^{-1}}_{=c_n^{-1}} \\
&= c_nt_nc_n^{-1}
\end{aligned}$$

Now, we will show (again using induction) that in a Coxeter system of rank m, and a Coxeter element $c = t_1 \cdots t_m$, performing m successive rotations on the leftmost reflection of the reduced T-factorization $\mathsf{t}_1 \cdots \mathsf{t}_m$ gives $\mathsf{t}_1^{\mathsf{c}} \cdots \mathsf{t}_m^{\mathsf{c}}$.

First consider the Coxeter element $c = t_1t_2$ and the reduced T-decomposition $\mathsf{t}_1\mathsf{t}_2$. On rotating t_1 we get $\mathsf{t}_2^{\mathsf{t}_1}\mathsf{t}_1$. Now again on rotating $t_2^{t_1}$ we get $\mathsf{t}_1^{\left(\mathsf{t}_2^{\mathsf{t}_1}\right)}\mathsf{t}_2^{\mathsf{t}_1}$.

$$t_1^{\left(t_2^{t_1}\right)} = t_1^{t_1t_2t_1^{-1}} = t_1t_2t_1^{-1}t_1t_1t_2^{-1}t_1^{-1} = t_1t_2t_1t_2^{-1}t_1^{-1} = ct_1c^{-1}.$$
$$t_2^{t_1} = t_1t_2t_1^{-1} = t_1t_2t_2t_2^{-1}t_1^{-1} = ct_2c^{-1}.$$

Let $c_{n-1} = t_1t_2 \cdots t_{n-1}$ be a Coxeter element of rank $n-1$ Coxeter system and let $\mathsf{t}_1\mathsf{t}_2 \cdots \mathsf{t}_{n-1}$ be a reduced T-decomposition of c. On performing rotations on the left most reflection, successively for $n-1$ times the final factorization can be written as:

$$\mathsf{t}_1^{P_1}\mathsf{t}_2^{P_2} \cdots \mathsf{t}_{n-1}{}^{P_{n-1}}$$

We assume here that $t_k^{P_k} = c_{n-1}t_kc_{n-1}^{-1}$, for each $0 < k < n-1$. With this as our induction hypothesis we will show that in a rank n Coxeter system, if $c_n = t_1t_2 \cdots t_{n-1}t_n$ is a Coxeter element then performing similar successive rotations on the left most reflection of the reduced T-factorization $\mathsf{t}_1\mathsf{t}_2\cdots\mathsf{t}_{n-1}\mathsf{t}_n$ for n times would give us $\mathsf{t}_1^{c_n}\mathsf{t}_2^{c_n}\cdots\mathsf{t}_{n-1}^{c_n}\mathsf{t}_n^{c_n}$.

If we just concatenate the reflection t_n to the right of the Coxeter element c_{n-1}, we obtain a Coxeter element c_n. Using the same notation as in the $n-1$ case we obtain the following factorization:

$$\left(\mathsf{t}_1^{P_1}\right)^{\left(\mathsf{t}_n^{P_n}\right)}\left(\mathsf{t}_2^{P_2}\right)^{\left(\mathsf{t}_n^{P_n}\right)}\cdots\left(\mathsf{t}_{n-1}{}^{P_{n-1}}\right)^{\left(\mathsf{t}_n^{P_n}\right)}\mathsf{t}_n^{P_n}$$

By induction hypothesis $t_k^{P_k}=c_{n-1}t_kc_{n-1}^{-1}$, $0<k<n$ and using the fact that $t_n^{P_n}=c_nt_nc_n^{-1}$ (from the first part of the proof) we have:

$$\begin{aligned}
\left(t_k^{P_k}\right)^{\left(t_n^{P_n}\right)} &= \left(c_{n-1}t_kc_{n-1}^{-1}\right)^{\left(c_nt_nc_n^{-1}\right)}\\
&= \left(c_nt_nc_n^{-1}\right)\left(c_{n-1}t_kc_{n-1}^{-1}\right)\left(c_nt_nc_n^{-1}\right)^{-1}\\
&= c_nt_n\underbrace{c_n^{-1}c_{n-1}}_{t_n^{-1}}t_k\underbrace{c_{n-1}^{-1}c_n}_{t_n}t_n^{-1}c_n^{-1}\\
&= c_n\underbrace{t_nt_n^{-1}}_{e}t_k\underbrace{t_nt_n^{-1}}_{e}c_n^{-1}\\
&= c_nt_kc_n^{-1}.
\end{aligned}$$

□

Theorem 4.3.4 (Proposition 1.6.1 in (Bessis, 2003))**.** *The set of reduced T-decompositions for a Coxeter element c is transitive under the action of Hurwitz moves.*

Proof. Let (W,S) be a Coxeter system of rank n with T being the set of reflections. We will prove this through induction on the rank n. It is vacuously true for Parabolic subgroups of rank 1. For the induction hypothesis we assume that the set of reduced T-decompositions for a Coxeter element in a parabolic Coxeter group of rank $n-1$ is transitive under the action of $\boldsymbol{B}_{n-1}$. Now let c be a Coxeter element in W. We can write

$$\bigcup_{t\in T} t\cdot \mathrm{Red}_T(tc)=\mathrm{Red}_T(c)$$

where $t \cdot \mathrm{Red}_T(tc)$ denotes the set of reduced T-words for tc with t concatenated to the front of each of those words. By Lemma 4.2.2, tc is a Coxeter element in a parabolic Coxeter group of rank $n-1$ therefore $\mathrm{Red}_T(tc)$ is transitive under the action of $\boldsymbol{B}_{n-1}$. If $u \in \mathrm{Red}_T(tc)$ then $t \cdot u \in \mathrm{Red}_T(c)$. Now since the $\boldsymbol{B}_{n-1}$-orbit of u contains the entire $\mathrm{Red}_T(tc)$ therefore the $\boldsymbol{B}_n$-orbit of $t \cdot u$ contains the entire $t \cdot \mathrm{Red}_T(tc)$ (since $\boldsymbol{B}_{n-1}$ is a restriction of $\boldsymbol{B}_n$). Thus it suffices to show that there exists an element $c' \in \mathrm{Red}_T(c)$ such that its $\boldsymbol{B}_n$-orbit contains at least one factorization of the form $t.u$ for every $t \in T$. Now let $c' = s_1 s_2 \cdots s_n$ be a reduced T-decomposition of c then $(s_1, \ldots, s_n) \in \mathrm{Red}_T(c)$. It is easy to see that $\sigma_1^{-1} \cdots \sigma_{i-1}^{-1}(s_1, \ldots, s_n)$ starts with the reflection s_i. Using Lemma 4.3.3 we can say that $((\sigma_{n_1} \cdots \sigma_1)^{nk} \sigma_1^{-1} \cdots \sigma_{i-1}^{-1})(s_1, \ldots, s_n)$ starts with $c^k s_i c^k$. By Lemma 4.1.6 every $t \in T$ can be written in the form $c^k s_i c^{-k}$ therefore we have demonstrated a series of Hurwitz moves on c' that will enable us to have a factorization that starts with our choice of $t \in T$. This completes the proof. □

In summary, for a given Coxeter group W and a Coxeter element $c \in W$, there exists a set $S \subset W$ such that (W, S) is a Coxeter system and c is a bipartite Coxeter element in (W, S). If a Coxeter element c' is not a bipartite Coxeter element in the Coxeter system (W, S) then we can find a different set $S' \subset W$ such that c' is a bipartite Coxeter element in (W, S'). Since the set of reduced T-decompositions for a Coxeter element c' is transitive under Hurwitz moves, therefore the reduced S'-decomposition of c' can be obtained from the reduced S-decomposition of c' by simply performing Hurwitz moves on it.

CHAPTER 5

A PRESENTATION FROM REDUCED T-DECOMPOSITIONS OF COXETER ELEMENTS.

This chapter introduces the main result—presentations for Artin groups arising from reduced T-decompositions of a Coxeter element.

5.1 The Presentation

Let W be a simply-laced Weyl group and let (W, S) be the corresponding Coxeter system. Let c be a Coxeter element of W and let $c = t_1 t_2 \cdots t_n$ be a reduced T-decomposition of c. We define a group presentation

$$\boldsymbol{B}(t_1, t_2, \ldots, t_n) := \langle \boldsymbol{t}_1, \boldsymbol{t}_2, \ldots, \boldsymbol{t}_n \big| \mathrm{Rel}(t_1, \ldots, t_n) \rangle_{\mathrm{group}} \tag{5.1}$$

where $\boldsymbol{t}_1, \boldsymbol{t}_2, \cdots, \boldsymbol{t}_n$ are a formal copy of the reflections $t_1, t_2, \cdots, t_n$, subject to the following relations

$$\mathrm{Rel}(t_1, \ldots, t_n) := \begin{cases} \boldsymbol{t}_i \boldsymbol{t}_j = \boldsymbol{t}_j \boldsymbol{t}_i & \text{if } t_i \parallel t_j, \\ \boldsymbol{t}_i \boldsymbol{t}_j \boldsymbol{t}_i = \boldsymbol{t}_j \boldsymbol{t}_i \boldsymbol{t}_j & \text{if } t_i \between t_j, \\ [\boldsymbol{t}_{i_1}, \boldsymbol{t}_{i_2} \cdots \boldsymbol{t}_{i_s} \cdots \boldsymbol{t}_{i_2}^{-1}] = e & \text{if } t_{i_1} \between t_{i_2} \between \cdots \between t_{i_{s-1}} \between t_{i_s} \between t_{i_1}, \\ & t_{i_j} \parallel t_{i_k} \text{ for } k \neq j-1, j+1. \end{cases} \tag{5.2}$$

Since we can perform Hurwitz moves on a reduced T-decomposition of c to obtain new reduced T-decompositions of c, therefore there are as many group presentations as the cardinality of the orbit of $(t_1, t_2, \ldots, t_n)$ under Hurwitz action (by Theorem 4.3.4 all the reduced T-decompositions of c are in the same orbit under Hurwitz action). Let $u_1 u_2 \cdots u_n$ be another reduced T decomposition of c obtained by performing Hurwitz move on $t_1 t_2 \cdots t_n$ at t_k, i.e. $\mu_k(t_1 t_2 \cdots t_n) = u_1 u_2 \cdots u_n$. We have the following group presentation

$$\boldsymbol{B}(u_1, u_2, \ldots, u_n) = \langle \boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n : \mathrm{Rel}(u_1, \ldots, u_n) \rangle_{\mathrm{group}}.$$

where

$$\mathrm{Rel}(u_1, u_2, \ldots, u_n) = \begin{cases} \boldsymbol{u}_i\boldsymbol{u}_j = \boldsymbol{u}_j\boldsymbol{u}_i & \text{if } u_i \parallel u_j, \\ \boldsymbol{u}_i\boldsymbol{u}_j\boldsymbol{u}_i = \boldsymbol{u}_j\boldsymbol{u}_i\boldsymbol{u}_j & \text{if } u_i \between u_j, \\ [\boldsymbol{u}_{i_1}, \boldsymbol{u}_{i_2} \cdots \boldsymbol{u}_{i_s} \cdots \boldsymbol{u}_{i_2}^{-1}] = e & \text{if } u_{i_1} \between u_{i_2} \between \cdots \between u_{i_{s-1}} \between u_{i_s} \between u_{i_1}, \\ & u_{i_j} \parallel u_{i_k} \text{ for } k \neq j-1, j+1. \end{cases}$$

Using Lemma 3.3.1 we can rewrite the last relation as

$$[\boldsymbol{u}_{i_1}, \boldsymbol{u}_{i_2} \cdots \boldsymbol{u}_{i_s} \cdots \boldsymbol{u}_{i_2}^{-1}] = e \text{ if } (u_{i_1}u_{i_2} \cdots\ u_{i_{s-1}}u_{i_s}u_{i_{s-1}} \cdots u_{i_2})^2 = e.$$

Since

$$t_1 \cdots t_{k-1}t_{k+1}^{t_k}t_k \cdots t_n = u_1 \cdots u_{k-1}u_ku_{k+1} \cdots u_n$$

therefore, $t_k = u_{k+1}$, $t_{k+1}^{t_k} = u_k$ and $u_i = t_i$ when $i \neq k$ or $k+1$. Using these we also get $t_{k+1} = u_{k+1}^{-1}u_ku_{k+1}$.

Theorem 5.1.1. *Let $t_1t_2\cdots t_n$ be a reduced T-decomposition of a Coxeter element c in a Coxeter group W. If $u_1u_2\cdots u_n$ be another reduced T-decomposition of c such that $\sigma_k(t_1, t_2, \ldots, t_n) = (u_1, u_2, \ldots, u_n)$. If $\boldsymbol{B}(t_1, t_2, \ldots, t_n)$ and $\boldsymbol{B}(u_1, u_2, \ldots, u_n)$ are the groups whose presentations are obtained using Equation* (5.1) *from the decompositions $t_1t_2\cdots t_n$ and $u_1u_2\cdots u_n$ respectively, then*

$$\boldsymbol{B}(t_1, t_2, \ldots, t_n) \cong \boldsymbol{B}(u_1, u_2, \ldots, u_n).$$

Proof. Define a map $\phi_k : \boldsymbol{B}(t_1, t_2, \ldots, t_n) \to \boldsymbol{B}(u_1, u_2, \ldots, u_n)$ by

$$\phi_k(\boldsymbol{t}_i) := \begin{cases} \boldsymbol{u}_{k+1} & \text{if } i = k \\ \boldsymbol{u}_{k+1}^{-1}\boldsymbol{u}_k\boldsymbol{u}_{k+1} & \text{if } i = k+1 \\ \boldsymbol{u}_i & \text{otherwise} \end{cases}$$

The definition of ϕ_k is motivated from how t_i and u_i are related in W, where $\sigma_k(t_1, t_2, \ldots, t_n) = (u_1, u_2, \ldots, u_n)$. We will show that the groups $\boldsymbol{B}(t_1, t_2, \ldots, t_n)$ and $\boldsymbol{B}(u_1, u_2, \ldots, u_n)$ are isomorphic by showing that ϕ_k is a bijective homomorphism. The proof for homomorphism will be shown in Chapter 6. For the bijection we will show that ϕ_k is invertible. Define $\psi_k : \boldsymbol{B}(u_1, u_2, \ldots, u_n) \to \boldsymbol{B}(t_1, t_2, \ldots, t_n)$ by[1]

$$\psi_k(\boldsymbol{u}_i) = \begin{cases} \boldsymbol{t}_k \boldsymbol{t}_{k+1} \boldsymbol{t}_k^{-1} & \text{if } i = k \\ \boldsymbol{t}_k & \text{if } i = k+1 \\ \boldsymbol{t}_i & \text{otherwise} \end{cases}$$

We will now show that ψ_k is the inverse of ϕ_k by showing that $\phi_k \circ \psi_k$ and $\psi_k \circ \phi_k$ are identity maps. Let $i \neq k, k+1$, then $\phi_k \circ \psi_k(\boldsymbol{u}_i) = \phi_k(\boldsymbol{t}_i) = \boldsymbol{u}_i$. For $i = k$ we obtain

$$\begin{aligned} \phi_k \circ \psi_k(\boldsymbol{u}_k) &= \phi_k(\boldsymbol{t}_k \boldsymbol{t}_{k+1} \boldsymbol{t}_k^{-1}) \\ &= \phi_k(\boldsymbol{t}_k)\phi_k(\boldsymbol{t}_{k+1})\phi_k(\boldsymbol{t}_k^{-1}) \\ &= \boldsymbol{u}_{k+1}\boldsymbol{u}_{k+1}^{-1}\boldsymbol{u}_k\boldsymbol{u}_{k+1}\boldsymbol{u}_{k+1}^{-1} \\ &= \boldsymbol{u}_k. \end{aligned}$$

Now for $i = k+1$

$$\begin{aligned} \phi_k \circ \psi_k(\boldsymbol{u}_{k+1}) &= \phi_k(\boldsymbol{t}_k) \\ &= \boldsymbol{u}_{k+1}. \end{aligned}$$

Therefore $\phi_k \circ \psi_k$ is the identity map of $\boldsymbol{B}(u_1, u_2, \ldots, u_n)$.

Next we will check $\psi_k \circ \phi_k$ is the identity map of $\boldsymbol{B}(t_1, t_2, \ldots, t_n)$. Again we check for $i \neq k, k+1$ and obtain $\psi_k \circ \phi_k(\boldsymbol{t}_i) = \psi_k(\boldsymbol{u}_i) = \boldsymbol{t}_i$. For $i = k$ we obtain

---

[1] This definition is motivated from the inverse Hurwitz action,

$$\sigma_k^{-1}(u_1, \ldots, u_{k-1}, u_k, u_{k+1}, \ldots, u_n) = (u_1, \ldots, u_{k-1}, u_{k+1}, {}^{u_{k+1}}u_k, \ldots, u_n)$$

Setting $(t_1, \ldots, t_{k-1}, t_k, t_{k+1}, \ldots, t_n) = (u_1, \ldots, u_{k-1}, u_{k+1}, {}^{u_{k+1}}u_k, \ldots, u_n)$ we get $u_{k+1} = t_k$, ${}^{u_{k+1}}u_k = t_{k+1}$ and $t_i = u_i$ when $i \neq k$ or $k+1$. Using these we also get $u_k = t_k t_{k+1} t_k^{-1}$.

$$\begin{aligned}\psi_k \circ \phi_k(\boldsymbol{t}_k) &= \psi_k(\boldsymbol{u}_{k+1}) \\ &= \boldsymbol{t}_k.\end{aligned}$$

When $i = k+1$, we obtain

$$\begin{aligned}\psi_k \circ \phi_k(\boldsymbol{t}_{k+1}) &= \psi_k(\boldsymbol{u}_{k+1}^{-1}\boldsymbol{u}_k\boldsymbol{u}_{k+1}) \\ &= \psi_k(\boldsymbol{u}_{k+1}^{-1})\psi_k(\boldsymbol{u}_k)\psi_k(\boldsymbol{u}_{k+1}) \\ &= \boldsymbol{t}_k^{-1}\boldsymbol{t}_k\boldsymbol{t}_{k+1}\boldsymbol{t}_k^{-1}\boldsymbol{t}_k^{-1} \\ &= \boldsymbol{t}_{k+1}\end{aligned}$$

Therefore, $\psi_k \circ \phi_k$ is the identity map of $\boldsymbol{B}(t_1, t_2, \ldots, t_n)$, and consequently ϕ_k is a bijection. This, along with the homomorphism implies that ϕ_k is an isomorphism from $\boldsymbol{B}(t_1, t_2, \ldots, t_n)$ to $\boldsymbol{B}(u_1, u_2, \ldots, u_n)$, so $\boldsymbol{B}(t_1, t_2, \ldots, t_n) \cong \boldsymbol{B}(u_1, u_2, \ldots, u_n)$ and our proof is complete. □

This isomorphism has been illustrated through Figure 5.1.

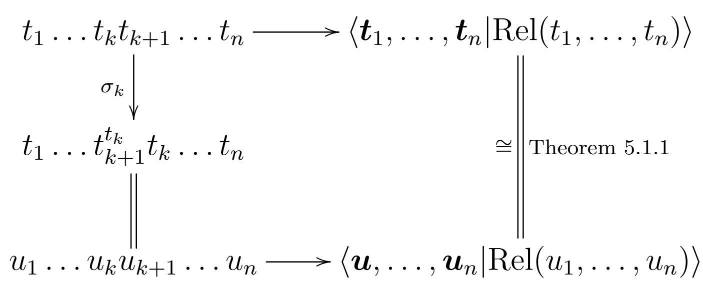


Figure 5.1. Diagram illustrating isomorphism between the two groups defined by presentations arising from reduced T-decompositions of Coxeter elements.

This brings us to our main theorem:

Theorem 5.1.2. *Let c be a Coxeter element of a simply-laced Weyl group W and suppose*

$$(t_1, t_2, \ldots, t_n)$$

is a reduced T-factorization of c. If

$$\boldsymbol{B}(t_1, t_2, \ldots, t_n) := \langle \boldsymbol{t}_1, \boldsymbol{t}_2, \ldots, \boldsymbol{t}_n \big| \mathrm{Rel}(t_1, \ldots, t_n) \rangle$$

where

$$\mathrm{Rel}(t_1, \ldots, t_n) = \begin{cases} \boldsymbol{t}_i\boldsymbol{t}_j = \boldsymbol{t}_j\boldsymbol{t}_i & \text{if } t_i \parallel t_j, \\ \boldsymbol{t}_i\boldsymbol{t}_j\boldsymbol{t}_i = \boldsymbol{t}_j\boldsymbol{t}_i\boldsymbol{t}_j & \text{if } t_i \between t_j, \\ [\boldsymbol{t}_{i_1}, \boldsymbol{t}_{i_2} \cdots \boldsymbol{t}_{i_s} \cdots \boldsymbol{t}_{i_2}^{-1}] = e & \text{if } t_{i_1} \between t_{i_2} \between \cdots \between t_{i_{s-1}} \between t_{i_s} \between t_{i_1}, \\ & t_{i_j} \parallel t_{i_k} \text{ for } k \neq j-1, j+1. \end{cases}$$

then

$$\boldsymbol{B}(t_1, t_2, \ldots, t_n) \cong \boldsymbol{B}(W).$$

Proof. Using induction along with Theorem 5.1.1, we can say that $\boldsymbol{B}(t_1, \ldots, t_n)$ and $\boldsymbol{B}(t'_1, \ldots, t'_n)$ are isomorphic if the two reduced T-decompositions—$t_1 \cdots t_n$ and $t'_1 \cdots t'_n$—of the Coxeter element c, are in a single orbit under Hurwitz move. By Theorem 4.3.4 all reduced T-decompositions of the Coxeter element c are in the same orbit under Hurwitz move, in particular any reduced S-decomposition of c, say $s_1 \cdots s_n$ is also in the same orbit therefore the group $\boldsymbol{B}(t_1, \ldots, t_n)$ is isomorphic to $\boldsymbol{B}(s_1, \ldots, s_n)$, consequently $\boldsymbol{B}(t_1, \ldots, t_n)$ is isomorphic to $\boldsymbol{B}(W)$. Notice that our presentation for $\boldsymbol{B}(s_1, \ldots, s_n)$ is exactly the same as the Artin-braid presentation $\boldsymbol{B}(W)$ because there are no cycles of the form $s_{i_1} \between s_{i_2} \between \cdots \between s_{i_k} \between s_{i_1}$ in (W, S) (W being simply-laced) where $s_{i_1}, s_{i_2}, \ldots, s_{i_n}$ are in S. □

CHAPTER 6

PROOF OF HOMOMORPHISM

For a finite, simply-laced Coxeter group W, and a Coxeter element $c \in W$ with a reduced T-decomposition $(t_1, t_2, \ldots, t_n)$, one can write a group presentation using Equation (5.1)

$$\boldsymbol{B}(t_1, t_2, \ldots, t_n) = \langle \boldsymbol{t}_1, \boldsymbol{t}_2, \ldots, \boldsymbol{t}_n : \mathrm{Rel}(t_1, \ldots, t_n) \rangle_{\text{group}}$$

and a presentation

$$\boldsymbol{B}(u_1, u_2, \ldots, u_n) = \langle \boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n : \mathrm{Rel}(u_1, \ldots, u_n) \rangle_{\text{group}}$$

where

$$\sigma_k(t_1, t_2, \ldots, t_n) = (u_1, u_2, \ldots, u_n)$$

which implies

$$t_i = \begin{cases} u_{k+1} & \text{if } i = k \\ u_{k+1}^{-1} u_k u_{k+1} & \text{if } i = k+1 \\ u_i & \text{otherwise} \end{cases}$$

or alternatively

$$u_i = \begin{cases} t_k t_{k+1} t_k^{-1} & \text{if } i = k \\ t_k & \text{if } i = k+1 \\ t_i & \text{otherwise.} \end{cases}$$

Define a map $\phi_k : \boldsymbol{B}(t_1, t_2, \ldots, t_n) \to \boldsymbol{B}(u_1, u_2, \ldots, u_n)$ by

$$\phi_k(\boldsymbol{t}_i) := \begin{cases} \boldsymbol{u}_{k+1} & \text{if } i = k \\ \boldsymbol{u}_{k+1}^{-1} \boldsymbol{u}_k \boldsymbol{u}_{k+1} & \text{if } i = k+1 \\ \boldsymbol{u}_i & \text{otherwise.} \end{cases}$$

Theorem 6.0.1. *ϕ_k is a homomorphism from $\boldsymbol{B}(t_1, t_2, \ldots, t_n)$ to $\boldsymbol{B}(u_1, u_2, \ldots, u_n)$.*

Proof. Define $\widetilde{\boldsymbol{t}}_i \in \boldsymbol{B}(u_1, u_2, \ldots, u_n)$

$$\tilde{\boldsymbol{t}}_i = \begin{cases} \boldsymbol{u}_{k+1} & i = k \\ \boldsymbol{u}_{k+1}^{-1}\boldsymbol{u}_k\boldsymbol{u}_{k+1} & i = k+1 \\ \boldsymbol{u}_i & \text{otherwise} \end{cases}$$

We will show that for any relation $R(\boldsymbol{t}_1, \ldots, \boldsymbol{t}_n)$ in $\mathrm{Rel}(t_1, \ldots, t_n)$, the relation $R(\widetilde{\boldsymbol{t}_1}, \ldots, \widetilde{\boldsymbol{t}_n})$ holds in $\boldsymbol{B}(u_1, u_2, \ldots, u_n)$. Let us first get the trivial cases out of the way. If $t_k t_{k+1} = t_{k+1} t_k$ then $t_i = u_i$ for all the i's, thus there is nothing to show. Therefore in this chapter we assume $t_k \between t_{k+1}$.

Another trivial case is when $\boldsymbol{t}_{k+1}$ is not involved in a relation. If $R(\boldsymbol{t}_1, \boldsymbol{t}_2, \ldots, \boldsymbol{t}_n) \in \mathrm{Rel}(t_1, \ldots, t_n)$ be a relation such that $\boldsymbol{t}_{k+1}$ is not involved then

$$R(\boldsymbol{t}_1, \ldots, \boldsymbol{t}_k, \boldsymbol{t}_{k+1}, \ldots, \boldsymbol{t}_n) = R(\boldsymbol{t}_1, \ldots, \boldsymbol{t}_k, \boldsymbol{e}, \ldots \boldsymbol{t}_n),$$

and also the relation $R(t_1, \ldots, t_k, t_{k+1}, \ldots t_n)$ holds in W, therefore

$$R(t_1, \ldots, t_k, t_{k+1}, \ldots t_n) = R(t_1, \ldots, t_k, e, \ldots, t_n).$$

Since

$$R(t_1, \ldots, t_k, t_{k+1}, \ldots t_n) = R(u_1, \ldots, u_{k-1}, u_{k+1}, u_{k+1}^{-1} u_k u_{k+1}, u_{k+2} \ldots, u_n)$$

and

$$R(t_1, \ldots, t_k, e, \ldots, t_n) = R(u_1, \ldots, u_{k-1}, u_{k+1}, e, u_{k+2}, \ldots, u_n)$$

therefore

$$R(u_1, \ldots, u_{k-1}, u_{k+1}, u_{k+1}^{-1} u_k u_{k+1}, u_{k+2} \ldots, u_n)$$

is the same as

$$R(u_1, \ldots, u_{k-1}, u_{k+1}, e, u_{k+2}, \ldots, u_n).$$

Consequently

$$R(\widetilde{\boldsymbol{t}}_1, \ldots, \widetilde{\boldsymbol{t}}_{k-1}, \widetilde{\boldsymbol{t}}_k, \widetilde{\boldsymbol{t}}_{k+1}, \widetilde{\boldsymbol{t}}_{k+2} \ldots, \widetilde{\boldsymbol{t}}_n)$$

holds in $\boldsymbol{B}(u_1, u_2, \ldots, u_n)$.

Therefore it suffices to show,

1. $\boldsymbol{t}_i\boldsymbol{t}_j = \boldsymbol{t}_j\boldsymbol{t}_i \implies \widetilde{\boldsymbol{t}}_i\widetilde{\boldsymbol{t}}_j = \widetilde{\boldsymbol{t}}_j\widetilde{\boldsymbol{t}}_i$

$\boldsymbol{t}_{k+1}\boldsymbol{t}_i = \boldsymbol{t}_i\boldsymbol{t}_{k+1} \implies \widetilde{\boldsymbol{t}}_{k+1}\widetilde{\boldsymbol{t}}_i = \widetilde{\boldsymbol{t}}_i\widetilde{\boldsymbol{t}}_{k+1}$ where $i \neq k$. See Lemma 6.0.3.

2. $\boldsymbol{t}_i\boldsymbol{t}_j\boldsymbol{t}_i = \boldsymbol{t}_j\boldsymbol{t}_i\boldsymbol{t}_j \implies \widetilde{\boldsymbol{t}}_i\widetilde{\boldsymbol{t}}_j\widetilde{\boldsymbol{t}}_i = \widetilde{\boldsymbol{t}}_j\widetilde{\boldsymbol{t}}_i\widetilde{\boldsymbol{t}}_j$

Case 1: $\boldsymbol{t}_i\boldsymbol{t}_{k+1}\boldsymbol{t}_i = \boldsymbol{t}_{k+1}\boldsymbol{t}_i\boldsymbol{t}_{k+1} \implies \widetilde{\boldsymbol{t}}_i\widetilde{\boldsymbol{t}}_{k+1}\widetilde{\boldsymbol{t}}_i = \widetilde{\boldsymbol{t}}_{k+1}\widetilde{\boldsymbol{t}}_i\widetilde{\boldsymbol{t}}_{k+1}$, where $i \neq k$. See Lemma 6.0.4.

Case 2: $\boldsymbol{t}_k\boldsymbol{t}_{k+1}\boldsymbol{t}_k = \boldsymbol{t}_{k+1}\boldsymbol{t}_k\boldsymbol{t}_{k+1} \implies \widetilde{\boldsymbol{t}}_k\widetilde{\boldsymbol{t}}_{k+1}\widetilde{\boldsymbol{t}}_k = \widetilde{\boldsymbol{t}}_{k+1}\widetilde{\boldsymbol{t}}_k\widetilde{\boldsymbol{t}}_{k+1}$. See Lemma 6.0.5.

3. $[\boldsymbol{t}_{i_1}, \boldsymbol{t}_{i_2}\cdots\boldsymbol{t}_{i_s}\cdots\boldsymbol{t}_{i_2}^{-1}] = e \implies [\widetilde{\boldsymbol{t}}_{i_1}, \widetilde{\boldsymbol{t}}_{i_2}\cdots\widetilde{\boldsymbol{t}}_{i_s}\cdots\widetilde{\boldsymbol{t}}_{i_2}^{-1}] = e$. See Lemma 6.0.6.

Case I: $[\boldsymbol{t}_k, \boldsymbol{t}_{k+1}\cdots\boldsymbol{t}_{i_s}\cdots\boldsymbol{t}_{k+1}^{-1}] = e$

$\implies [\widetilde{\boldsymbol{t}}_k, \widetilde{\boldsymbol{t}}_{k+1}\cdots\widetilde{\boldsymbol{t}}_{i_s}\cdots\widetilde{\boldsymbol{t}}_{k+1}^{-1}] = e$ (t_k and t_{k+1} appear right at the beginning)

Case II: $[\boldsymbol{t}_{i_1}, \boldsymbol{t}_{i_2}\cdots\boldsymbol{t}_k\boldsymbol{t}_{k+1}\cdots\boldsymbol{t}_{i_s}\cdots\boldsymbol{t}_{k+1}^{-1}\boldsymbol{t}_k^{-1}\cdots\boldsymbol{t}_{i_2}] = e$

$\implies [\widetilde{\boldsymbol{t}}_{i_1}, \widetilde{\boldsymbol{t}}_{i_2}\cdots\widetilde{\boldsymbol{t}}_k\widetilde{\boldsymbol{t}}_{k+1}\cdots\widetilde{\boldsymbol{t}}_{i_s}\cdots\widetilde{\boldsymbol{t}}_{k+1}^{-1}\widetilde{\boldsymbol{t}}_k^{-1}\cdots\widetilde{\boldsymbol{t}}_{i_2}] = e$ (t_k and t_{k+1} don't appear at the beginning or at the end, but somewhere in between)

Case III: $[\boldsymbol{t}_{i_1}, \boldsymbol{t}_{i_2}\cdots\boldsymbol{t}_k\boldsymbol{t}_{k+1}\boldsymbol{t}_k^{-1}\cdots\boldsymbol{t}_{i_2}^{-1}] = e$

$\implies [\widetilde{\boldsymbol{t}}_{i_1}, \widetilde{\boldsymbol{t}}_{i_2}\cdots\widetilde{\boldsymbol{t}}_k\widetilde{\boldsymbol{t}}_{k+1}\widetilde{\boldsymbol{t}}_k^{-1}\cdots\widetilde{\boldsymbol{t}}_{i_2}^{-1}] = e$ (t_k and t_{k+1} appear at the end)

Case IV: $[\boldsymbol{t}_{k+1}, \boldsymbol{t}_{i_2}\cdots\boldsymbol{t}_{i_s}\cdots\boldsymbol{t}_{i_2}^{-1}] = e$

$\implies [\widetilde{\boldsymbol{t}}_{k+1}, \widetilde{\boldsymbol{t}}_{i_2}\cdots\widetilde{\boldsymbol{t}}_{i_s}\cdots\widetilde{\boldsymbol{t}}_{i_2}^{-1}] = e$ (t_{k+1} appears at the beginning and t_k is absent)

Case V: $[\boldsymbol{t}_{i_1}, \boldsymbol{t}_{i_2}\cdots\boldsymbol{t}_{k+1}\cdots\boldsymbol{t}_{i_s}\cdots\boldsymbol{t}_{k+1}\cdots\boldsymbol{t}_{i_2}^{-1}] = e$

$\implies [\widetilde{\boldsymbol{t}}_{i_1}, \widetilde{\boldsymbol{t}}_{i_2}\cdots\widetilde{\boldsymbol{t}}_{k+1}\cdots\widetilde{\boldsymbol{t}}_{i_s}\cdots\widetilde{\boldsymbol{t}}_{k+1}\cdots\widetilde{\boldsymbol{t}}_{i_2}^{-1}] = e$ (t_{k+1} doesn't appear at the beginning or at the end, but somewhere in between and t_k is absent)

Case VI: $[\boldsymbol{t}_{i_1}, \boldsymbol{t}_{i_2}\cdots\boldsymbol{t}_{k+1}\cdots\boldsymbol{t}_{i_2}^{-1}] = e$

$\implies [\widetilde{\boldsymbol{t}}_{i_1}, \widetilde{\boldsymbol{t}}_{i_2}\cdots\widetilde{\boldsymbol{t}}_{k+1}\cdots\widetilde{\boldsymbol{t}}_{i_2}^{-1}] = e$ (t_{k+1} appears at the end and t_k is absent) □

The next lemma will help us in the proofs of the following lemmas.

Lemma 6.0.2. $\boldsymbol{t}_i\boldsymbol{t}_j^{-1}\boldsymbol{t}_k\boldsymbol{t}_j = \boldsymbol{t}_j^{-1}\boldsymbol{t}_k\boldsymbol{t}_j\boldsymbol{t}_i \iff \boldsymbol{t}_i\boldsymbol{t}_j\boldsymbol{t}_k\boldsymbol{t}_j^{-1} = \boldsymbol{t}_j\boldsymbol{t}_k\boldsymbol{t}_j^{-1}\boldsymbol{t}_i$.

Proof.

$$\begin{aligned}
&\boldsymbol{t}_i\boldsymbol{t}_j\boldsymbol{t}_k\boldsymbol{t}_j^{-1} = \boldsymbol{t}_j\boldsymbol{t}_k\boldsymbol{t}_j^{-1}\boldsymbol{t}_i\\
\implies &\boldsymbol{t}_i\boldsymbol{t}_j\boldsymbol{t}_k\boldsymbol{t}_j^{-1}\boldsymbol{t}_i^{-1} = \boldsymbol{t}_j\boldsymbol{t}_k\boldsymbol{t}_j^{-1}\\
\implies &\boldsymbol{t}_i\boldsymbol{t}_j\boldsymbol{t}_k\boldsymbol{t}_j^{-1}\boldsymbol{t}_i^{-1} = \boldsymbol{t}_j\boldsymbol{t}_k\boldsymbol{t}_i\boldsymbol{t}_j\boldsymbol{t}_j^{-1}\boldsymbol{t}_i^{-1}\boldsymbol{t}_j^{-1}\\
\implies &\boldsymbol{t}_i\boldsymbol{t}_j\boldsymbol{t}_k\boldsymbol{t}_j^{-1}\boldsymbol{t}_i^{-1} = \boldsymbol{t}_j\boldsymbol{t}_k\boldsymbol{t}_i\boldsymbol{t}_j\boldsymbol{t}_i^{-1}\boldsymbol{t}_j^{-1}\boldsymbol{t}_i^{-1}\\
\implies &\boldsymbol{t}_i\boldsymbol{t}_j\boldsymbol{t}_k = \boldsymbol{t}_j\boldsymbol{t}_k\boldsymbol{t}_i\boldsymbol{t}_j\boldsymbol{t}_i^{-1}\\
\implies &\boldsymbol{t}_i\boldsymbol{t}_j\boldsymbol{t}_k\boldsymbol{t}_i = \boldsymbol{t}_j\boldsymbol{t}_k\boldsymbol{t}_i\boldsymbol{t}_j\\
\implies &\boldsymbol{t}_j^{-1}\boldsymbol{t}_i\boldsymbol{t}_j\boldsymbol{t}_k\boldsymbol{t}_i = \boldsymbol{t}_k\boldsymbol{t}_i\boldsymbol{t}_j\\
\implies &\boldsymbol{t}_j^{-1}\boldsymbol{t}_i\boldsymbol{t}_j\boldsymbol{t}_k = \boldsymbol{t}_k\boldsymbol{t}_i\boldsymbol{t}_j\boldsymbol{t}_i^{-1}\\
\implies &\boldsymbol{t}_j^{-1}\boldsymbol{t}_i\boldsymbol{t}_j\boldsymbol{t}_k = \boldsymbol{t}_k\boldsymbol{t}_j^{-1}\boldsymbol{t}_i\boldsymbol{t}_j
\end{aligned}$$

□

Lemma 6.0.3. $\boldsymbol{t}_{k+1}\boldsymbol{t}_i = \boldsymbol{t}_i\boldsymbol{t}_{k+1} \implies \widetilde{\boldsymbol{t}}_{k+1}\widetilde{\boldsymbol{t}}_i = \widetilde{\boldsymbol{t}}_i\widetilde{\boldsymbol{t}}_{k+1}$ *where* $i \neq k$.

Proof. Since $\boldsymbol{t}_{k+1}\boldsymbol{t}_i = \boldsymbol{t}_i\boldsymbol{t}_{k+1}$ therefore $t_{k+1} \parallel t_i$. There are two cases to consider $t_i \between t_k$ and $t_i \parallel t_k$.

Case I: Assume $t_i \between t_k$. First, we observe that since $u_i = t_i$ and $u_{k+1} = t_k$, thus $u_{k+1}u_i = u_iu_{k+1} \implies t_it_k = t_kt_i$, a contradiction, thus we get

$$u_i \between u_{k+1} \tag{6.1}$$

Next, we observe that

$$\begin{aligned}
u_ku_{k+1} &= t_kt_{k+1}t_k^{-1}t_k = t_kt_{k+1},\\
u_{k+1}u_k &= t_kt_kt_{k+1}t_k^{-1} = t_{k+1}t_k.
\end{aligned}$$

Therefore $u_ku_{k+1} = u_{k+1}u_k \implies t_kt_{k+1} = t_{k+1}t_k$, a contradiction, thus get

$$u_{k+1} \between u_k \tag{6.2}$$

And finally, $u_ku_i = u_iu_k \implies t_kt_{k+1}t_k^{-1}t_i = t_it_kt_{k+1}t_k^{-1}$

$$
\begin{aligned}
&\implies t_i t_k t_{k+1} t_k = t_k t_{k+1} t_k t_i \\
&\implies t_i t_k t_{k+1} = t_k t_{k+1} t_k t_i t_k \\
&\implies t_i t_k t_{k+1} = t_k t_{k+1} t_i t_k t_i \\
&\implies t_k t_i t_k t_{k+1} = t_{k+1} t_i t_k t_i \\
&\implies t_i t_k t_i t_{k+1} = t_i t_{k+1} t_k t_i \\
&\implies t_k t_{k+1} t_i = t_{k+1} t_k t_i \\
&\implies t_k t_{k+1} = t_{k+1} t_k.
\end{aligned}
$$

Therefore $u_k u_i = u_i u_k \implies t_i t_k = t_i t_k$, a contradiction, thus we get

$$u_k \between u_i \tag{6.3}$$

Equation (6.1), Equation (6.2) and Equation (6.3) together with Lemma 6.0.2 imply

$$
\begin{aligned}
&u_i \between u_{k+1} \between u_k \between u_i \\
\implies &[\boldsymbol{u}_i, \boldsymbol{u}_{k+1}^{-1} \boldsymbol{u}_k \boldsymbol{u}_{k+1}] = e \\
\implies &\boldsymbol{u}_i \boldsymbol{u}_{k+1}^{-1} \boldsymbol{u}_k \boldsymbol{u}_{k+1} = \boldsymbol{u}_{k+1}^{-1} \boldsymbol{u}_k \boldsymbol{u}_{k+1} \boldsymbol{u}_i \\
\implies &\widetilde{\boldsymbol{t}}_i \widetilde{\boldsymbol{t}}_{k+1} = \widetilde{\boldsymbol{t}}_{k+1} \widetilde{\boldsymbol{t}}_i
\end{aligned}
$$

Case II: Assume $t_i \parallel t_k$. We observe that

$$
\begin{aligned}
\widetilde{\boldsymbol{t}}_{k+1} \widetilde{\boldsymbol{t}}_i &= \boldsymbol{u}_{k+1}^{-1} \boldsymbol{u}_k \boldsymbol{u}_{k+1} \boldsymbol{u}_i, \\
\widetilde{\boldsymbol{t}}_i \widetilde{\boldsymbol{t}}_{k+1} &= \boldsymbol{u}_i \boldsymbol{u}_{k+1}^{-1} \boldsymbol{u}_k \boldsymbol{u}_{k+1}
\end{aligned}
$$

therefore to show that $\widetilde{\boldsymbol{t}}_{k+1} \widetilde{\boldsymbol{t}}_i = \widetilde{\boldsymbol{t}}_i \widetilde{\boldsymbol{t}}_{k+1}$, it suffices to show that $\boldsymbol{u}_k$ and $\boldsymbol{u}_i$ commute and $\boldsymbol{u}_{k+1}$ and $\boldsymbol{u}_i$ commute. It is clear that $\boldsymbol{u}_{k+1}$ and $\boldsymbol{u}_i$ commute as $u_{k+1} = t_k$ and $u_i = t_i$ and by assumption $t_k t_i = t_i t_k$. So we only need to show that $\boldsymbol{u}_k$ and $\boldsymbol{u}_i$ commute. We will do so by showing that u_i and u_k commute.

$$\begin{aligned} u_k u_i &= t_k t_{k+1} t_k t_i \\ &= t_i t_k t_{k+1} t_k \\ &= u_i u_k \end{aligned}$$

As t_i commutes with both t_{k+1} and t_k. Therefore, $\boldsymbol{u}_i$ commutes with both $\boldsymbol{u}_{k+1}$ and $\boldsymbol{u}_k$. So $\widetilde{\boldsymbol{t}}_{k+1}\widetilde{\boldsymbol{t}}_i = \widetilde{\boldsymbol{t}}_i\widetilde{\boldsymbol{t}}_{k+1}$ when t_k and t_i commute. □

Lemma 6.0.4. $\boldsymbol{t}_i\boldsymbol{t}_{k+1}\boldsymbol{t}_i = \boldsymbol{t}_{k+1}\boldsymbol{t}_i\boldsymbol{t}_{k+1} \implies \widetilde{\boldsymbol{t}}_i\widetilde{\boldsymbol{t}}_{k+1}\widetilde{\boldsymbol{t}}_i = \widetilde{\boldsymbol{t}}_{k+1}\widetilde{\boldsymbol{t}}_i\widetilde{\boldsymbol{t}}_{k+1}$ *where* $t_i \neq t_k$.

Proof. Since $\boldsymbol{t}_i\boldsymbol{t}_{k+1}\boldsymbol{t}_i = \boldsymbol{t}_{k+1}\boldsymbol{t}_i\boldsymbol{t}_{k+1}$ therefore $t_i \between t_{k+1}$.

Case I: Assume $t_k \between t_i$. First, we will argue that $\boldsymbol{u}_i$ and $\boldsymbol{u}_k$ commute by showing that u_k and u_i commute. Since $u_i u_k = t_i t_k t_{k+1} t_k$ and $u_k u_i = t_k t_{k+1} t_k t_i$ it suffices to show that t_i and $t_k t_{k+1} t_k$ commute. Now using our assumptions $t_k \between t_{k+1}$ and $t_k \between t_i$ we have $t_i \between t_k \between t_{k+1} \between t_i$, thus invoking Lemma 3.3.1 we get $[t_i, t_k t_{k+1} t_k] = e$, consequently $\boldsymbol{u}_i$ and $\boldsymbol{u}_k$ commute.

Next we note that $\boldsymbol{u}_i$ and $\boldsymbol{u}_{k+1}$ do not commute since $u_i = t_i$ and $u_{k+1} = t_k$ and we assumed that $t_k \between t_i$ thus we have $u_i \between u_{k+1}$, consequently $\boldsymbol{u}_i$ and $\boldsymbol{u}_{k+1}$ do not commute.

Using these two results we obtain,

$$\begin{aligned} \widetilde{\boldsymbol{t}}_{k+1}\widetilde{\boldsymbol{t}}_i\widetilde{\boldsymbol{t}}_{k+1} &= \boldsymbol{u}_{k+1}^{-1}\boldsymbol{u}_k\boldsymbol{u}_{k+1}\boldsymbol{u}_i\boldsymbol{u}_{k+1}^{-1}\boldsymbol{u}_k\boldsymbol{u}_{k+1} \\ &= \boldsymbol{u}_k\boldsymbol{u}_{k+1}\boldsymbol{u}_k^{-1}\boldsymbol{u}_i\boldsymbol{u}_k\boldsymbol{u}_{k+1}\boldsymbol{u}_k^{-1} \\ &= \boldsymbol{u}_k\boldsymbol{u}_{k+1}\boldsymbol{u}_i\boldsymbol{u}_{k+1}\boldsymbol{u}_k^{-1} \\ &= \boldsymbol{u}_k\boldsymbol{u}_i\boldsymbol{u}_{k+1}\boldsymbol{u}_i\boldsymbol{u}_k^{-1} \end{aligned}$$

Similarly

$$\begin{aligned} \widetilde{\boldsymbol{t}}_i\widetilde{\boldsymbol{t}}_{k+1}\widetilde{\boldsymbol{t}}_i &= \boldsymbol{u}_i\boldsymbol{u}_{k+1}^{-1}\boldsymbol{u}_k\boldsymbol{u}_{k+1}\boldsymbol{u}_i \\ &= \boldsymbol{u}_i\boldsymbol{u}_k\boldsymbol{u}_{k+1}\boldsymbol{u}_k^{-1}\boldsymbol{u}_i \\ &= \boldsymbol{u}_k\boldsymbol{u}_i\boldsymbol{u}_{k+1}\boldsymbol{u}_i\boldsymbol{u}_k^{-1} \end{aligned}$$

Therefore, $\widetilde{\boldsymbol{t}}_i\widetilde{\boldsymbol{t}}_{k+1}\widetilde{\boldsymbol{t}}_i = \widetilde{\boldsymbol{t}}_{k+1}\widetilde{\boldsymbol{t}}_i\widetilde{\boldsymbol{t}}_{k+1}$.

Case II: Assume $t_k \parallel t_i$. We will show that $\boldsymbol{u}_i$ and $\boldsymbol{u}_{k+1}$ commute whereas $\boldsymbol{u}_i$ and $\boldsymbol{u}_k$ do not. First, we will show that $\boldsymbol{u}_k$ and $\boldsymbol{u}_i$ don't commute by showing $u_k \between u_i$.

$$\begin{aligned}
&\text{If } u_i u_k = u_k u_i \\
&\implies t_i t_k t_{k+1} t_k = t_k t_{k+1} t_k t_i \\
&\implies t_i t_k t_{k+1} = t_k t_{k+1} (t_k t_i t_k) \\
&\implies t_i t_k t_{k+1} = t_k t_{k+1} t_i \\
&\implies (t_k t_i t_k) t_{k+1} = t_{k+1} t_i \\
&\implies t_i t_{k+1} = t_{k+1} t_i
\end{aligned}$$

which is a contradiction, thus $u_i \between u_k$, consequently $\boldsymbol{u}_k$ and $\boldsymbol{u}_i$ do not commute. Next we observe that $\boldsymbol{u}_{k+1}$ and $\boldsymbol{u}_i$ commute as u_i and u_{k+1} commute (as $u_i = t_i$ and $u_{k+1} = t_k$ and $t_i \parallel t_k$ by assumption).

By establishing these relations, we observe:

$$\begin{aligned}
\widetilde{\boldsymbol{t}}_{k+1}\widetilde{\boldsymbol{t}}_i\widetilde{\boldsymbol{t}}_{k+1} &= \boldsymbol{u}_{k+1}^{-1}\boldsymbol{u}_k(\boldsymbol{u}_{k+1}\boldsymbol{u}_i\boldsymbol{u}_{k+1}^{-1})\boldsymbol{u}_k\boldsymbol{u}_{k+1} \\
&= \boldsymbol{u}_{k+1}^{-1}(\boldsymbol{u}_k\boldsymbol{u}_i\boldsymbol{u}_k)\boldsymbol{u}_{k+1} \\
&= (\boldsymbol{u}_{k+1}^{-1}\boldsymbol{u}_i)\boldsymbol{u}_k(\boldsymbol{u}_i\boldsymbol{u}_{k+1}) \\
&= \boldsymbol{u}_i\boldsymbol{u}_{k+1}^{-1}\boldsymbol{u}_k\boldsymbol{u}_{k+1}\boldsymbol{u}_i \\
&= \widetilde{\boldsymbol{t}}_i\widetilde{\boldsymbol{t}}_{k+1}\widetilde{\boldsymbol{t}}_i.
\end{aligned}$$

Therefore $\widetilde{\boldsymbol{t}}_i\widetilde{\boldsymbol{t}}_{k+1}\widetilde{\boldsymbol{t}}_i = \widetilde{\boldsymbol{t}}_{k+1}\widetilde{\boldsymbol{t}}_i\widetilde{\boldsymbol{t}}_{k+1}$. □

Lemma 6.0.5. $\boldsymbol{t}_k\boldsymbol{t}_{k+1}\boldsymbol{t}_k = \boldsymbol{t}_{k+1}\boldsymbol{t}_k\boldsymbol{t}_{k+1} \implies \widetilde{\boldsymbol{t}}_k\widetilde{\boldsymbol{t}}_{k+1}\widetilde{\boldsymbol{t}}_k = \widetilde{\boldsymbol{t}}_{k+1}\widetilde{\boldsymbol{t}}_k\widetilde{\boldsymbol{t}}_{k+1}$

Proof. Since $\boldsymbol{t}_k\boldsymbol{t}_{k+1}\boldsymbol{t}_k = \boldsymbol{t}_{k+1}\boldsymbol{t}_k\boldsymbol{t}_{k+1}$ therefore $t_k \between t_{k+1}$. First, we will show that $\boldsymbol{u}_k$ and $\boldsymbol{u}_{k+1}$ don't commute by showing that u_k and u_{k+1} don't.

$$\begin{aligned} u_k u_{k+1} u_k &= t_k t_{k+1}(t_k t_k) t_k t_{k+1} t_k \\ &= t_k(t_{k+1} t_k t_{k+1}) t_k \\ &= t_k(t_k t_{k+1} t_k) t_k \\ &= u_{k+1} u_k u_{k+1} \end{aligned}$$

Thus we have the relation $\boldsymbol{u}_k\boldsymbol{u}_{k+1}\boldsymbol{u}_k = \boldsymbol{u}_{k+1}\boldsymbol{u}_k\boldsymbol{u}_{k+1} \implies \boldsymbol{u}_k\boldsymbol{u}_{k+1}\boldsymbol{u}_k^{-1} = \boldsymbol{u}_{k+1}^{-1}\boldsymbol{u}_k\boldsymbol{u}_{k+1}$. Now,

$$\begin{aligned} \widetilde{\boldsymbol{t}}_{k+1}\widetilde{\boldsymbol{t}}_k\widetilde{\boldsymbol{t}}_{k+1} &= \boldsymbol{u}_{k+1}^{-1}\boldsymbol{u}_k\boldsymbol{u}_{k+1}(\boldsymbol{u}_{k+1}\boldsymbol{u}_{k+1}^{-1})\boldsymbol{u}_k\boldsymbol{u}_{k+1} \\ &= (\boldsymbol{u}_{k+1}^{-1}\boldsymbol{u}_k\boldsymbol{u}_{k+1})\boldsymbol{u}_k\boldsymbol{u}_{k+1} \\ &= \boldsymbol{u}_k\boldsymbol{u}_{k+1}(\boldsymbol{u}_k^{-1}\boldsymbol{u}_k)\boldsymbol{u}_{k+1} \\ &= \boldsymbol{u}_k\boldsymbol{u}_{k+1}\boldsymbol{u}_{k+1} \\ &= \boldsymbol{u}_{k+1}\boldsymbol{u}_{k+1}^{-1}\boldsymbol{u}_k\boldsymbol{u}_{k+1}\boldsymbol{u}_{k+1} \\ &= \widetilde{\boldsymbol{t}}_k\widetilde{\boldsymbol{t}}_{k+1}\widetilde{\boldsymbol{t}}_k \end{aligned}$$

Therefore $\widetilde{\boldsymbol{t}}_k\widetilde{\boldsymbol{t}}_{k+1}\widetilde{\boldsymbol{t}}_k = \widetilde{\boldsymbol{t}}_{k+1}\widetilde{\boldsymbol{t}}_k\widetilde{\boldsymbol{t}}_{k+1}$. □

Lemma 6.0.6. $[\boldsymbol{t}_{i_1}, \boldsymbol{t}_{i_2} \cdots \boldsymbol{t}_{i_s} \cdots \boldsymbol{t}_{i_2}^{-1}] = e \implies [\widetilde{\boldsymbol{t}}_{i_1}, \widetilde{\boldsymbol{t}}_{i_2} \cdots \widetilde{\boldsymbol{t}}_{i_s} \cdots \widetilde{\boldsymbol{t}}_{i_2}^{-1}] = e$

Proof. We will use several cases to address all the possible positions of $\boldsymbol{t}_k$ and $\boldsymbol{t}_{k+1}$ in the relation $\left[\boldsymbol{t}_{i_1}, \boldsymbol{t}_{i_2} \cdots \boldsymbol{t}_{i_s} \cdots \boldsymbol{t}_{i_2}^{-1}\right] = e$. In all the following cases we assume that in the sequence $(t_{i_1}, t_{i_2}, \ldots, t_{i_{s-1}}, t_{i_s})$ each reflection commutes with every other reflections except for the two reflections that appear adjacent to it in the sequence. Also t_{i_1} doesn't commute with t_{i_s}.

Case I: Assume that $t_{i_1} = t_k$ and $t_{i_2} = t_{k+1}$. First we observe that since $u_k = t_k t_{k+1} t_k^{-1}$, $u_{i_s} = t_{i_s}$ and $u_{i_3} = t_{i_3}$ therefore u_k^{-1} doesn't commute with u_{i_3} and u_{i_s}, but commutes with every other reflection u_{i_l}, which implies $\left[\boldsymbol{u}_k^{-1}, \boldsymbol{u}_{i_3} \cdots \boldsymbol{u}_{i_s} \cdots \boldsymbol{u}_{i_3}\right] = e$. Using these results we will show that $\left[\widetilde{\boldsymbol{t}}_k, \widetilde{\boldsymbol{t}}_{k+1} \cdots \widetilde{\boldsymbol{t}}_{i_s} \cdots \widetilde{\boldsymbol{t}}_{k+1}^{-1}\right] = e$. To this end

$$
\begin{aligned}
\widetilde{\boldsymbol{t}}_{k+1} \cdots \widetilde{\boldsymbol{t}}_{i_s} \cdots \widetilde{\boldsymbol{t}}_{k+1}^{-1} \widetilde{\boldsymbol{t}}_k &= \boldsymbol{u}_{k+1}^{-1} \boldsymbol{u}_k \boldsymbol{u}_{k+1} \cdots \boldsymbol{u}_{i_s} \cdots \boldsymbol{u}_{k+1}^{-1} \boldsymbol{u}_k^{-1} \boldsymbol{u}_{k+1} \boldsymbol{u}_{k+1} \\
&= \boldsymbol{u}_k \boldsymbol{u}_{k+1} \boldsymbol{u}_k^{-1} \boldsymbol{u}_{i_3} \cdots \boldsymbol{u}_{i_s} \cdots \boldsymbol{u}_{i_3} \boldsymbol{u}_{k+1}^{-1} \boldsymbol{u}_k^{-1} \boldsymbol{u}_{k+1} \boldsymbol{u}_{k+1} \\
&= \boldsymbol{u}_k \boldsymbol{u}_{k+1} \boldsymbol{u}_{i_3} \cdots \boldsymbol{u}_{i_s} \cdots \boldsymbol{u}_{i_3} \boldsymbol{u}_k^{-1} \boldsymbol{u}_{k+1}^{-1} \boldsymbol{u}_k^{-1} \boldsymbol{u}_{k+1} \boldsymbol{u}_{k+1}
\end{aligned}
$$

Now, it can be easily seen that $\boldsymbol{u}_k \boldsymbol{u}_{k+1} \boldsymbol{u}_k = \boldsymbol{u}_{k+1} \boldsymbol{u}_k \boldsymbol{u}_{k+1}$ by checking that $u_k u_{k+1} u_k = u_{k+1} u_k u_{k+1}$, and using this result we may replace $\boldsymbol{u}_k^{-1} \boldsymbol{u}_{k+1}^{-1} \boldsymbol{u}_k^{-1}$ by $\boldsymbol{u}_{k+1}^{-1} \boldsymbol{u}_k^{-1} \boldsymbol{u}_{k+1}^{-1}$, therefore

$$
\begin{aligned}
&\boldsymbol{u}_k \boldsymbol{u}_{k+1} \boldsymbol{u}_{i_3} \cdots \boldsymbol{u}_{i_s} \cdots \boldsymbol{u}_{i_3} \boldsymbol{u}_k^{-1} \boldsymbol{u}_{k+1}^{-1} \boldsymbol{u}_k^{-1} \boldsymbol{u}_{k+1} \boldsymbol{u}_{k+1} \\
&= \boldsymbol{u}_k \boldsymbol{u}_{k+1} \cdots \boldsymbol{u}_{i_s} \cdots \boldsymbol{u}_{i_3} \boldsymbol{u}_{k+1}^{-1} \boldsymbol{u}_k^{-1} \boldsymbol{u}_{k+1}^{-1} \boldsymbol{u}_{k+1} \boldsymbol{u}_{k+1} \\
&= \boldsymbol{u}_k \boldsymbol{u}_{k+1} \cdots \boldsymbol{u}_{i_s} \cdots \boldsymbol{u}_{k+1}^{-1} \boldsymbol{u}_k^{-1} \boldsymbol{u}_{k+1} \\
&= \boldsymbol{u}_k \boldsymbol{u}_{k+1} \cdots \boldsymbol{u}_{i_s} \cdots \boldsymbol{u}_{k+1}^{-1} \boldsymbol{u}_k^{-1} \boldsymbol{u}_{k+1} \\
&= \boldsymbol{u}_{k+1} \boldsymbol{u}_{k+1}^{-1} \boldsymbol{u}_k \boldsymbol{u}_{k+1} \cdots \boldsymbol{u}_{i_s} \cdots \boldsymbol{u}_{k+1}^{-1} \boldsymbol{u}_k^{-1} \boldsymbol{u}_{k+1} \\
&= \widetilde{\boldsymbol{t}}_k \widetilde{\boldsymbol{t}}_{k+1} \cdots \widetilde{\boldsymbol{t}}_{i_s} \cdots \widetilde{\boldsymbol{t}}_{k+1}^{-1}
\end{aligned}
$$

Thus we have shown:

$$
\widetilde{\boldsymbol{t}}_k \widetilde{\boldsymbol{t}}_{k+1} \cdots \widetilde{\boldsymbol{t}}_{i_s} \cdots \widetilde{\boldsymbol{t}}_{k+1}^{-1} = \widetilde{\boldsymbol{t}}_{k+1} \cdots \widetilde{\boldsymbol{t}}_{i_s}, \cdots \widetilde{\boldsymbol{t}}_{k+1}^{-1} \widetilde{\boldsymbol{t}}_k
$$

Case II: We assume that $t_{i_r} = t_k$ and $t_{i_{r+1}} = t_{k+1}$, where $1 < r < n$. We intend to show that

$$
\left[\widetilde{\boldsymbol{t}}_{i_1}, \widetilde{\boldsymbol{t}}_{i_2} \cdots \widetilde{\boldsymbol{t}}_k \widetilde{\boldsymbol{t}}_{k+1} \cdots \widetilde{\boldsymbol{t}}_{i_s} \cdots \widetilde{\boldsymbol{t}}_{k+1}^{-1} \widetilde{\boldsymbol{t}}_k^{-1} \cdots \widetilde{\boldsymbol{t}}_{i_2}\right] = e
$$

To this end

$$
\begin{aligned}
&\widetilde{\boldsymbol{t}}_{i_1} \widetilde{\boldsymbol{t}}_{i_2} \cdots \widetilde{\boldsymbol{t}}_k \widetilde{\boldsymbol{t}}_{k+1} \cdots \widetilde{\boldsymbol{t}}_{i_s} \cdots \widetilde{\boldsymbol{t}}_{k+1}^{-1} \widetilde{\boldsymbol{t}}_k^{-1} \cdots \widetilde{\boldsymbol{t}}_{i_2} \\
&= \boldsymbol{u}_{i_1} \boldsymbol{u}_{i_2} \cdots \boldsymbol{u}_{k+1} \boldsymbol{u}_{k+1}^{-1} \boldsymbol{u}_k \boldsymbol{u}_{k+1} \cdots \boldsymbol{u}_{i_s} \cdots \boldsymbol{u}_{k+1}^{-1} \boldsymbol{u}_k^{-1} \boldsymbol{u}_{k+1} \boldsymbol{u}_{k+1}^{-1} \cdots \boldsymbol{u}_{i_2}^{-1}
\end{aligned}
$$

Again, invoking the relation $\boldsymbol{u}_k\boldsymbol{u}_{k+1}\boldsymbol{u}_k = \boldsymbol{u}_{k+1}\boldsymbol{u}_k\boldsymbol{u}_{k+1}$ in a slightly different form we obtain

$$\begin{aligned}&\boldsymbol{u}_{i_1}\boldsymbol{u}_{i_2}\cdots\boldsymbol{u}_{k+1}\boldsymbol{u}_{k+1}^{-1}\boldsymbol{u}_k\boldsymbol{u}_{k+1}\cdots\boldsymbol{u}_{i_s}\cdots\boldsymbol{u}_{k+1}^{-1}\boldsymbol{u}_k^{-1}\boldsymbol{u}_{k+1}\boldsymbol{u}_{k+1}^{-1}\cdots\boldsymbol{u}_{i_2}^{-1}\\ &=\boldsymbol{u}_{i_1}\boldsymbol{u}_{i_2}\cdots\boldsymbol{u}_{i_{r-1}}\boldsymbol{u}_k\boldsymbol{u}_{k+1}\boldsymbol{u}_{i_{r+2}}\cdots\boldsymbol{u}_{i_s}\cdots\boldsymbol{u}_{i_{r+2}}^{-1}\boldsymbol{u}_{k+1}^{-1}\boldsymbol{u}_k^{-1}\boldsymbol{u}_{i_{r-1}}^{-1}\cdots\boldsymbol{u}_{i_2}^{-1}\\ &=\boldsymbol{u}_{i_1}\boldsymbol{u}_{i_2}\cdots\boldsymbol{u}_{i_{r-1}}\boldsymbol{u}_k\boldsymbol{u}_{i_{r+2}}\cdots\boldsymbol{u}_{i_s}\cdots\boldsymbol{u}_{i_{r+2}}^{-1}\boldsymbol{u}_k^{-1}\boldsymbol{u}_{i_{r-1}}^{-1}\cdots\boldsymbol{u}_{i_2}^{-1}\end{aligned}$$

Here we used the fact that $\boldsymbol{u}_{k+1}$ commutes with $\boldsymbol{u}_{i_{r+2}}, \boldsymbol{u}_{i_{r+3}}, \ldots, \boldsymbol{u}_{i_s}$, this is because $u_{k+1} = t_k$ and $u_{i_{r+2}} = t_{i_{r+2}}, u_{i_{r+3}} = t_{i_{r+3}}, \ldots, u_{i_s} = t_{i_s}$. Now we also observe that $u_k = t_k t_{k+1} t_k$ does not commute with $u_{i_{r-1}} = t_{i_{r-1}}$ and $u_{i_{r+2}} = t_{i_{r+2}}$ but commutes with u_{i_l} for $l \neq r_1, r+2$. Therefore, invoking the relation $\left[\boldsymbol{u}_{i_1}, \boldsymbol{u}_{i_2}\cdots\boldsymbol{u}_{i_{r-1}}\boldsymbol{u}_k\boldsymbol{u}_{i_{r+2}}\cdots\boldsymbol{u}_{i_s}\cdots\boldsymbol{u}_{i_{r+2}}^{-1}\boldsymbol{u}_k^{-1}\boldsymbol{u}_{i_{r-1}}^{-1}\cdots\boldsymbol{u}_{i_2}^{-1}\right] = e$ we get

$$\begin{aligned}&\boldsymbol{u}_{i_1}\boldsymbol{u}_{i_2}\cdots\boldsymbol{u}_{i_{r-1}}\boldsymbol{u}_k\boldsymbol{u}_{i_{r+2}}\cdots\boldsymbol{u}_{i_s}\cdots\boldsymbol{u}_{i_{r+2}}^{-1}\boldsymbol{u}_k^{-1}\boldsymbol{u}_{i_{r-1}}^{-1}\cdots\boldsymbol{u}_{i_2}^{-1}\\ &=\boldsymbol{u}_{i_2}\cdots\boldsymbol{u}_{i_{r-1}}\boldsymbol{u}_k\boldsymbol{u}_{i_{r+2}}\cdots\boldsymbol{u}_{i_s}\cdots\boldsymbol{u}_{i_{r+2}}^{-1}\boldsymbol{u}_k^{-1}\boldsymbol{u}_{i_{r-1}}^{-1}\cdots\boldsymbol{u}_{i_2}^{-1}\boldsymbol{u}_{i_1}\\ &=\widetilde{\boldsymbol{t}}_{i_2}\cdots\widetilde{\boldsymbol{t}}_k\widetilde{\boldsymbol{t}}_{k+1}\cdots\widetilde{\boldsymbol{t}}_{i_s}\cdots\widetilde{\boldsymbol{t}}_{k+1}^{-1}\widetilde{\boldsymbol{t}}_k^{-1}\cdots\widetilde{\boldsymbol{t}}_{i_2}\widetilde{\boldsymbol{t}}_{i_1}\end{aligned}$$

Therefore we have shown that

$$\widetilde{\boldsymbol{t}}_{i_1}\widetilde{\boldsymbol{t}}_{i_2}\cdots\widetilde{\boldsymbol{t}}_k\widetilde{\boldsymbol{t}}_{k+1}\cdots\widetilde{\boldsymbol{t}}_{i_s}\cdots\widetilde{\boldsymbol{t}}_{k+1}^{-1}\widetilde{\boldsymbol{t}}_k^{-1}\cdots\widetilde{\boldsymbol{t}}_{i_2}$$

$$=\widetilde{\boldsymbol{t}}_{i_2}\cdots\widetilde{\boldsymbol{t}}_k\widetilde{\boldsymbol{t}}_{k+1}\cdots\widetilde{\boldsymbol{t}}_{i_s}\cdots\widetilde{\boldsymbol{t}}_{k+1}^{-1}\widetilde{\boldsymbol{t}}_k^{-1}\cdots\widetilde{\boldsymbol{t}}_{i_2}\widetilde{\boldsymbol{t}}_{i_1}$$

Case III: Here we assume that $t_{i_{s-1}} = t_k$ and $t_{i_s} = t_{k+1}$ and we intend to show that

$$\left[\widetilde{\boldsymbol{t}}_{i_1}, \widetilde{\boldsymbol{t}}_{i_2}\cdots\widetilde{\boldsymbol{t}}_k\widetilde{\boldsymbol{t}}_{k+1}\widetilde{\boldsymbol{t}}_k^{-1}\cdots\widetilde{\boldsymbol{t}}_{i_2}^{-1}\right] = e.$$

To this end

$$\begin{aligned}\widetilde{\boldsymbol{t}}_{i_1}\widetilde{\boldsymbol{t}}_{i_2}\cdots\widetilde{\boldsymbol{t}}_k\widetilde{\boldsymbol{t}}_{k+1}\widetilde{\boldsymbol{t}}_k^{-1}\cdots\widetilde{\boldsymbol{t}}_{i_2}^{-1} &= \boldsymbol{u}_{i_1}\boldsymbol{u}_{i_2}\cdots\boldsymbol{u}_{k+1}\boldsymbol{u}_{k+1}^{-1}\boldsymbol{u}_k\boldsymbol{u}_{k+1}\boldsymbol{u}_{k+1}^{-1}\cdots\boldsymbol{u}_{i_2}^{-1}\\ &= \boldsymbol{u}_{i_1}\boldsymbol{u}_{i_2}\cdots\boldsymbol{u}_{k-1}\boldsymbol{u}_k\boldsymbol{u}_{k-1}^{-1}\cdots\boldsymbol{u}_{i_2}^{-1}\end{aligned}$$

We observe that $u_k = t_k t_{k+1} t_k^{-1}$ doesn't commute with $u_{i_1} = t_{i_1}$ and $u_{k-1} = t_{k-1}$, thus we can invoke the relation $\left[\boldsymbol{u}_{i_1}, \boldsymbol{u}_{i_2} \cdots \boldsymbol{u}_{k-1} \boldsymbol{u}_k \boldsymbol{u}_{k-1}^{-1} \cdots \boldsymbol{u}_{i_2}^{-1}\right] = e$ to obtain

$$\begin{aligned}
\boldsymbol{u}_{i_1} \boldsymbol{u}_{i_2} \cdots \boldsymbol{u}_{k-1} \boldsymbol{u}_k \boldsymbol{u}_{k-1}^{-1} \cdots \boldsymbol{u}_{i_2}^{-1} &= \boldsymbol{u}_{i_2} \cdots \boldsymbol{u}_{k-1} \boldsymbol{u}_k \boldsymbol{u}_{k-1}^{-1} \cdots \boldsymbol{u}_{i_2}^{-1} \boldsymbol{u}_{i_1} \\
&= \boldsymbol{u}_{i_2} \cdots \boldsymbol{u}_{k-1} (\boldsymbol{u}_{k+1} \boldsymbol{u}_{k+1}^{-1}) \boldsymbol{u}_k (\boldsymbol{u}_{k+1} \boldsymbol{u}_{k+1}^{-1}) \boldsymbol{u}_{k-1}^{-1} \cdots \boldsymbol{u}_{i_2}^{-1} \boldsymbol{u}_{i_1} \\
&= \widetilde{\boldsymbol{t}}_{i_2} \cdots \widetilde{\boldsymbol{t}}_k \widetilde{\boldsymbol{t}}_{k+1} \widetilde{\boldsymbol{t}}_k^{-1} \cdots \widetilde{\boldsymbol{t}}_{i_2}^{-1} \widetilde{\boldsymbol{t}}_{i_1}
\end{aligned}$$

Thus, we have shown $\widetilde{\boldsymbol{t}}_{i_1} \widetilde{\boldsymbol{t}}_{i_2} \cdots \widetilde{\boldsymbol{t}}_k \widetilde{\boldsymbol{t}}_{k+1} \widetilde{\boldsymbol{t}}_k^{-1} \cdots \widetilde{\boldsymbol{t}}_{i_2}^{-1} = \widetilde{\boldsymbol{t}}_{i_2} \cdots \widetilde{\boldsymbol{t}}_k \widetilde{\boldsymbol{t}}_{k+1} \widetilde{\boldsymbol{t}}_k^{-1} \cdots \widetilde{\boldsymbol{t}}_{i_2}^{-1} \widetilde{\boldsymbol{t}}_{i_1}$

Case IV: Here we assume that $t_{i_1} = t_{k+1}$ and $t_{i_l} \neq t_k$ for $1 \leq l \leq s$. Thus we intend to prove that $\left[\widetilde{\boldsymbol{t}}_{k+1}, \widetilde{\boldsymbol{t}}_{i_2} \cdots \widetilde{\boldsymbol{t}}_{i_s} \cdots \widetilde{\boldsymbol{t}}_{i_2}^{-1}\right] = e$. Now,

$$\widetilde{\boldsymbol{t}}_{k+1} \widetilde{\boldsymbol{t}}_{i_2} \cdots \widetilde{\boldsymbol{t}}_{i_s} \cdots \widetilde{\boldsymbol{t}}_{i_2}^{-1} = \boldsymbol{u}_{k+1}^{-1} \boldsymbol{u}_k \boldsymbol{u}_{k+1} \boldsymbol{u}_{i_2} \cdots \boldsymbol{u}_{i_s} \cdots \boldsymbol{u}_{i_2}^{-1}$$

As $u_{k+1} = t_k$, therefore u_{k+1} commutes with every u_{i_l}, except u_k, therefore

$$\boldsymbol{u}_{k+1}^{-1} \boldsymbol{u}_k \boldsymbol{u}_{k+1} \boldsymbol{u}_{i_2} \cdots \boldsymbol{u}_{i_s} \cdots \boldsymbol{u}_{i_2}^{-1} = \boldsymbol{u}_{k+1}^{-1} \boldsymbol{u}_k \boldsymbol{u}_{i_2} \cdots \boldsymbol{u}_{i_s} \cdots \boldsymbol{u}_{i_2}^{-1} \boldsymbol{u}_{k+1}$$

Now, we observe that $u_k = t_k t_{k+1} t_k^{-1}$ does not commute with $u_{i_2} = t_{i_2}$ and $u_{i_s} = t_{i_s}$, but commutes with the rest of the u_{i_l} thus allowing us to invoke the relation

$$\left[\boldsymbol{u}_k, \boldsymbol{u}_{i_2} \cdots \boldsymbol{u}_{i_s} \cdots \boldsymbol{u}_{i_2}^{-1}\right] = e.$$

Using this and the fact that u_{k+1}^{-1} commutes with every u_{i_l} except u_k we get

$$\begin{aligned}
\boldsymbol{u}_{k+1}^{-1} \boldsymbol{u}_k \boldsymbol{u}_{i_2} \cdots \boldsymbol{u}_{i_s} \cdots \boldsymbol{u}_{i_2}^{-1} \boldsymbol{u}_{k+1} &= \boldsymbol{u}_{k+1}^{-1} \boldsymbol{u}_{i_2} \cdots \boldsymbol{u}_{i_s} \cdots \boldsymbol{u}_{i_2}^{-1} \boldsymbol{u}_k \boldsymbol{u}_{k+1} \\
&= \boldsymbol{u}_{i_2} \cdots \boldsymbol{u}_{i_s} \cdots \boldsymbol{u}_{i_2}^{-1} \boldsymbol{u}_{k+1}^{-1} \boldsymbol{u}_k \boldsymbol{u}_{k+1} \\
&= \widetilde{\boldsymbol{t}}_{i_2} \cdots \widetilde{\boldsymbol{t}}_{i_s} \cdots \widetilde{\boldsymbol{t}}_{i_2}^{-1} \widetilde{\boldsymbol{t}}_{k+1}
\end{aligned}$$

Thus we have shown that $\widetilde{\boldsymbol{t}}_{k+1} \widetilde{\boldsymbol{t}}_{i_2} \cdots \widetilde{\boldsymbol{t}}_{i_s} \cdots \widetilde{\boldsymbol{t}}_{i_2}^{-1} = \widetilde{\boldsymbol{t}}_{i_2} \cdots \widetilde{\boldsymbol{t}}_{i_s} \cdots \widetilde{\boldsymbol{t}}_{i_2}^{-1} \widetilde{\boldsymbol{t}}_{k+1}$.

Case V: Here we assume that $t_{i_r} = t_{k+1}$ where $1 < r < s$ and $t_{i_l} \neq t_k$ for $1 \leq l \leq s$. We intend to show that

$$\left[\widetilde{\boldsymbol{t}}_{i_1}, \widetilde{\boldsymbol{t}}_{i_2} \cdots \widetilde{\boldsymbol{t}}_{k+1} \cdots \widetilde{\boldsymbol{t}}_{i_s} \cdots \widetilde{\boldsymbol{t}}_{k+1} \cdots \widetilde{\boldsymbol{t}}_{i_2}^{-1}\right] = e$$

$$\begin{aligned}
\widetilde{\boldsymbol{t}}_{i_1}\widetilde{\boldsymbol{t}}_{i_2} \cdots \widetilde{\boldsymbol{t}}_{k+1} \cdots \widetilde{\boldsymbol{t}}_{i_s} \cdots \widetilde{\boldsymbol{t}}_{k+1} \cdots \widetilde{\boldsymbol{t}}_{i_2}^{-1} &= \boldsymbol{u}_{i_1}\boldsymbol{u}_{i_2} \cdots \boldsymbol{u}_{k+1}^{-1}\boldsymbol{u}_k\boldsymbol{u}_{k+1} \cdots \boldsymbol{u}_{i_s} \cdots \boldsymbol{u}_{k+1}^{-1}\boldsymbol{u}_k^{-1}\boldsymbol{u}_{k+1} \cdots \boldsymbol{u}_{i_2}^{-1} \\
&= \boldsymbol{u}_{k+1}^{-1}\boldsymbol{u}_{i_1}\boldsymbol{u}_{i_2} \cdots \boldsymbol{u}_k \cdots \boldsymbol{u}_{i_s} \cdots \boldsymbol{u}_k^{-1} \cdots \boldsymbol{u}_{i_2}^{-1}\boldsymbol{u}_{k+1}
\end{aligned}$$

As u_{k+1} commutes with u_{i_l} for all l. Also we use the fact that $u_k = t_k t_{k+1} t_k^{-1}$ doesn't commute with $u_{i_{r+1}}$ and $u_{i_{r-1}}$ but commutes with every other u_{i_l}'s, to invoke the relation

$$\boldsymbol{u}_{i_1}\boldsymbol{u}_{i_2} \cdots \boldsymbol{u}_k \cdots \boldsymbol{u}_{i_s} \cdots \boldsymbol{u}_k^{-1} \cdots \boldsymbol{u}_{i_2}^{-1} = \boldsymbol{u}_{i_2} \cdots \boldsymbol{u}_k \cdots \boldsymbol{u}_{i_s} \cdots \boldsymbol{u}_k^{-1} \cdots \boldsymbol{u}_{i_2}^{-1}\boldsymbol{u}_{i_1}$$

to obtain

$$\begin{aligned}
&\boldsymbol{u}_{k+1}^{-1}\boldsymbol{u}_{i_1}\boldsymbol{u}_{i_2} \cdots \boldsymbol{u}_k \cdots \boldsymbol{u}_{i_s} \cdots \boldsymbol{u}_k^{-1} \cdots \boldsymbol{u}_{i_2}\boldsymbol{u}_{k+1} \\
=&\boldsymbol{u}_{k+1}^{-1}\boldsymbol{u}_{i_2} \cdots \boldsymbol{u}_k \cdots \boldsymbol{u}_{i_s} \cdots \boldsymbol{u}_k^{-1} \cdots \boldsymbol{u}_{i_2}^{-1}\boldsymbol{u}_{i_1}\boldsymbol{u}_{k+1} \\
=&\boldsymbol{u}_{i_2} \cdots \boldsymbol{u}_{k+1}^{-1}\boldsymbol{u}_k\boldsymbol{u}_{k+1} \cdots \boldsymbol{u}_{i_s} \cdots \boldsymbol{u}_{k+1}^{-1}\boldsymbol{u}_k^{-1}\boldsymbol{u}_{k+1} \cdots \boldsymbol{u}_{i_2}^{-1}\boldsymbol{u}_{i_1} \\
=&\widetilde{\boldsymbol{t}}_{i_2} \cdots \widetilde{\boldsymbol{t}}_{k+1} \cdots \widetilde{\boldsymbol{t}}_{i_s} \cdots \widetilde{\boldsymbol{t}}_{k+1} \cdots \widetilde{\boldsymbol{t}}_{i_2}^{-1}\widetilde{\boldsymbol{t}}_{i_1}
\end{aligned}$$

Therefore the relation holds.

Case VI: Assume $t_{i_s} = t_k$ and $t_{i_l} \neq t_k$ for $1 \leq l \leq s$, therefore we intend to prove that

$$\left[\widetilde{\boldsymbol{t}}_{i_1}, \widetilde{\boldsymbol{t}}_{i_2} \cdots \widetilde{\boldsymbol{t}}_{k+1} \cdots \widetilde{\boldsymbol{t}}_{i_2}^{-1}\right] = e$$

$$\begin{aligned}
\widetilde{\boldsymbol{t}}_{i_1}\widetilde{\boldsymbol{t}}_{i_2} \cdots \widetilde{\boldsymbol{t}}_{k+1} \cdots \widetilde{\boldsymbol{t}}_{i_2}^{-1} &= \boldsymbol{u}_{i_1}\boldsymbol{u}_{i_2} \cdots \boldsymbol{u}_{k+1}^{-1}\boldsymbol{u}_k\boldsymbol{u}_{k+1} \cdots \boldsymbol{u}_{i_2}^{-1} \\
&= \boldsymbol{u}_{k+1}^{-1}\boldsymbol{u}_{i_1}\boldsymbol{u}_{i_2} \cdots \boldsymbol{u}_k \cdots \boldsymbol{u}_{i_2}^{-1}\boldsymbol{u}_{k+1}
\end{aligned}$$

as u_{k+1} commutes with all the u_{i_l}'s except u_k. Similar to the previous cases we can invoke the relation

$$\boldsymbol{u}_{i_1}\boldsymbol{u}_{i_2}\cdots\boldsymbol{u}_k\cdots\boldsymbol{u}_{i_2}^{-1} = \boldsymbol{u}_{i_2}\cdots\boldsymbol{u}_k\cdots\boldsymbol{u}_{i_2}^{-1}\boldsymbol{u}_{i_1}$$

to obtain

$$\begin{aligned}
\boldsymbol{u}_{k+1}^{-1}\boldsymbol{u}_{i_1}\boldsymbol{u}_{i_2}\cdots\boldsymbol{u}_k\cdots\boldsymbol{u}_{i_2}^{-1}\boldsymbol{u}_{k+1} &= \boldsymbol{u}_{k+1}^{-1}\boldsymbol{u}_{i_2}\cdots\boldsymbol{u}_k\cdots\boldsymbol{u}_{i_2}^{-1}\boldsymbol{u}_{i_1}\boldsymbol{u}_{k+1} \\
&= \boldsymbol{u}_{i_2}\cdots\boldsymbol{u}_{k+1}^{-1}\boldsymbol{u}_k\boldsymbol{u}_{k+1}\cdots\boldsymbol{u}_{i_2}^{-1}\boldsymbol{u}_{i_1} \\
&= \widetilde{\boldsymbol{t}}_{i_2}\cdots\widetilde{\boldsymbol{t}}_{k+1}\cdots\widetilde{\boldsymbol{t}}_{i_2}^{-1}\widetilde{\boldsymbol{t}}_{i_1}
\end{aligned}$$

Thus the relation holds. □

CHAPTER 7

BESSIS' PRESENTATION AND ITS CONNECTION WITH OUR BRAID GROUP PRESENTATION

Recall that in a Coxeter system (W, S), relations of the form $s_i s_j s_i \cdots = s_j s_i s_j \cdots$, where s_i and s_j are in S, are called braid relations. Invoking a braid relation to rewrite a word in S is called a *braid move*.

Theorem 7.0.1 (Matsumoto's lemma)**.** *Let (W, S) be a Coxeter system. Let $s_{i_1} \cdots s_{i_k}$ and $s_{j_1} \cdots s_{j_k}$ be any two reduced S-decompositions of an element $w \in W$, then $s_{i_1} \cdots s_{i_k}$ can be transformed into $s_{j_1} \cdots s_{j_k}$ by successive braid move.*

We observe here that this theorem only takes the S-expressions into consideration.

Definition 7.0.2. Let (W, T, c) be a dual Coxeter system. Relations of the form $st = t^s s$, where $s, t \in T$ and $st \leq_T c$ are called *dual-braid relations*. Invoking a dual-braid relations to rewrite a word in T is called *dual-braid move*.

Theorem 7.0.3 (Dual to Matsumoto's Lemma (Bessis, 2003))**.** *Let (W, T, c) be a dual Coxeter system. Let $w \in W$ be such that $w \leq_T c$. Let $t_{i_1} \cdots t_{i_n}$ and $t_{j_1} \cdots t_{j_n}$ be two T-decompositions of w, then $t_{i_1} \cdots t_{i_n}$ can be transformed into $t_{j_1} \cdots t_{j_n}$ by successive dual-braid moves.*

Proof. By Lemma 4.2.3, since $w \leq_T c$, therefore w is a Coxeter element in some parabolic subgroup of (W, T). Thus using Theorem 4.3.4 we have our result. □

Matsumoto's lemma states that all reduced S-decompositions of any word w are connected under braid relations, whereas Bessis' result states that for a given Coxeter element c the reduced T-decompositions of any word $w \leq_T c$ are connected under dual-braid relations.

Building on work of Birman-Ko-Lee (Birman et al., 1998), Bessis (Bessis, 2003) gave a second, different presentation for Artin groups associated with finite Coxeter groups, by replacing the set of simple reflections S by the set of all the reflections T, leading to a 'dual' presentation for the Artin group $\boldsymbol{B}(W)$ called the *dual braid presentation*.

Theorem 7.0.4 (Dual-Braid presentation (Bessis, 2003)). *Let W be a Coxeter group and let c be a Coxeter element in W, then the corresponding Artin group $\boldsymbol{B}(W)$ has a presentation*

$$\boldsymbol{B}(W) \cong \langle \boldsymbol{T} \big| \boldsymbol{t}_i\boldsymbol{t}_j = \boldsymbol{t}_k\boldsymbol{t}_i, \text{ for } t_i, t_j, t_k \in T \text{ with } t_it_j = t_kt_i \text{ and } t_it_j \leq_T c\rangle_{group} \tag{7.1}$$

where $\boldsymbol{T}$ is a formal copy of the set of all the reflections T and $\boldsymbol{t}_i\boldsymbol{t}_j = \boldsymbol{t}_k\boldsymbol{t}_i$ are dual-braid relations.

Theorem 7.0.5. *Let c be a Coxeter element in W and let $t_1 \cdots t_n$ be a reduced T-decomposition of c. If $R(\boldsymbol{t}_1, \ldots, \boldsymbol{t}_n)$ is a relation in* $\mathrm{Rel}(t_1, \ldots, t_n)$ *in Equation* (5.1)*, then $R(\boldsymbol{t}_1, \ldots, \boldsymbol{t}_n)$ is a dual-braid relation.*

Proof. The relations in $\mathrm{Rel}(t_1, \ldots, t_n)$ are of the following three types

1. $\boldsymbol{t}_i\boldsymbol{t}_j = \boldsymbol{t}_j\boldsymbol{t}_i$,
2. $\boldsymbol{t}_i\boldsymbol{t}_j\boldsymbol{t}_i = \boldsymbol{t}_j\boldsymbol{t}_i\boldsymbol{t}_j$ and
3. $[\boldsymbol{t}_{i_1}, \boldsymbol{t}_{i_2} \cdots \boldsymbol{t}_{i_s} \cdots \boldsymbol{t}_{i_2}^{-1}]$.

For a relation $\boldsymbol{t}_i\boldsymbol{t}_j = \boldsymbol{t}_j\boldsymbol{t}_i$ in $\mathrm{Rel}(t_1, \ldots, t_n)$, $t_it_j = t_jt_i$ holds in the Coxeter group W. Since t_i and t_j are reflections appearing in the reduced T-decomposition $t_1 \cdots t_n$ and they commute, therefore both t_it_j and $t_jt_i \leq_T c$.

For a relation $\boldsymbol{t}_i\boldsymbol{t}_j\boldsymbol{t}_i = \boldsymbol{t}_j\boldsymbol{t}_i\boldsymbol{t}_j$ in $\mathrm{Rel}(t_1, \ldots, t_n)$, $t_it_jt_i = t_jt_it_j$ holds in the Coxeter group W, which can also be written as either $t_it_j = t_i^{t_j}t_i$ or $t_jt_i = t_j^{t_i}t_j$. Since t_i and t_j are reflections appearing in the reduced T-decomposition $t_1 \cdots t_n$, therefore either t_it_j or $t_jt_i \leq_T c$, making $t_it_jt_i = t_jt_it_j$ a dual braid relation.

For a relation $[\boldsymbol{t}_{i_1}, \boldsymbol{t}_{i_2} \cdots \boldsymbol{t}_{i_{s-1}}\boldsymbol{t}_{i_s}\boldsymbol{t}_{i_{s-1}}^{-1} \cdots \boldsymbol{t}_{i_2}^{-1}] = e$ in $\mathrm{Rel}(t_1, \ldots, t_n)$, the following is true in the Coxeter group W

$$t_{i_1} \between t_{i_2} \between \cdots \between t_{i_{s-1}} \between t_{i_s} \between t_{i_1} \text{ and } t_{i_j} \parallel t_{i_k} \text{ for } k \neq j-1, j+1$$

where the reflections $t_{i_1}, t_{i_2}, \ldots, t_{i_{s-1}}, t_{i_s}$ belong to the set $\{t_1, \cdots, t_n\}$. Using the s equations arising from Lemma 3.3.1 we can rewrite the reduced T-decomposition $t_1 \cdots t_n$ to show that $t_{i_1}t \leq_T c$. □

The next result readily follows.

Theorem 7.0.6. *Let c be a Coxeter element in W and let $m = |Red_T(c)|$. Let $Red_T(c) = \{c_1, c_2, \ldots, c_m\}$ where c_i is a reduced T-decomposition of c. Let T_{c_i} denote the set consisting of the reflections present in the reduced T-word for c_i and $\boldsymbol{T}_{c_i}$ denote the corresponding set in the Artin group. We write down the m presentations of $\boldsymbol{B}$ using Equation* (5.1) *as follows*

$$\begin{gathered}
\left\langle \boldsymbol{T}_{c_1} \middle| \mathrm{Rel}(T_{c_1}) \right\rangle \\
\left\langle \boldsymbol{T}_{c_2} \middle| \mathrm{Rel}(T_{c_2}) \right\rangle \\
\vdots \\
\left\langle \boldsymbol{T}_{c_m} \middle| \mathrm{Rel}(T_{c_n}) \right\rangle
\end{gathered}$$

Then Bessis' dual-braid presentation for $\boldsymbol{B}$ is

$$\left\langle \bigcup_{i=1}^{m} \boldsymbol{T}_{c_i} \middle| \begin{array}{cc} \boldsymbol{t}_i\boldsymbol{t}_j = \boldsymbol{t}_j\boldsymbol{t}_i, & \textit{if } \boldsymbol{t}_i\boldsymbol{t}_j = \boldsymbol{t}_j\boldsymbol{t}_i \in \mathrm{Rel}(T_{c_l}) \textit{ for some } 1 \in [m], \\ \boldsymbol{t}_i\boldsymbol{t}_j\boldsymbol{t}_i = \boldsymbol{t}_j\boldsymbol{t}_i\boldsymbol{t}_j, & \textit{if } \boldsymbol{t}_i\boldsymbol{t}_j\boldsymbol{t}_i = \boldsymbol{t}_j\boldsymbol{t}_i\boldsymbol{t}_j \in \mathrm{Rel}(T_{c_l}) \textit{ for some } 1 \in [m], \\ [\boldsymbol{t}_{i_1}, \boldsymbol{t}_{i_2} \cdots \boldsymbol{t}_{i_s} \cdots \boldsymbol{t}_{i_2}^{-1}] = e, & \textit{if } [\boldsymbol{t}_{i_1}, \boldsymbol{t}_{i_2} \cdots \boldsymbol{t}_{i_s} \cdots \boldsymbol{t}_{i_2}^{-1}] = e \in \mathrm{Rel}(T_{c_l}) \textit{ for some } 1 \in [m]. \end{array} \right\rangle \tag{7.2}$$

CHAPTER 8

CLUSTER ALGEBRA AND QUIVER MUTATION

8.1 Quivers

Definition 8.1.1. A graph (denoted by the pair $\mathcal{Q} = (V, E)$) is called a *quiver* if it is a finite directed graph with integer weights assigned to its edges, where V and E are the sets of vertices and edges respectively. Additionally if a quiver does not allow any 1-cycle (loop) or 2-cycle, then we call it *cluster quivers*.

Since we will be looking at quivers only from a cluster algebra perspective therefore in this text we will avoid 1-cycles and 2-cycles and we will refer to them as just "quivers". Sometimes we will denote the set of vertices and the set of edges for a quiver $\mathcal{Q}$ by $V(\mathcal{Q})$ and $E(\mathcal{Q})$ respectively, when the quiver in question is ambiguous. We will also assume that the quivers in this text contains only finitely many vertices i.e. $|V(\mathcal{Q})| < \infty$. A directed edge from a vertex a to a vertex b with an assigned weight 'm' in a quiver will be denoted by $a \xrightarrow{m} b$ (or equivalently as $a \xleftarrow{-m} b$).

Definition 8.1.2 (Quiver Mutation)**.** Let $\mathcal{Q}$ be a quiver and let i, j and k be vertices in $\mathcal{Q}$. A *quiver mutation on $\mathcal{Q}$ at k*, transforms $\mathcal{Q}$ into a new quiver $\mathcal{Q}' := \mu_k^{\text{quiv}}(\mathcal{Q})$ obtained by

1. reversing the direction of all the edges incident to the vertex k, while keeping the assigned weights to the edges unchanged, in the quiver $\mathcal{Q}$.

2. replacing every 3-cycle of the form

$$i \xrightarrow{p} k \xrightarrow{q} j \xrightarrow{r} i$$

   in $\mathcal{Q}$ (where p and q are non-zero integers, either both positive, or both negative and r is any integer) with a 3-cycle of the form

$$i \xleftarrow{p} k \xleftarrow{q} j \xleftarrow{pq-r} i$$

3. keeping all the other edges and their assigned weights unchanged.

The second bullet in Definition 8.1.2 is illustrated in Figure 8.1.

Remark 8.1.3. In Definition 8.1.2 the second bullet is consistent with the first, since replacing $i \xrightarrow{p} k \xrightarrow{q} j \xrightarrow{r} i$ with $i \xrightarrow{p} k \xrightarrow{q} j \xrightarrow{r} i$ takes care of the first bullet by reversing the arrows incident to k. Notice that the cycle $i \xleftarrow{p} k \xleftarrow{q} j \xleftarrow{pq-r} i$ can also be written as $i \xrightarrow{-p} k \xrightarrow{-q} j \xrightarrow{r-pq} i$ and reversing the direction of the edge incident to the vertex k is equivalent to replacing the assigned weight to the edge incident to k by its negative. Also notice that, the notation $\mathcal{Q} \xrightarrow{\mu_k^{\text{quiv}}} \mu_k^{\text{quiv}}(\mathcal{Q})$ as used in Figure 8.1 denotes that the quiver $\mu_k^{\text{quiv}}(\mathcal{Q})$ is obtained as a result of a quiver mutation at k on $\mathcal{Q}$. However, a more befitting notation would be $\mathcal{Q} \xleftrightarrow{\mu_k^{\text{quiv}}} \mu_k^{\text{quiv}}(\mathcal{Q})$ taking Proposition 8.1.4 into account.

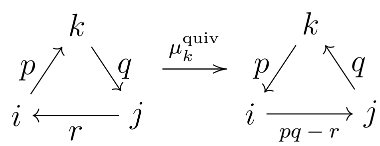


Figure 8.1. Quiver mutation on a 3-cycle.

Proposition 8.1.4. *For a quiver $\mathcal{Q}$, $\mu_k^{\text{quiv}}(\mu_k^{\text{quiv}}(\mathcal{Q})) = \mathcal{Q}$.*

Proof. We observe that $pq - (pq - r) = r$, and reversing the direction of an edge twice brings us to the initial orientation of the edge. Thus the claim follows. □

Definition 8.1.5. A quiver $\mathcal{Q}$ is said to be *mutation equivalent* to another quiver $\mathcal{Q}'$ if one can be obtained from the other by a finite number of quiver mutations. The set of all quivers mutation equivalent to the quiver $\mathcal{Q}$ is called the *mutation class of $\mathcal{Q}$*.

Definition 8.1.6. Let $\mathcal{Q}$ be a quiver, and $|V(\mathcal{Q})| = n$. The *exchange matrix of $\mathcal{Q}$* is the $n \times n$ skew symmetric matrix $B(\mathcal{Q}) := (b_{ij})$

$$b_{ij} = \begin{cases} r & \text{if } \mathcal{Q} \text{ contains the edge } i \xrightarrow{r} j \\ -r & \text{if } \mathcal{Q} \text{ contains the edge } i \xleftarrow{r} j \\ 0 & \text{otherwise} \end{cases}$$

for vertices i and j in $\mathcal{Q}$.

Remark 8.1.7. There is a much general notion allowing exchange matrices to be skew symmetrizable. However, since we are only dealing with a simply-laced case, we don't need that level of generality.

Proposition 8.1.8. *Let $\mathcal{Q}$ be a quiver and let $B(\mathcal{Q}) = (b_{ij})$ be the corresponding exchange matrix. If $B' = B(\mu_k^{\text{quiv}}(\mathcal{Q})) = (b'_{ij})$ is the exchange matrix corresponding to $\mu_k^{\text{quiv}}(\mathcal{Q})$, then*

$$b'_{ij} = \begin{cases} -b_{ij} & i = k \text{ or } j = k \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \text{otherwise} \end{cases}$$

For a quiver $\mathcal{Q}$ we will denote the associated exchange matrix by $B(\mathcal{Q})$ and for a skew symmetric matrix B we will denote the associated quiver by $\Gamma(B)$. With some abuse of notation we will denote $B(\mu_k^{\text{quiv}}(\mathcal{Q}))$ by $\mu_k^{\text{quiv}}(B(\mathcal{Q}))$ or simply $\mu_k^{\text{quiv}}(B)$ when it is understood that B is an exchange matrix.

Definition 8.1.9. A quiver $\mathcal{Q}$ is of *finite mutation type* if there are finitely many quivers in the mutation class of $\mathcal{Q}$, otherwise it is called *mutation infinite*.

Lemma 8.1.10. *Let $\mathcal{Q}$ be a connected quiver with more than 2 vertices. If $\mathcal{Q}$ contains an edge with assigned weight of 3 or more then $\mathcal{Q}$ is mutation infinite.*

Proof. Let $1 \xrightarrow{w_{12}} 2$ be a sub-quiver of $\mathcal{Q}$ such that $w_{12} \geq 3$. Since $\mathcal{Q}$ is connected and contains more than 2 vertices therefore $1 \xrightarrow{w_{12}} 2 \subset \mathcal{Q}_3 \subset \mathcal{Q}$ such that $\mathcal{Q}_3 = 1 \xrightarrow{w_{12}} 2 \xrightarrow{w_{23}} 3 \xrightarrow{w_{31}} 1$, where we can assume that $w_{31} \leq w_{23} \leq w_{12}$ and $w_{23} > 0$ (if the relations between w_{12}, w_{23}, and w_{31} are different then they can be altered to fit our description by a series of quiver mutations on $\mathcal{Q}$). Now,

$$\mu_2^{\text{quiv}}(\mathcal{Q}_3) = 1 \xleftarrow{w_{12}} 2 \xleftarrow{w_{23}} 3 \xleftarrow{w'_{31}} 1$$

Since $w'_{31} = w_{12}w_{23} - w_{31} > 2w_{23} - w_{31} \geq w_{31}$ therefore in this way we can increase the assigned weight to the edge $\{1, 3\}$ without affecting the assigned weights to $\{1, 2\}$ and $\{2, 3\}$. Thus we can increase the weights endlessly creating infinitely many quivers in the mutation class of $\mathcal{Q}$. $\square$

8.2 Cluster Algebra

Definition 8.2.1. For a quiver $\widetilde{\mathcal{Q}}$ with n vertices a *seed* is a pair $((u_1, \ldots, u_n), \widetilde{\mathcal{Q}})$ where $(u_1, \ldots, u_n)$ is an n-tuple of algebraically independent, rational functions in n indeterminates—$x_1, \ldots, x_n$ (where the u_i's are in bijection with the n vertices of the quiver $\widetilde{\mathcal{Q}}$). The n tuple, $(u_1, \ldots, u_n)$ is called a *cluster (variable) of rank n*. Define *seed mutation* (μ_k) on a seed $\left((u_1, \ldots, u_n), \widetilde{\mathcal{Q}}\right)$ by

$$\mu_k\left((u_1, \ldots, u_n), \widetilde{\mathcal{Q}}\right) = ((u_1', \ldots, u_n'), \mu_k^{\text{quiv}}(\widetilde{\mathcal{Q}}))$$

where

$$u_i' = \begin{cases} u_i & \text{if } i \neq k \\ \frac{1}{u_k}\left\{\prod\limits_{i \in k_{\mathrm{T}}} u_i + \prod\limits_{j \in k_{\mathrm{A}}} u_j\right\} & \text{if } i = k \end{cases}$$

where if $l \in k_T$ then the vertex associated to u_l is incident to the vertex associated to u_k with a directed edge from the vertex associated to u_l to the vertex associated to u_k, and if $l \in k_A$ then the vertex associated to u_l incident to the vertex associated to u_k with a directed edge from the vertex associated to u_k to the vertex associated to u_l in $\widetilde{\mathcal{Q}}$.

Definition 8.2.2. A *cluster algebra* $\mathcal{A}(\mathcal{Q})$ is the algebra generated by all the cluster variables that can be constructed from an initial seed $((u_1, \ldots, u_n), \mathcal{Q})$ (generally $((x_1, \ldots, x_n), \mathcal{Q})$) by performing repeated seed mutations on the initial seed in all possible ways.

Definition 8.2.3. An *exchange graph* associated to a cluster algebra $\mathcal{A}(\mathcal{Q})$ is a graph with seeds as its vertices where two seeds are adjacent if one can be obtained from another by seed mutation.

Definition 8.2.4. Let $\mathcal{Q}$ be a quiver. A cluster algebra $\mathcal{A}(\mathcal{Q})$ is of *finite type* if there are only finitely many seeds. A quiver $\mathcal{Q}$ is of *finite type* if the associated cluster algebra $\mathcal{A}(\mathcal{Q})$ is of finite type.

Lemma 8.2.5. *Let $\mathcal{Q}$ be a quiver and let $\mathcal{Q}' \subset \mathcal{Q}$ be a sub-quiver of $\mathcal{Q}$ such that $\mathcal{Q}'$ is not of finite type then $\mathcal{Q}$ is not of finite type.*

Proof. If $\mathcal{Q}'$ is not of finite type then the associated cluster algebra $\mathcal{A}(\mathcal{Q}')$ contains infinitely many cluster variables. Since the vertices in $\mathcal{Q}'$ are also vertices in $\mathcal{Q}$ therefore it is easy to see that seed mutations at these vertices will construct infinitely many cluster variables of $\mathcal{A}(\mathcal{Q})$. □

Theorem 8.2.6 ((Fomin and Zelevinsky, 2003))**.** *For a quiver $\mathcal{Q}$, mutation equivalent to an orientation of a simply-laced Dynkin diagram, the cluster algebra $\mathcal{A}(\mathcal{Q})$ is of finite type. Conversely if a cluster algebra $\mathcal{A}(\mathcal{Q})$ is of finite type then $\mathcal{Q}$ is mutation equivalent to some quiver whose underlying unoriented graph is a simply-laced type Dynkin diagram.*

Theorem 8.2.7 ((Fomin and Zelevinsky, 2003))**.** *For a cluster algebra $\mathcal{A}(\mathcal{Q})$ of finite type there is a bijection between the initial cluster variables of $\mathcal{A}(\mathcal{Q})$ and the negative simple roots of the corresponding root system and a bijection between the non-initial cluster variables of $\mathcal{A}(\mathcal{Q})$ and the positive roots of the corresponding root system.*

Lemma 8.2.8. *If a quiver $\mathcal{Q}$ is of finite type then $\mathcal{Q}$ is of finite mutation type.*

Proof. If there are infinitely many quivers in the mutation class of $\mathcal{Q}$, then there is an edge in $\mathcal{Q}$ whose assigned weight is at least 3. Seed mutation at the sink of that edge constructs infinitely many cluster variables. □

Example 8.2.9. In Figure 8.2 we begin with the initial seed $((x_1, x_2),\ \bullet \longrightarrow \bullet\ )$ and perform repeated seed mutations in all possible ways to obtain all the 5 seeds and 5 cluster variables. Since the underlying quiver is of A_2 type, therefore this is a cluster algebra of type A_2.

Lemma 8.2.10. *If $\mathcal{Q}$ is a connected quiver of finite type and $|V(\mathcal{Q})| > 2$ then every 3-cycle in $\mathcal{Q}$, is oriented in a cyclic way.*

Proof. First we recall that any quiver (in this case, a 3-cycle) with an edge weight greater than 2 is mutation infinite by Lemma 8.1.10, consequently it is also of non-finite type by Lemma 8.2.8, thus we only need to check for those 3-cycles whose edge weights are either 1 or 2. Let $\mathcal{Q}_3 \subset \mathcal{Q}$ be a connected sub-quiver, with $|V(\mathcal{Q}_3)| = 3$ such that $\mathcal{Q}_3$ is a 3-cycle graph. Let the weights assigned to the edges of $\mathcal{Q}_3$ be either 1 or 2, with the restriction that not all of them are 1. Also we assume that the edges of $\mathcal{Q}_3$ are not oriented in a cyclic way. We will show that $\mathcal{Q}_3$ is mutation infinite.

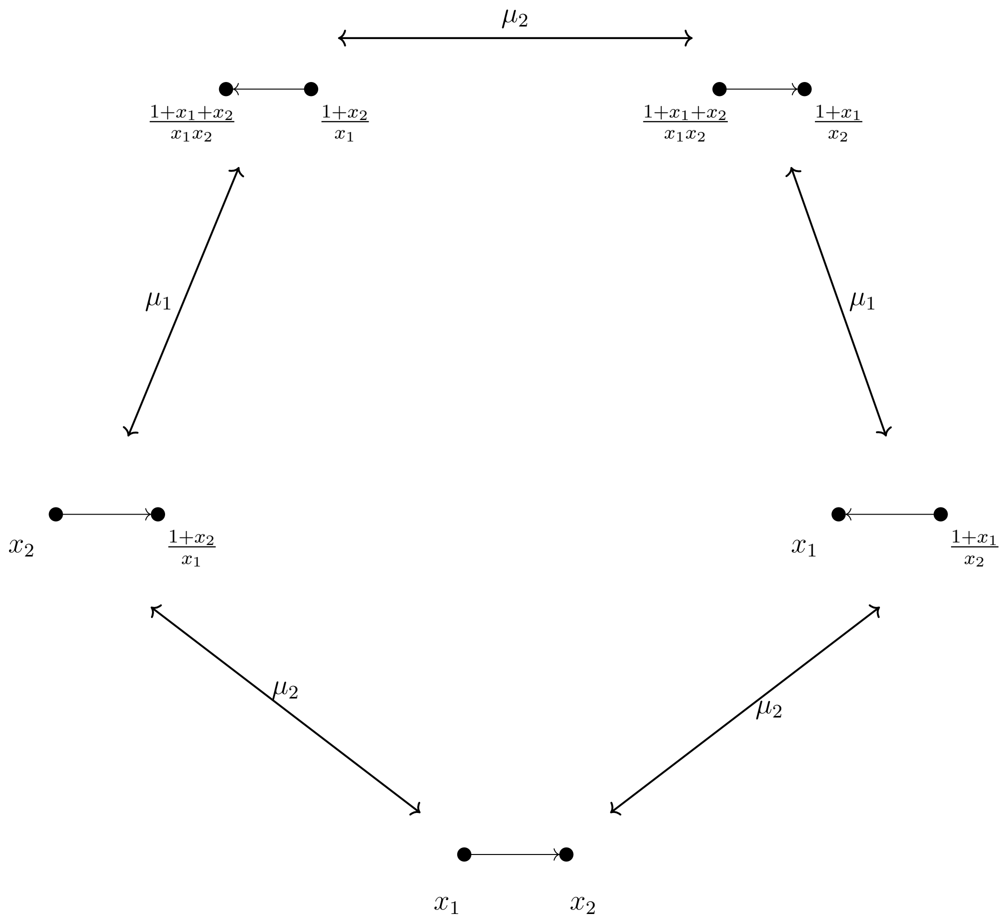


Figure 8.2. Exchange graph of $\mathcal{A}(A_2)$.

1. $v_{i_1} \longrightarrow v_{i_2} \xleftarrow{2} v_{i_3} \longrightarrow v_{i_1} \xleftrightarrow{\boldsymbol{\mu}^{\mathbf{quiv}}_{\boldsymbol{v_{i_1}}}} v_{i_1} \longleftarrow v_{i_2} \xleftarrow{3} v_{i_3} \longleftarrow v_{i_1}$

2. $v_{i_1} \longrightarrow v_{i_2} \longleftarrow v_{i_3} \xrightarrow{2} v_{i_1} \xleftrightarrow{\boldsymbol{\mu}^{\mathbf{quiv}}_{\boldsymbol{v_{i_1}}}} v_{i_1} \longleftarrow v_{i_2} \xleftarrow{3} v_{i_3} \xleftarrow{2} v_{i_1}$

3. $v_{i_1} \xrightarrow{2} v_{i_2} \longleftarrow v_{i_3} \longrightarrow v_{i_1} \xleftrightarrow{\boldsymbol{\mu}^{\mathbf{quiv}}_{\boldsymbol{v_{i_1}}}} v_{i_1} \xleftarrow{2} v_{i_2} \xleftarrow{3} v_{i_3} \longleftarrow v_{i_1}$

4. $v_{i_1} \xrightarrow{2} v_{i_2} \xleftarrow{2} v_{i_3} \longrightarrow v_{i_1} \xleftrightarrow{\boldsymbol{\mu}^{\mathbf{quiv}}_{\boldsymbol{v_{i_1}}}} v_{i_1} \xleftarrow{2} v_{i_2} \xleftarrow{4} v_{i_3} \longleftarrow v_{i_1}$

5. $v_{i_1} \longrightarrow v_{i_2} \xleftarrow{2} v_{i_3} \xrightarrow{2} v_{i_1} \xleftrightarrow{\boldsymbol{\mu}^{\mathbf{quiv}}_{\boldsymbol{v_{i_1}}}} v_{i_1} \longleftarrow v_{i_2} \xleftarrow{4} v_{i_3} \xleftarrow{3} v_{i_1}$

6. $v_{i_1} \xrightarrow{2} v_{i_2} \longleftarrow v_{i_3} \xrightarrow{2} v_{i_1} \xleftrightarrow{\boldsymbol{\mu}^{\mathbf{quiv}}_{\boldsymbol{v_{i_1}}}} v_{i_1} \xleftarrow{2} v_{i_2} \xleftarrow{5} v_{i_3} \xleftarrow{2} v_{i_1}$

7. $v_{i_1} \xrightarrow{2} v_{i_2} \xleftarrow{2} v_{i_3} \xrightarrow{2} v_{i_1} \xleftrightarrow{\boldsymbol{\mu}^{\mathbf{quiv}}_{\boldsymbol{v_{i_1}}}} v_{i_1} \xleftarrow{2} v_{i_2} \xleftarrow{6} v_{i_3} \xleftarrow{2} v_{i_1}$

Since each of these quivers is mutation equivalent to a quiver that has an edge with an assigned weight of 3 or more, therefore they are all mutation infinite. Thus by Lemma 8.2.8, $\mathcal{Q}_3 \not\subset \mathcal{Q}$.

Now we will show that every 3-cycle in $\mathcal{Q}$, whose edge weights are 1, are oriented in a cyclic way. To see this consider a quiver $\widetilde{\mathcal{Q}}_3$ which is a 3-cycle with edge weights equal to 1 and whose edges are not oriented in a cyclic way. This quiver is of the form $1 \longrightarrow 2 \longleftarrow 3 \longrightarrow 1$ (up to relabeling of the vertices) and is not of finite type. Since this mutation class doesn't contain any quiver whose underlying unoriented graph is a simply-laced type Dynkin diagram, therefore $\widetilde{\mathcal{Q}}_3$ is not a quiver of finite type by Theorem 8.2.6. Consequently any quiver $\mathcal{Q}$ such that $\widetilde{\mathcal{Q}}_3 \subset \mathcal{Q}$ is not of finite type. □

Lemma 8.2.11. *If $\mathcal{Q}$ is a quiver, mutation-equivalent to an orientation of a simply-laced Dynkin diagram then all its edges have an assigned weight of 1.*

Proof. Consider the sequence of quivers $\{\mathcal{Q}_i\}_{i=1\ldots n}$ such that $\mathcal{Q}_{i+1} = \mu^{\text{quiv}}_{k_i}(\mathcal{Q}_i)$, where k_i is any node in $\mathcal{Q}_i$ and $\mathcal{Q}_1 = \mathcal{Q}$. It suffices to show that none of the elements of this sequence contain edges with labels greater than one. This can be proved through induction. The base case is true since all simply-laced Dynkin diagram have edges with weights never more than 1. We will show that $\mathcal{Q}_n$ which is the same as $\mu_k(\mathcal{Q}_{n-1})$ does not have edges with labels greater than one.

Let $\mathcal{Q}_n = \mu_k(\mathcal{Q}_{n-1})$. From the definition of quiver mutation we know that the mutation on the quiver $\mathcal{Q}_{n-1}$ at k affects only those edges which are incident to the node k and edges between nodes i and j whenever there is a path involving two other nodes i to j of the form $i \to k \to j$ (up to relabeling of nodes). So if an edge with a label greater than 1 shows up in $\mathcal{Q}_n$ due to mutation on $\mathcal{Q}_{n-1}$ it has to be an edge that is either incident to the node k or in between the nodes i and j.

If $\mathcal{Q}_n$ has an edge with the label $p \geq 1$ of the form $i \overset{p}{\to} k$, (or $i \overset{p}{\leftarrow} k$) then $\mathcal{Q}_{n-1}$ has an oriented edge of the form $i \overset{p}{\leftarrow} k$ (or $i \overset{p}{\to} k$). By induction hypothesis $p = 1$.

Now, let there be an oriented path of the form $i \to k \to j$ in $\mathcal{Q}_n$ with an edge oriented from j to i, with a label $p \geq 0$, see Figure 8.3. As we have already shown, the edges $i \to k$ and $k \to j$ can have a label of at most one since these edges are incident to the node k. Note that the edge between the nodes i and j must be oriented from j to i i.e. $i \overset{p}{\leftarrow} j$ (see Figure 8.3) such that all the edges in

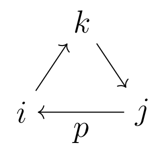


Figure 8.3. A 3-cycle in the quiver $\mathcal{Q}_n$.

the 3-cycle are oriented in a cyclic way, as required by Lemma 8.2.10. Now, $\mu_k(\mathcal{Q}_n) = \mathcal{Q}_{n-1}$. Thus mutating $\mathcal{Q}_n$ at k we obtain $\mathcal{Q}_{n-1}$ as shown in Figure 8.4.

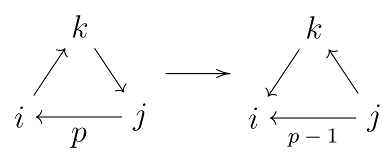


Figure 8.4. A quiver mutation on the sub-quiver given in Figure 8.3 at k.

By our induction hypothesis and Lemma 8.2.10 either $p-1=0$ or $p-1=-1 \implies p=1$ or $p=0$, which completes the proof. $\square$

CHAPTER 9

ROOT ORDER, REFLECTION ORDER AND COXETER ORDER

For a given word $\mathsf{x}_1\mathsf{x}_2\cdots\mathsf{x}_n$ a *subword* is an expression $\mathsf{x}_{i_1}\mathsf{x}_{i_2}\cdots\mathsf{x}_{i_\mathsf{m}}$ where the sequence $\{i_1, i_2, \ldots, i_m\}$ is a sub-sequence of the sequence $\{1, 2, \ldots, n\}$. Recall that in a Coxeter system (W, S) the element $c = s_{i_1}\cdots s_{i_n}$ where $s_{i_j} \in S$, $s_{i_j} \neq s_{i_\ell}$ for $j \neq \ell$ and $|S| = l_S(c)$, is called a Coxeter element. Fix a reduced S-expression $\mathsf{c} = \mathsf{s}_1\cdots\mathsf{s}_n$ for a Coxeter element c in W. Let c^∞ be the infinite word

$$\mathsf{c}^\infty := \mathsf{s}_1\cdots\mathsf{s}_n|\mathsf{s}_1\cdots\mathsf{s}_n|\mathsf{s}_1\cdots\mathsf{s}_n|\cdots$$

Definition 9.0.1. For an element $w \in W$ the *$\mathsf{c}-$sorting word* for w for a given reduced S-expression c for a Coxeter element $c \in W$ is the lexicographically first subword of c^∞ that is a reduced expression for w.

Notice that the $\mathsf{c}-$sorting word for an element w depends on the choice of the particular reduced S-word c for c. However, for two reduced S-words, c_1 and c_2 for c, the c_1-sorting word is commutation equivalent to the c_2-sorting word for w. This is because all reduced S-words for c are commutation equivalent. We will denote the $\mathsf{c}-$sorting word for w by $\mathsf{w}(\mathsf{c})$. When the actual ordering of the letters in the word $\mathsf{w}(\mathsf{c})$ is not important we will use the notation $w(c)$ to denote any reduced S-word that is commutation equivalent to $\mathsf{w}(\mathsf{c})$. For $s \in S$ if $s \leq_S c$, then the $\mathsf{c}-$sorting word for an element $w \in W$ begins with s if and only if $s \leq_S w$. We can keep repeating this argument iteratively as follows, to compute the entire $\mathsf{c}-$sorting word for w.

$$\mathsf{w}(\mathsf{c}) = \begin{cases} \mathsf{sw}'(\bar{\mathsf{s}}\mathsf{cs}) & \text{if } s \leq_S w, \text{ where } w = sw' \text{ and it is an } S\text{-factorization} \\ \mathsf{w}(\bar{\mathsf{s}}\mathsf{cs}) & \text{if } s \not\leq_S w \end{cases}$$

The following lemma follows immediately from this discussion

Lemma 9.0.2. *Let $u = s_1\cdots s_l \in \boldsymbol{B}^+$ and $u \leq_S w \in \boldsymbol{B}^+$. If $\mathsf{s}_1\cdots\mathsf{s}_\mathsf{l}$ is initial in the word c^∞, for a Coxeter element c in W, then $\mathsf{s}_1\cdots\mathsf{s}_\mathsf{l}$ is also initial in the word $\mathsf{w}(\mathsf{c})$.*

Example 9.0.3. The $\mathsf{c}-$sorting word for the permutation 4231 in $\mathfrak{S}_3$ is the word $\mathsf{s}_1\mathsf{s}_2\mathsf{s}_3|\mathsf{s}_2|\mathsf{s}_1$ for $\mathsf{c} = \mathsf{s}_1\mathsf{s}_2\mathsf{s}_3$, where $s_1 = (12), s_2 = (23)$ and $s_3 = (13)$ are transpositions in $\mathfrak{S}_3$. Observe that subjecting the permutation 4231 to its $\mathsf{c}-$sorting word—

$$4231 \xrightarrow{s_1} 4132 \xrightarrow{s_2} 4123 \xrightarrow{s_3} 3124 \xrightarrow{s_2} 2134 \xrightarrow{s_1} 1234$$

'sorts' the permutation in the numerical order (think bubble sorting), hence the name '$\mathsf{c}-$sorting'.

9.1 Geometric Interpretation of the Length Function l_S

Recall, from Chapter 3 every root system R in a standard Euclidean space E has a base Δ—Δ is a basis for E and every vector $v \in R$ can be expressed as a linear combination of roots in Δ with either all positive (and 0) or all negative (and 0) coefficients. Recall that the roots in R can be partitioned into positive roots R^+ and negative roots R^- i.e. $R = R^+ \sqcup R^-$ such that $R^- = -R^+$. Also recall that the group generated by the reflections $t_v : R \mapsto R$ defined by $t_v(u) = u - 2\frac{\langle u,v\rangle}{\langle v,v\rangle}v$, where $u, v \in R$ is called a Weyl group $W(R)$. We will call the pair $(W(R), S(\Delta))$ (or simply (W, S)), where $S(\Delta) = \{t_{v_i} | v_i \in \Delta\}$ a *Weyl system*. Since a Weyl group is also a Coxeter system, therefore we will use Coxeter terminology while talking about Weyl groups.

Consider the set $R_w := R^+ \cap w^{-1}(R^-)$. If $A = w^{-1}(R^-)$ then $w(A) = R^-$, i.e. $w^{-1}(R^-)$ are the roots that become negative roots when acted on by $w \in W$, consequently $R^+ \cap w^{-1}(R^-)$ is the set of positive roots that when acted on by w become negative roots. Let us find out the different characteristics of these sets for different elements in W, starting with the simple reflections

Lemma 9.1.1. *For a Weyl system $(W(R), S(R))$ if $s_v \in S(R)$ where v be the root in Δ corresponding to s_v then $R_{s_v} = \{v\}$.*

Proof. Since $s_v(v) = -v$ therefore $v \in R_{s_v}$. If possible let $u \in R_{s_v}$ and $u \neq v$. $u \in R^+$ therefore u can be written as a linear combination of roots in Δ with positive coefficients

$$u = \sum_{\delta_i \in \Delta} c_i \delta_i, \text{ where } c_i \geq 0$$

Since $v \in \Delta$ so we can write

$$u = cv + \sum_{\delta_i \in \Delta \setminus v} c_i \delta_i, \text{ where } c, c_i \geq 0$$

Since $s_v(u) = u - 2\frac{\langle u,v \rangle}{\langle v,v \rangle} v \implies u = s_v(u) + 2\frac{\langle u,v \rangle}{\langle v,v \rangle} v$

$$\begin{aligned} u &= cv + \sum_{\delta_i \in \Delta \setminus v} c_i \delta_i \\ \implies s_v(u) + 2\frac{\langle u, v \rangle}{\langle v, v \rangle} v &= cv + \sum_{\delta_i \in \Delta \setminus v} c_i \delta_i \\ \implies s_v(u) &= -2\frac{\langle u, v \rangle}{\langle v, v \rangle} v + cv + \sum_{\delta_i \in \Delta \setminus v} c_i \delta_i \\ \implies s_v(u) &= \left(-2\frac{\langle u, v \rangle}{\langle v, v \rangle} + c \right) v + \sum_{\delta_i \in \Delta \setminus v} c_i \delta_i \end{aligned}$$

Thus we have expressed $s_v(u)$ as a linear combination of roots in Δ. Since $s_v(u)$ is a root therefore either all the coefficients are positive (or zero) or all coefficients are negative (or zero). Since all the c_i's are positive (or zero) therefore all the coefficients are positive (or zero), consequently $s_v(u)$ is a positive root.

Thus $R_{s_v} = \{v\}$ and $|R_{s_v}| = 1$. □

Since the number of roots in R_{s_v} is just one and there is a bijection between Δ and S, therefore the next result follows.

Corollary 9.1.2. *The root $v \in \Delta$ corresponds to the simple reflection $s \in S(\Delta)$, of the Weyl group $W(\Delta)$, if and only if $s(v) = -v$, where $S(\Delta)$ is the set of generators in $W(\Delta)$ that corresponds to the roots in Δ.*

Theorem 9.1.3. *Let (W, S) be a Weyl system, then $l_S(wt_v) > l_S(w) \iff w(v) \in R^+$ and $l_S(wt_v) < l_S(w) \iff w(v) \in R^-$, where $t_v \in T$ (T is closure of S under conjugation) and $v \in R^+$ is the root corresponding to the reflection t_v.*

Proof. First we will use induction on $l_S(w)$ to prove that $l_S(wt_v) > l_S(w) \implies w(v) \in R^+$. $l_S(w) = 0$ is a trivial case. Let us assume that the statement is true for $k < l_S(w)$. If $l_S(w) > 0$

then there exists $s \in S$ such that $l_S(sw) < l_S(w)$ (observe that if $w = s_1 \cdots s_n$ is a reduced S-decomposition then setting $s_n = s$ implies $l_S(sw) < l_S(w)$), in particular $l_S(sw)+1 = l_S(w)$. Which implies

$$l_S(sw) < l_S(w) < l_S(wt_v)$$
$$\implies l_S(sw) < l_S(w) - 1 < l_S(wt_v) - 1 \leq l_S(swt_v)$$

where $l_S(w) - 1 < l_S(wt_v) - 1$ follows from the inequality $l_S(w) < l_S(wt_v)$ (the hypothesis of the proof) and for the inequality $l_S(wt_v) - 1 \leq l_S(swt_v)$ we observe that if s is S-initial in wt_v then $l_S(wt_v) - 1 = l_S(swt_v)$ and if s is not S-initial in wt_v then $l_S(wt_v) + 1 = l_S(swt_v)$, combining these we get our desired inequality. Using our induction hypothesis we get $l_S(sw) < l_S(swt_v) \implies sw(v) \in R^+$, (since $l_S(sw) = k$). If possible let $w(v) \in R^-$, then $w(v) = \alpha$ where α is the root in Δ that corresponds to the simple reflection $s \in S$. $w(v) = \alpha \implies wt_v = sw$ (because $wt_v w^{-1}(w(v)) = wt_v(v) = w(-v) = -w(v)$ and $s(\alpha) = -\alpha$, therefore $wt_v w^{-1} = s$), which contradicts the inequality $l_S(sw) < l_S(wt_v)$, thus $w(v) \in R^+$.

Now we will show that $l_S(wt_v) < l_S(w) \implies w(v) \in R^-$. To this end we observe that if $l_S(wt_v) < l_S(w)$ then $l_S((wt_v)t_v) > l_S(wt_v)$ which implies (by the previous part of this proof) $wt_v(v) \in R^+$, which in turn implies that $w(-v) \in R^- \implies -w(v) \in R^- \implies w(v) \in R^-$. Now if $w(v) \in R^- \implies l_S(wt_v) > l_S(w)$ then $l_S(wt_v) < l_S(w) \implies w(v) \in R^+$ which is a contradiction (since we have just proved $l_S(wt_v) < l_S(w) \implies w(v) \in R^-$) therefore $w(v) \in R^- \implies l_S(wt_v) < l_S(w)$. This completes the proof. □

Lemma 9.1.4. *Let (W, S) be a Weyl system, for $w \in W$, $l_S(w) = |R_w|$.*

Proof. We will use induction to prove this. If $l_S(w) = 1$ for some $w \in W$ then $|R_w| = l_S(w)$, as shown in Lemma 9.1.1. Let us assume that if $l_S(w) = k$ for some $w \in W$ then $l_S(w) = |R_w|$. Now let $w \in W$ such that $l_S(w) = k+1$, then there exists some $s_v \in S$ (v is the corresponding positive root such that $s_v(v) = -v$) such that $l_S(ws_v) = k$. By our induction hypothesis $|R_{ws_v}| = l_S(ws_v) = k$. By Theorem 9.1.3 since $|R_{ws_v}| < |R_w|$ therefore $v \in R_w$. But since $ws_v(v) = w(-v) = -w(v) \in R^+$, therefore $v \notin R_{ws_v}$. However, for all the other roots $u \in R^+$, $u \neq v$, $u \in R_w \iff u \in R_{ws_v}$, this is because s_v permutes the set $R^+ \setminus \{v\}$ (by Lemma 9.1.1), therefore

$$s_v\left(R^+ \setminus \{v\}\right) = R^+ \setminus \{v\}$$
$$\implies ws_v\left(R^+ \setminus \{v\}\right) = w\left(R^+ \setminus \{v\}\right)$$

which implies

$$R^- \cap ws_v(R^+ \setminus \{v\}) = R^- \cap w(R^+ \setminus \{v\})$$

but since $ws_v(v) \in R^+$ therefore $R^- \cap ws_v(R^+ \setminus \{v\}) = R^- \cap ws_v(R^+)$, thus

$$R^- \cap ws_v(R^+) = R^- \cap w(R^+ \setminus \{v\})$$

Now $v \in R_w$, therefore $R^- \cap w(R^+ \setminus \{v\}) = \{R^- \cap w(R^+)\} \setminus \{-v\}$, as a result we have

$$R^- \cap ws_v(R^+) = [R^- \cap w(R^+)] \setminus \{-v\}$$
$$\implies [R^- \cap ws_v(R^+)] \cup \{-v\} = R^- \cap w(R^+)$$

Observe that $|R^- \cap w(R^+)| = |R^+ \cap w^{-1}(R^-)| = |R_w|$ and $|R^- \cap ws_v(R^+)| = |R^+ \cap (ws_v)^{-1}(R^-)| = |R_{ws_v}|$, thus we have

$$|R^- \cap ws_v(R^+)| + |\{-v\}| = |R^- \cap w(R^+)|$$
$$\implies k + 1 = |R^- \cap w(R^+)| = |R_w| \qquad \square$$

Lemma 9.1.5. *If $w \in W$, where (W, S) is a Weyl system. Let $s_1 \cdots s_k$ be a reduced S-decomposition of w, and the reflection s_i corresponds to the root v_i. Then*

$$R^+ \cap w(R^-) = \{v_1, s_1(v_2), \ldots, s_1 \cdots s_{k-2}(v_{k-1}), s_1 \cdots s_{k-1}(v_k)\}$$

Proof. First we will show that

$$R^+ \cap w(R^-) \subseteq \{v_1, s_1(v_2), \ldots, s_1 \cdots s_{k-2}(v_{k-1}), s_1 \cdots s_{k-1}(v_k)\}$$

$v \in R^+ \cap w(R^-) \iff v \in R^+$ and $w^{-1}(v) \in R^-$. Since $w^{-1} = s_k \cdots s_1$ therefore $s_k \cdots s_1(v) \in R^-$. Since $v \in R^+$ therefore there exists a minimum number $j \leq k$ such that $s_j \cdots s_1(v) \in R^-$ and $s_{j-1} \cdots s_1(v), s_{j-2} \cdots s_1(v), \ldots, s_2 s_1(v), s_1(v) \in R^+$. Thus $s_{j-1} \cdots s_1(v) = v_j$ (since $s_j(\widetilde{v}) = -\widetilde{v} \iff \widetilde{v} = v_j$ by Lemma 9.1.1), as a result $s_1 \cdots s_{j-1}(s_{j-1} \cdots s_1(v)) = s_1 \cdots s_{j-1}(v_j)$ which is also the same as v, thus $v \in \{v_1, s_1(v_2), \ldots, s_1 \cdots s_{k-2}(v_{k-1}), s_1 \cdots s_{k-1}(v_k)\}$.

Note that the set $R^+ \cap w(R^-) = R_{w^{-1}}$ and by Lemma 9.1.4 we have $l_S(w^{-1}) = |R_{w^{-1}}|$. Also since $l_S(w^{-1}) = l_S(w)$ thus $|R_{w^{-1}}| = l_S(w)$. Now, clearly the set

$$v_1, s_1(v_2), \ldots, s_1 \cdots s_{k-2}(v_{k-1}), s_1 \cdots s_{k-1}(v_k)\}$$

has exactly $l_S(w)$ distinct elements, therefore

$$|R_{w^{-1}}| = |\{v_1, s_1(v_2), \ldots, s_1 \cdots s_{k-2}(v_{k-1}), s_1 \cdots s_{k-1}(v_k)\}|$$

as a result (in view of the inclusion proved earlier)

$$R^+ \cap w(R^-) = \{v_1, s_1(v_2), \ldots, s_1 \cdots s_{k-2}(v_{k-1}), s_1 \cdots s_{k-1}(v_k)\}$$

□

Definition 9.1.6 (Inversion sets and inversion sequences)**.** Let (W, S) be a Coxeter system, and let $w \in W$ (whose reduced S-decomposition $w = s_1 \cdots s_k$), then the *inversion set of w* is the set

$$\text{inv}(w) := R^+ \cap w(R^-) = \{v_1, s_1(v_2), \ldots, s_1 \cdots s_{k-2}(v_{k-1}), s_1 \cdots s_{k-1}(v_k)\}$$

where v_i is the root corresponding to the simple reflection s_i. The sequence

$$\text{inv}(s_1 \cdots s_k) := (v_1, s_1(v_2), \ldots, s_1 \cdots s_{k-1}(v_k))$$

is called an *inversion sequence* of $s_1 \cdots s_k$ for the element w. This inversion sequence (for a particular choice of a reduced S-decomposition) specifies a total order on the set $\text{inv}(w)$ called the *induced order*.

For a chosen reduced S-decomposition $w = s_1 \cdots s_2$ define $\beta_1 = v_1, \beta_i = s_1 \cdots s_{i-1}(v_i),$ where $i = 2, \ldots, k$. Clearly $\beta_1, \ldots, \beta_k \in R^+$ and $\beta_i \neq \beta_j$, if $i \neq j$, the induced ordering $(<)$ on the set $\text{inv}(w) = \{\beta_1, \ldots, \beta_k\}$, is given by $\beta_i < \beta_j \iff i < j$. Note that this ordering depends on the choice of the reduced S-decomposition of w.

Corollary 9.1.7. *Let $w \in W$ where W is a Weyl group, and let $w = s_1 \cdots s_k$ be a reduced S-decomposition. If the root v_i corresponds to the reflection s_i then*

$$s_{s_1 \cdots s_{j-1}(v_j)} = s_1 \cdots s_{j-1} s_j s_{j-1} \cdots s_1$$

Proof. We know that $s_{s_1 \cdots s_{j-1}(v_j)}(s_1 \cdots s_{j-1}(v_j)) = -s_1 \cdots s_{j-1}(v_j)$ by Lemma 9.1.1, therefore it suffices to show that $s_1 \cdots s_{j-1} s_j s_{j-1} \cdots s_1 (s_1 \cdots s_{j-1}(v_j)) = -s_1 \cdots s_{j-1}(v_j)$.

$$\begin{aligned} s_1 \cdots s_{j-1} s_j s_{j-1} \cdots s_1 (s_1 \cdots s_{j_1}(v_j)) &= s_1 \cdots s_{j-1} s_j (v_j) \\ &= s_1 \cdots s_{j-1}(-v_j) \\ &= -s_1 \cdots s_{j-1}(v_j) \end{aligned}$$ □

Lemma 9.1.8 (Matsumoto's (left) strong Exchange condition)**.** *Let (W, S) be a Coxeter system and let T be the closure of S under conjugation. If $l_S(tw) < l_S(w)$ for some $t \in T$ and let $w = s_1 \cdots s_k$ be an S-decomposition (not necessarily reduced), then there exists an index j such that*

$$tw = s_1 \cdots \hat{s}_j \cdots s_k$$

where $s_1 \cdots \hat{s}_j \cdots s_k$ is the S-decomposition of w with the j^{th} entry deleted. Furthermore if the decomposition $s_1 \cdots s_k$ is reduced then j is unique.

Proof. Here, we will prove this for a Weyl group but it is true in general for finite Coxeter groups. In fact any group generated by the a set of involutions is a Coxeter if and only if it satisfies the exchange condition. Let (W, S) be a Weyl system and let $w \in W$ whose S-decomposition is $s_1 \cdots s_k$. We will first show that the set $\text{inv}(w) = \{\beta_1, \ldots, \beta_k\}$ is the same as the set

$$\text{exch}(w) := \{\gamma \in R^+ | t_\gamma s_1 \cdots \hat{s}_i \cdots s_k = w, \text{ for some } 1 \leq i \leq k\}$$

where $t_\gamma \in T$ is the reflection corresponding to the root $\gamma \in R^+$. Let $\beta_j \in \mathrm{inv}(w)$, $\beta_j = s_1 \cdots s_{j-1}(v_j)$, where v_j is the root corresponding to the reflection s_j. Then

$$\begin{aligned} s_{\beta_j} s_1 \cdots \hat{s}_j \cdots s_k &= s_{s_1 \cdots s_{j-1}(v_j)} s_1 \cdots \hat{s}_j \cdots s_k \\ &= s_1 \cdots s_{j-1} s_j s_{j-1} \cdots s_1 s_1 \cdots \hat{s}_j \cdots s_k \\ &= s_1 \cdots s_k = w \end{aligned}$$

Furthermore since $|l_S(w)| = |\mathrm{inv}(w)|$ (in a similar way as in Lemma 9.1.4) the set $\mathrm{inv}(w) = \mathrm{exch}(w)$. To prove the statement of this lemma all we need to show is that for any $t_\gamma \in T$ if $l_S(t_\gamma w) \leq l_S(w)$ then $\gamma \in \mathrm{inv}(w)$. To this end we observe that if $l_S(t_\gamma w) < l_S(w)$ which implies $l_S(w^{-1} t_\gamma^{-1}) = l_S(w^{-1} t_\gamma) < l_S(w^{-1})$ and that in turn implies $w^{-1}(\gamma) \in R^-$. Thus $\gamma \in R^+ \cap w(R^-) \implies \gamma \in \mathrm{inv}(w)$.

For the uniqueness part we observe that if $w = s_1 \cdots s_k$ is a reduced S-decomposition with $l_S(w) = k$ and if $tw = s_1 \cdots \hat{s}_i \cdots s_j \cdots s_k = s_1 \cdots s_i \cdots \hat{s}_j \cdots s_k$ then that implies $s_{i+1} \cdots s_j = s_i \cdots s_{j-1} \implies s_{i+1} \cdots s_{j-1} = s_i \cdots s_j$. Therefore $w = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_k$ is another reduced S-decomposition of w, which contradicts the assumption that $l_S(w) = k$. □

We have a similar 'right' version of this lemma as follows.

Lemma 9.1.9 (Matsumoto's (right) strong Exchange condition)**.** *Let (W, S) be a Coxeter system and let T be the closure of S under conjugation. If $l_S(wt) < l_S(w)$ for some $t \in T$ and let $w = s_1 \cdots s_k$ be an S-decomposition (not necessarily reduced), then there exists an index j such that*

$$wt = s_1 \cdots \hat{s}_j \cdots s_k$$

where $s_1 \cdots \hat{s}_j \cdots s_k$ is the S-decomposition of w with the j^{th} entry deleted. Furthermore if the decomposition $s_1 \cdots s_k$ is reduced then j is unique.

Proof. If $l_S(wt) < l_S(w)$ then $l_S(tw^{-1}) < l_S(w^{-1})$, therefore by Lemma 9.1.8 there exists an index i such that $tw^{-1} = s_k \cdots \hat{s}_i \cdots s_1$, which in turn implies that $wt = s_1 \cdots \hat{s}_i \cdots s_k$. The uniqueness part can be proved in a similar way as in *Lemma* 9.1.8. □

9.2 Root Order

Definition 9.2.1. For a Coxeter system (W, S), the *longest element* $w_0 \in W$ is defined by any one of these equivalent definitions.

1. $l_S(w_0) \geq l_S(w)$ for all $w \in W$ i.e. w_0 has maximal length.
2. $l_S(w_0 w) = l_S(w_0) - l_S(w)$, for all $w \in W$
3. $l_S(w_0 s) < l_S(w_0)$, for all $s \in S$

Proposition 9.2.2. *The three definitions of the longest element—1, 2 and 3, are equivalent.*

Proof. It is easy to see that $2 \implies 1$, (since $l_S(w_0 w) = l_S(w_0) - l_S(w) \implies l_S(w_0 w) + l_S(w) = l_S(w_0) \implies l_S(w) \leq l_S(w_0)$) and $1 \implies 3$ (since $w_0 s \in W$ for all $s \in S$ therefore by 1 $l_S(w_0 s) < l_S(w_0)$). To see that $3 \implies 2$, it suffices to show that w^{-1} is S-final in w_0, i.e. $w_0 = \widetilde{w} w^{-1}$, with $\widetilde{w} \in W$ and $l_S(w_0) = l_S(\widetilde{w}) + l_S(w^{-1})$, because if $w_0 = \widetilde{w} w^{-1}$ then $l_S(w_0 w) = l_S(\widetilde{w} w^{-1} w) = l_S(\widetilde{w}) = l_S(w_0) - l_S(w^{-1}) = l_S(w_0) - l_S(w)$. We will use induction on the length $l_S(w)$ of $w \in W$ (the same w as in 3). If $l_S(w) = 1$ then let $w = \widetilde{s} \in S$. By the hypothesis of the proof $l_S(w_0 \widetilde{s}) < l_S(w_0)$, therefore using a similar argument as in Lemma 9.1.8 we can show that

$$v_{\widetilde{s}} \in \{\gamma \in R^+ | s_1 \cdots \hat{s}_i \cdots s_k s_\gamma = w_0 \text{ for some } 1 \leq i \leq k\}$$

where γ is the root corresponding to the reflection s_γ, $v_{\widetilde{s}}$ is the root corresponding to the reflection $\widetilde{s}$ and $s_1 \cdots s_k$ is any reduced S-decomposition of w_0. Thus $w_0 = u \widetilde{s}^{-1}$ and $l_S(w_0) = l_S(u) + l_S(\widetilde{s}^{-1})$. Now for the induction step we will show that $w_0 = \widetilde{w} w^{-1}$. There exists $v \in W$ and $s \in S$ such that $w^{-1} = v^{-1} s$ and $l_S(w^{-1}) = l_S(v^{-1}) + l_S(s)$. Since $l_S(v^{-1}) < l_S(w)$ by induction hypothesis there exists $\widetilde{v} \in W$ such that $w_0 = \widetilde{v} v^{-1}$. By our assumption $l_S(w_0 s) < l_S(w_0)$, therefore by Lemma 9.1.9

$$w_0 s = \widetilde{v} v^{-1} s = \widetilde{v}' v^{-1} \text{ (because } v^{-1} s \text{ is reduced)}$$
$$\implies w_0 s s = w_0 = \widetilde{v}' v^{-1} s = \widetilde{v}' w^{-1}$$

where $\widetilde{v}'$ represents v with a letter omitted in its decomposition. □

The longest element is an involution—$l_S(w_0^2) = l_S(w_0) - l_S(w_0) = 0$, and is unique—if w is of the same length as w_0 then $l_S(w_0 w) = l_S(w_0) - l_S(w) = 0$, which implies that w and w_0 are inverse of each other, and by the involution property we just showed $w_0 = w$. It is also easy to see that the 'left' versions of these statements are also true—$l_S(ww_0) = l_S(w_0) - l_S(w)$ for all $w \in W$, and $l_S(sw_0) < l_S(w_0)$, for all $s \in S$.

So, in summary, there is a longest element in W, (denoted by w_0) i.e. for every $w \in W$, $w \leq_S w_0$ ($\leq_S$ being the right weak order)— a similar argument as in the proof of the last proposition could be used to show that for any $w \in W$, w_0 can be written as $w_0 = w^{-1}\widetilde{w}$ such that $l_S(w_0) = l_S(w^{-1}) + l_S(\widetilde{w})$ for some $\widetilde{w} \in W$, consequently it follows that $w_0 = w\hat{w}$ such that $l_S(w_0) = l_S(w) + l_S(\hat{w})$ for some $\hat{w} \in W$. Similarly for every $w \in W$, $w_0 \geq_S w$ where $\geq_S$ is the left weak order i.e. there exists $w' \in W$ such that $w_0 = w'w$ and $l_S(w_0) = l_S(w') + l_S(w)$.

As we have already seen for a given element $w \in W$ a particular choice of a reduced S-decomposition of w induces a total order on the set $\text{inv}(w)$—if $w = s_1 \cdots s_k$ is a reduced S-decomposition of w then the induced order on the set $\text{inv}(w)$ is given by

$$s_1 < s_1(v_2) < \cdots < s_1 s_2 \cdots s_{i-1}(v_i).$$

It is easy to notice that the weak right order $\leq_S$ implies the inclusion order on the set $\text{inv}(w)$.

Theorem 9.2.3. *$w \leq_S w' \iff inv(w) \subseteq inv(w')$*

Proof. If $w \leq_S w'$ then there exists a reduced S-decomposition of w', say $w' = s_1 \cdots s_k$ such that $w = s_1 \cdots s_i$ where $i < k$. It follows that

$$\text{inv}(w) = \{v_1, s_1(v_2), \ldots, s_1 \cdots s_{i-1}(v_i)\} \subseteq \text{inv}(w') = \{s_1, s_1(v_2), \ldots, s_1 \cdots s_{k-1}(v_k)\}. \qquad \square$$

Since for every $w \in W$, $w \leq_S w_0$, therefore $\text{inv}(w) \subseteq \text{inv}(w_0)$, thus $R^+ \subseteq \text{inv}(w_0)$, on the other hand, all inversion sets are subsets of R^+, we have $R^+ = \text{inv}(w_0)$. Therefore defining an order on the set $\text{inv}(w_0)$ defines an order on all the roots in R^+, and we already have an order defined on the inversion sets—the induced order. Now we will define an order on the positive roots and correspondingly define an order on the set of all reflections.

Definition 9.2.4 (Root order)**.** The total order induced (on the set of positive roots) by an inversion sequence (of a particular reduced S-decomposition) for the longest element, is called the *root order*. If $s_1 \cdots s_k$ is a particular reduced S-decomposition of w_0 then we will denote the *associated root order* by inversion sequence $\text{inv}(s_1 \cdots s_k)$

Since $\text{inv}(w_0) = R^+$, therefore root order is a total order on all the positive roots R^+ however since the longest element has multiple non-commutation equivalent reduced S-decompositions, therefore there are multiple root orders. An interesting question to ask here would be—is any arbitrary ordering of R^+, a root order? The answer in general is - no. The next theorem characterizes roots orders.

Theorem 9.2.5 ((Papi, 1994))**.** *An order* $(<')$ *on the set* R^+ *is a root order if and only if for every* $v_1, v_2 \in R^+$ *if* $av_1 + bv_2 \in R^+$*, (where* a *and* b *are positive real numbers) then either*

$$v_1 <' av_1 + bv_2 <' v_2$$

or

$$v_2 <' av_1 + bv_2 <' v_1$$

In (Dyer, 1993) M. Dyer defined a reflection order as follows.

Definition 9.2.6. An order $<^*$ on the set of reflections T is called a *reflection order* if for any two reflections t_1 and t_2 in T the positive roots $v_{t_1}, v_{t_1 t_2 t_1}, \ldots, v_{t_2 t_1 t_2}, v_{t_2}$ are of the form $av_{t_1} + bv_{t_2}$ (where a, b are positive real numbers), and either

$$t_1 <^* t_1 t_2 t_1 <^* \cdots <^* t_2 t_1 t_2 <^* t_2$$

or

$$t_2 <^* t_2 t_1 t_2 <^* \cdots <^* t_1 t_2 t_1 <^* t_1$$

where v_{t_i} is the root corresponding to the reflection t_i.

For any given total order $(<)$ on the set R^+ we can determine a total order $(<')$ on the set of reflections T in the following way

$$v_1 < v_2 \iff t_{v_1} <' t_{v_2}$$

where v_1 and v_2 are positive roots and t_{v_1} and t_{v_2} are the corresponding reflections. It has been shown in (Dyer, 1993) that any total order $(<')$ on T is a reflection order if and only if the corresponding order $(<)$ of R^+ is a root order. Now finally we make the definition that we have been building up to!

Definition 9.2.7 (Coxeter order)**.** Let c denote a reduced S-word for a Coxeter element in W then the total order given by $\text{inv}(w_0(\mathsf{c}))$ on the positive roots (and correspondingly on the set of reflections) is the *Coxeter order* and is denoted by $\leq_{\mathsf{c}}$.

If c_1 and c_2 are two reduced S-decomposition of a Coxeter element c then $\mathsf{c}_1 \equiv \mathsf{c}_2$. However, the Coxeter order $\leq_{\mathsf{c}_1}$ is different from the Coxeter order $\leq_{\mathsf{c}_2}$. This gives us a preorder on the set of reflections.

Definition 9.2.8 (Coxeter preorder)**.** Let c be a Coxeter element, and let $\text{Red}_S(c)$ be the set of reduced S-decompositions of c, then we define a *Coxeter preorder* on the set of reflection (denoted by $\leq_c$) as $t_1 \leq_c t_2$ if $t_1 \leq_{\mathsf{c}} t_2$ for some reduced S-decomposition $\mathsf{c} \in \text{Red}_S(c)$.

Observe that we can have $t_1 \leq_c t_2$ and $t_2 \leq_c t_1$ hold together if for a reduced S-decomposition c_1 of c we have $t_1 \leq_{\mathsf{c}_1} t_2$ and for another reduced S-decomposition c_2 of c we have $t_1 \leq_{\mathsf{c}_2} t_2$. In the corresponding proset t_1 will be considered equivalent to t_2.

CHAPTER 10

TWO-PART FACTORIZATIONS UNDER FACTORIZATION MUTATION AND QUIVERS UNDER MUTATION.

A set partition of the set $\{1, \ldots, n\}$ into blocks is called *noncrossing* if the convex hulls of its blocks don't overlap each other when placed on a circle in a cyclic order as vertices of a regular n-gon. In other words a partition is called noncrossing if for every $0 \leq u < v < w < x \leq n$ if $u \sim w$ and $v \sim x$ then $u \sim v \sim w \sim x$, where $a \sim b$ indicates that a and b are in the same block of the partition, and u, v, w, x are in $\{1, \ldots, n\}$.

Noncrossing partitions are also defined for finite Coxeter systems (Brady and Watt, 2002; Bessis, 2003). For any Coxeter element c in a Coxeter group W an element $w \in W$ is a *$c-$noncrossing partition* if $w \leq_T c$, where T is the set of reflections in W. We will denote the set of $c-$noncrossing partition by $\mathrm{NC}(W, c)$.

Theorem 10.0.1. *The set $NC(W, c)$ is a lattice under the absolute order.*

This theorem has been independently and differently proven in (Bessis, 2003; Brady and Watt, 2008; Ingalls and Thomas, 2009; Reading, 2011). The $c-$noncrossing lattice is denoted by

$$\mathrm{NCL}(W, c) := [e, c]_{\mathrm{Abs}(W)}$$

where e (the identity element) and c are the lattice's minimal and maximal, respectively.

Theorem 10.0.2 ((Reading, 2008))**.** *Let (W, S) be a Coxeter system and let c be a Coxeter element in W. Let $W_{\langle s\rangle}$ denote the parabolic subgroup generated by $S\backslash\{s\}$ and let $c_{\langle s\rangle}$ be the Coxeter element for $W_{\langle s\rangle}$ obtained by deleting s from the defining word for c then $[e, c_{\langle s\rangle}]$ in $W_{\langle s\rangle}$ is isomorphic to $[e, c_{\langle s\rangle}]$ in W. In particular every element below $c_{\langle s\rangle}$ in W is in $W_{\langle s\rangle}$ and vice versa.*

Lemma 10.0.3. *Let s_1 be S-initial in a Coxeter element c and let $c = s_1 r_2 \cdots r_n$ be a T-factorization of c then the reflections $r_2, \ldots, r_n$ are in $W_{\langle s_1\rangle}$.*

Proof. If s is S-initial in c then there exists a reduced S-factorization of c, say, $c = s_1 s_2 \cdots s_n$ such that s_i's are all simple reflections. This implies that the element $r_2 \cdots r_n$ is actually the parabolic Coxeter element $c_{\langle s_1 \rangle} = s_2 \cdots s_n$. By Theorem 10.0.2 therefore the interval $[e, c_{\langle s_1 \rangle}]_{\mathrm{Abs}(W)}$ is an exact copy of the interval $[e, c_{\langle s_1 \rangle}]_{\mathrm{Abs}(W_{\langle s_1 \rangle})}$. Since the reflections $r_2, \ldots, r_n$ are all $T-$initial in $r_2 \cdots r_n$, therefore appear in the lattice $[e, c_{\langle s_1 \rangle}]_{\mathrm{Abs}(W)}$, and consequently must appear in the lattice $[e, c_{\langle s_1 \rangle}]_{\mathrm{Abs}(W_{\langle s_1 \rangle})}$. Now since every element in the lattice $[e, c_{\langle s_1 \rangle}]_{\mathrm{Abs}(W_{\langle s_1 \rangle})}$ are in $W_{\langle s_1 \rangle}$ thus $r_2, \ldots, r_n$ are in $W_{\langle s_1 \rangle}$. □

We can prove a similar result for a simple reflection that is S-final in c.

Lemma 10.0.4. *Let s_n be S-final in a Coxeter element c and let $c = r_1 \cdots r_{n-1} s_n$ be a T-factorization of c then the reflections $r_1, \ldots, r_{n-1}$ are in $W_{\langle s_n \rangle}$.*

Lemma 10.0.5 ((Speyer, 2007))**.** *Let w_0 be the longest element in W then $\mathsf{w}_0(\mathsf{c})$ is initial in c^∞, where c denotes an S-word for a Coxeter element c.*

It must be noted here that $\mathsf{w}_0(\mathsf{c})$ may not be a prefix in the word c^∞, for example for the Coxeter element $c = s_1 s_2 s_3$, $\mathsf{w}_0(\mathsf{s}_1\mathsf{s}_2\mathsf{s}_3)$ is the following underlined subword in $(\mathsf{s}_1\mathsf{s}_2\mathsf{s}_3)^\infty$

$$\underline{\mathsf{s}_1}\ \underline{\mathsf{s}_2}\ \underline{\mathsf{s}_3}|\underline{\mathsf{s}_1}\ \underline{\mathsf{s}_2}\ \mathsf{s}_3|\underline{\mathsf{s}_1}\ \mathsf{s}_2\ \mathsf{s}_3|\cdots$$

Lemma 10.0.6. *Let c denote a reduced S-word for a Coxeter element c, then c is a prefix in $\mathsf{w}_0(\mathsf{c})$.*

Proof. Any element in W is S-initial in the element w_0, in particular any Coxeter element is S-initial in w_0. Also the word c is lexicographically minimal in the word c^∞. Since the word $\mathsf{w}_0(\mathsf{c})$ is defined to be the lexicographically first subword of the word c^∞ that is a reduced S-expression for w_0, therefore c is a prefix in $\mathsf{w}_0(\mathsf{c})$. □

Lemma 10.0.7. *Let $\mathsf{s}\mathsf{s}_2 \cdots \mathsf{s}_n$ be a reduced S-word for a Coxeter element c, then $\mathsf{w}_0(\mathsf{s}_2 \cdots \mathsf{s}_n \mathsf{s})$ is commutation equivalent to $\bar{\mathsf{s}}\mathsf{w}_0(\mathsf{s}\mathsf{s}_2 \cdots \mathsf{s}_n)\mathsf{s}^{\mathsf{w}_0^{-1}}$. However, if s is the last letter in an S-word, say $\mathsf{s}_1 \cdots \mathsf{s}_{n-1}\mathsf{s}$ for c, then $\mathsf{w}_0(\mathsf{s}\mathsf{s}_1 \cdots \mathsf{s}_{n-1})$ is commutation equivalent to $\mathsf{s}\mathsf{w}_0(\mathsf{s}_1 \cdots \mathsf{s}_{n-1}\mathsf{s})\overline{\mathsf{s}^{\mathsf{w}_0^{-1}}}$. Here $\bar{\mathsf{s}}\mathsf{w}$ (or $\mathsf{w}\bar{\mathsf{s}}$) denotes the word obtained by removing an initial (or the final) letter s from a word w.*

Proof. By Lemma 10.0.6, let $\mathsf{w}_0(\mathsf{s}\mathsf{s}_2\cdots\mathsf{s}_n) = \mathsf{s}\mathsf{s}_2\cdots\mathsf{s}_N$. Using Lemma 10.0.5 we can say that the word $\mathsf{w}_0(\mathsf{s}\mathsf{s}_2\cdots\mathsf{s}_n) = \mathsf{s}\mathsf{s}_2\cdots\mathsf{s}_N$ is initial in the word $(\mathsf{s}\mathsf{s}_2\cdots\mathsf{s}_n)^\infty$, therefore the word $\mathsf{s}_2\cdots\mathsf{s}_N$ is initial in $(\mathsf{s}_2\cdots\mathsf{s}_n\mathsf{s})^\infty$, thus by Lemma 9.0.2, $\mathsf{s}_2\cdots\mathsf{s}_N$ is initial in $\mathsf{w}_0(\mathsf{s}_2\cdots\mathsf{s}_n\mathsf{s})$. However $l_S(s_2\cdots s_N)+1 = l_S(w_0(s_2\cdots s_n s))$, therefore $\mathsf{w}_0(\mathsf{s}_2\cdots\mathsf{s}_n\mathsf{s}) \equiv \mathsf{s}_2\cdots\mathsf{s}_N\mathsf{s}_i$ for some $s_i \in S$. Here the '$\equiv$' (instead of '$=$') is due to the fact that $\mathsf{s}_2\cdots\mathsf{s}_N$ is initial in $\mathsf{w}_0(\mathsf{s}_2\cdots\mathsf{s}_n\mathsf{s})$ and not necessarily a prefix.

$$\begin{aligned}
&\mathsf{s}_2\cdots\mathsf{s}_N\mathsf{s}_i = \mathsf{s}\mathsf{s}_2\cdots\mathsf{s}_N\\
\implies &\mathsf{s}_i = \mathsf{s}_N^{-1}\cdots\mathsf{s}_2^{-1}\mathsf{s}\mathsf{s}_2\cdots\mathsf{s}_N\\
\implies &\mathsf{s}_i = \mathsf{s}_N^{-1}\cdots\mathsf{s}_2^{-1}\mathsf{s}^{-1}\mathsf{s}\mathsf{s}\mathsf{s}_2\cdots\mathsf{s}_N\\
\implies &\mathsf{s}_i = \mathsf{w}_0^{-1}\mathsf{s}\mathsf{w}_0
\end{aligned}$$

Consequently, $\mathsf{w}_0(\mathsf{s}_2\cdots\mathsf{s}_n\mathsf{s}) \equiv \bar{\mathsf{s}}\mathsf{w}_0(\mathsf{s}\mathsf{s}_2\cdots\mathsf{s}_n)\mathsf{s}^{\mathsf{w}_0^{-1}}$.

On the other hand if $\mathsf{s}_1\cdots\mathsf{s}_{n-1}\mathsf{s}$ is a reduced S-word for c, (here s is the final letter of the word), then there exists a Coxeter element $c' = scs^{-1}$ whose S-factorization is $\mathsf{s}\mathsf{s}_1\cdots\mathsf{s}_{n-1}$, with s as its first letter. Therefore $\mathsf{w}_0(\mathsf{s}_1\cdots\mathsf{s}_{n-1}\mathsf{s}) \equiv \bar{\mathsf{s}}\mathsf{w}_0(\mathsf{s}\mathsf{s}_1\cdots\mathsf{s}_{n-1})\mathsf{s}^{\mathsf{w}_0^{-1}} \iff \mathsf{s}\mathsf{w}_0(\mathsf{s}_1\cdots\mathsf{s}_{n-1}\mathsf{s})\overline{\mathsf{s}^{\mathsf{w}_0^{-1}}} \equiv \mathsf{w}_0(\mathsf{s}\mathsf{s}_1\cdots\mathsf{s}_{n-1})$. This will make sense if $\mathsf{s}^{\mathsf{w}_0^{-1}}$ is final in $\mathsf{w}_0(\mathsf{s}_1\cdots\mathsf{s}_{n-1}\mathsf{s})$ which is shown in the next lemma. ☐

Lemma 10.0.8. *For a Coxeter element $c \in W$, let $\mathsf{s}_1\mathsf{s}_2\cdots\mathsf{s}_n$ be a reduced S-factorization for c, then the word $\mathsf{s}_1^{\mathsf{w}_0^{-1}}\mathsf{s}_2^{\mathsf{w}_0^{-1}}\cdots\mathsf{s}_n^{\mathsf{w}_0^{-1}}$ is final in the word $\mathsf{w}_0(\mathsf{s}_1\mathsf{s}_2\cdots\mathsf{s}_n)$.*

Proof. By Lemma 10.0.6 any $\mathsf{s}_1\mathsf{s}_2\cdots\mathsf{s}_n$ is a prefix of the word $\mathsf{w}_0(\mathsf{s}_1\mathsf{s}_2\cdots\mathsf{s}_n)$. By the first part of Lemma 10.0.7 if we remove the letters $\mathsf{s}_1, \mathsf{s}_2, \ldots, \mathsf{s}_n$, sequentially, from the beginning of the word $\mathsf{s}_1\mathsf{s}_2\cdots\mathsf{s}_n$ and then add them to the end of the word then we get back $\mathsf{s}_1\mathsf{s}_2\cdots\mathsf{s}_n$ and therefore obtain

$$\bar{\mathsf{s}}_n\cdots\bar{\mathsf{s}}_2\bar{\mathsf{s}}_1\mathsf{w}_0(\mathsf{s}_1\mathsf{s}_2\cdots\mathsf{s}_n)\mathsf{s}_1^{\mathsf{w}_0^{-1}}\mathsf{s}_2^{\mathsf{w}_0^{-1}}\cdots\mathsf{s}_n^{\mathsf{w}_0^{-1}} \equiv \mathsf{w}_0(\mathsf{s}_1\mathsf{s}_2\cdots\mathsf{s}_n)$$

This proves the desired result. ☐

Lemma 10.0.9 ((Reading and Speyer, 2011))**.** *Let c be a Coxeter element in a Coxeter group W and let c be a reduced S-word for c. Let s be initial in c then for any two reflections r_1 and r_2 in $T \cap W_{\langle s\rangle}$, $r_1 \leq_{\mathsf{c}} r_2 \iff r_1 \leq_{\bar{\mathsf{s}}\mathsf{c}\mathsf{s}} r_2$, where T is the set of reflections.*

Definition 10.0.10 (Cambrian rotation)**.** Let c be a Coxeter element in a Coxeter system (W, S). For any $s \in S$ such that s is S-initial in c we define a *left Cambrian rotation* to be the conjugation of c by s to obtain a new Coxeter element $s^{-1}cs$. Similarly for any $s \in S$ such that s is S-final in c we define the *right Cambrian rotation* to be the conjugation of c by s to obtain a new Coxeter element scs^{-1}.

Lemma 10.0.11. *Let c be a Coxeter element and let $\mathsf{s}_1 \cdots \mathsf{s}_n$ be a reduced S-word for c and let the word $\mathsf{s}_1 \cdots \mathsf{s}_j$ be a prefix of the word $\mathsf{w}_0(\mathsf{s}_1 \cdots \mathsf{s}_n)$. Then the first letter in the word*

$$\bar{\mathsf{s}}_j \cdots \bar{\mathsf{s}}_1 \mathsf{w}_0(\mathsf{s}_1 \cdots \mathsf{s}_n)\mathsf{s}_1^{\mathsf{w}_0^{-1}} \cdots \mathsf{s}_j^{\mathsf{w}_0^{-1}}$$

is S-initial in the Coxeter element $s_j^{-1} \cdots s_1^{-1} c s_1 \cdots s_j$. Similarly, if $\mathsf{s}_j \cdots \mathsf{s}_N$ is a suffix of the word $\mathsf{w}_0(\mathsf{s}_1 \cdots \mathsf{s}_n)$ then the last letter in the word

$$\mathsf{s}_j^{\mathsf{w}_0} \cdots \mathsf{s}_N^{\mathsf{w}_0} \mathsf{w}_0(\mathsf{s}_1 \cdots \mathsf{s}_n)\bar{\mathsf{s}}_N \cdots \bar{\mathsf{s}}_j$$

is S-final in the Coxeter element $s_j^{w_0} \cdots s_N^{w_0} c \big(s_N^{w_0}\big)^{-1} \cdots \left(s_j^{w_0}\right)^{-1}$.

Proof. For any word w let $\mathrm{rev}(\mathsf{w})$ denote the reverse of the word w. Since $\mathsf{s}_1 \cdots \mathsf{s}_n$ is a prefix of the word $\mathsf{w}_0(\mathsf{s}_1 \cdots \mathsf{s}_n)$ by Lemma 10.0.6 therefore if $j < n$ then it follows directly. To see that it is in general true for any j, recall that the word $\mathsf{w}_0(\mathsf{s}_1 \cdots \mathsf{s}_n)$ is initial in the word $(\mathsf{s}_1 \cdots \mathsf{s}_n)^\infty$ by Lemma 10.0.5. For the second part of the lemma, we observe that by Lemma 10.0.5, $\mathsf{w}_0\left(\mathrm{rev}\left(\mathsf{s}_1^{\mathsf{w}_0^{-1}} \cdots \mathsf{s}_n^{\mathsf{w}_0^{-1}}\right)\right)$ is initial in $\left(\mathrm{rev}\left(\mathsf{s}_1^{\mathsf{w}_0^{-1}} \cdots \mathsf{s}_n^{\mathsf{w}_0^{-1}}\right)\right)^\infty$. Since $\mathsf{s}_1^{\mathsf{w}_0^{-1}} \mathsf{s}_2^{\mathsf{w}_0^{-1}} \cdots \mathsf{s}_n^{\mathsf{w}_0^{-1}}$ is final in the word $\mathsf{w}_0(\mathsf{s}_1\mathsf{s}_2 \cdots \mathsf{s}_n)$ we can conclude that $\mathsf{w}_0\left(\mathrm{rev}\left(\mathsf{s}_1^{\mathsf{w}_0^{-1}} \cdots \mathsf{s}_n^{\mathsf{w}_0^{-1}}\right)\right) \equiv \mathrm{rev}\left(\mathsf{w}_0(\mathsf{s}_1 \cdots \mathsf{s}_n)\right)$. This gives us the desired result. ☐

Lemma 10.0.12. *Fix a reduced S-word $\mathsf{c} = \mathsf{s}_1 \cdots \mathsf{s}_n$ for a Coxeter element c in W. Given any reflection t, there exists a word w which is a prefix of the word $\mathsf{w}_0(\mathsf{s}_1 \cdots \mathsf{s}_n)$ and a reduced S-word c' for the Coxeter element $w^{-1}cw$ such that t^w is a simple reflection and is the minimal element in the order $\leq_{\mathsf{c}'}$ and if $t \leq_{\mathsf{c}} t_1 \leq_{\mathsf{c}} t_2$ then $t_1^w \leq_{\mathsf{c}'} t_2^w$ and if $t_1 \leq_{\mathsf{c}} t \leq_{\mathsf{c}} t_2$ then $t_2^w \leq_{\mathsf{c}'} t_1^w$.*

Similarly there exists a word $\widetilde{\mathsf{w}}$ which is a suffix of the word $\mathrm{w}_0(\mathrm{s}_1 \cdots \mathrm{s}_n)$ and a reduced S-word $\hat{\mathrm{c}}$ for the Coxeter element $\widetilde{w} c \widetilde{w}^{-1}$ such that $t^{\widetilde{w}}$ is a simple reflection and is the maximal in the order $\leq_{\hat{\mathsf{c}}}$ and if $t_1 \leq_{\mathsf{c}} t_2 \leq_{\mathsf{c}} t$ then $t_1^{\widetilde{w}} \leq_{\hat{\mathsf{c}}} t_2^{\widetilde{w}}$ and if $t_1 \leq_{\mathsf{c}} t \leq_{\mathsf{c}} t_2$ then $t_2^{\widetilde{w}} \leq_{\hat{\mathsf{c}}} t_1^{\widetilde{w}}$.

Proof. Let $\mathsf{w}_0(\mathsf{s}_1\cdots\mathsf{s}_n)=\mathsf{s}_1\cdots\mathsf{s}_N$. Then there exists $\mathsf{w}=\mathsf{s}_1\cdots\mathsf{s}_{m-1}$ a prefix of $\mathsf{w}_0(\mathsf{s}_1\cdots\mathsf{s}_n)$ such that $t=s_m^w$. This existence is guaranteed by the fact that the set R^+ and the set $\mathrm{inv}(\mathsf{w}_0(\mathsf{c}))$ are the same. Using Lemma 10.0.7 and Lemma 10.0.11 there exists a reduced S-word c' for the Coxeter element $w^{-1}cw$ such that $\mathsf{s}_m\cdots\mathsf{s}_N\mathsf{s}_1^{\mathsf{w}_0^{-1}}\cdots\mathsf{s}_{m-1}^{\mathsf{w}_0^{-1}}$ is commutation equivalent to $\mathsf{w}_0(\mathsf{c}')$.

Now, if $t\le_{\mathsf{c}} t_1\le_{\mathsf{c}} t_2$ then $t=s_1\cdots s_m\cdots s_1$, $t_1=s_1\cdots s_m\cdots s_{m+a}\cdots s_m\cdots s_1$ and $t_2=s_1\cdots s_m\cdots s_{m+a}\cdots s_{m+b}\cdots s_{m+a}\cdots s_m\cdots s_1$, for some a,b such that $N\ge m+b>m+a>m$ and $s_1\cdots s_m\cdots s_{m+a}\cdots s_{m+b}$ is a prefix of the word $w_0(s_1\cdots s_n)$. Therefore conjugating t,t_1 and t_2 by w gives $t^w=s_m, t_1^w=s_m\cdots s_{m+a}\cdots s_m$ and $t_2^w=s_m\cdots s_{m+a}\cdots s_{m+b}\cdots s_{m+a}\cdots s_m$, which clearly respects the order $t^w\le_{\mathsf{c}'} t_1^w\le_{\mathsf{c}'} t_2^w$.

Now, for the case $t_1\le_{\mathsf{c}} t\le_{\mathsf{c}} t_2$, it suffices to show that for any reduced S-word $\widetilde{\mathsf{c}}$ for a Coxeter element and for a word $u_1\cdots u_N$ commutation equivalent to $w(\tilde{c})$, u_1 is the maximal reflection in the order $\le_{\tilde{c}'}$, and for all the other reflections if $v_i\le_{\tilde{c}} v_j$ then $v_i^{u_1}\le_{\tilde{c}'} v_j^{u_1}$ where $\tilde{c}'$ is a reduced S-word for the Coxeter element $u_1^{-1}\tilde{c}u_1$. This follows from the following observations. u_1 is S-initial in $\tilde{c}$ by Lemma 10.0.11 thus using Lemma 10.0.7 we obtain $w_0(\tilde{c}')=u_2\cdots u_N u_1^{w_0^{-1}}$ and we also have

$$u_1\le_{\tilde{c}} u_1u_2u_1\le_{\tilde{c}}\cdots\le_{\tilde{c}} u_1u_2\cdots u_{N-1}u_Nu_{N-1}\cdots u_2u_1.$$

$$u_2\le_{\tilde{c}'}\cdots\le_{\tilde{c}'} u_2\cdots u_{N-1}u_Nu_1^{w_0^{-1}}u_Nu_{N-1}\cdots u_2$$

where $u_2\cdots u_{N-1}u_Nu_1^{w_0^{-1}}u_Nu_{N-1}\cdots u_2=u_1$ and $c'=u_2\cdots u_nu_1$.

The other part can be similarly proved. □

Definition 10.0.13. Let c be a Coxeter element in a Coxeter system (W,S), and $|S|=n$. Any reduced T-decomposition: $c=r_1\cdots r_n$ will be called a *two-part factorization* if there exists an index i such that, $r_1\le_c\cdots\le_c r_i$ and $r_{i+1}\le_c\cdots\le_c r_n$ where $\le_c$ is Coxeter order. We may use a divider '|' to indicate the index of partition—$r_1\cdots r_i|r_{i+1}\cdots r_n$. We will call the word $r_1\cdots r_i$ the left part and the word $r_{i+1}\cdots r_n$ the right part of the two-part factorization.

We will denote the set of all two-part factorizations of c by *$Fact_2(c)$*.

Definition 10.0.14. Let c be a Coxeter element, and let $F_c = r_1 \cdots r_i | r_{i+1} \cdots r_n$ be a two-part factorization of c. A *factorization mutation* on F_c at r_k $(1 \leq k \leq i)$ denoted by $\mu_k^{\text{fact}}(F_c)$ is the action $\mu_{l-1}\mu_{l-2}\cdots\mu_{k+1}\mu_k(F_c)$, where l is the index $(i+1 \leq l \leq n)$ such that $r_l \leq_c r_k \leq_c r_{l+1}$.

Essentially, we choose a reflection say r_k from the left part of the two-part factorization and through a series of Hurwitz move we 'push' it to a new position on the right part of the two-part factorization such that the resulting decomposition is still a two-part factorization.

Since any Coxeter elements can be written as a product of all the simple reflections, each appearing exactly once, therefore we can start by writing a reduced S-word for a Coxeter element and add a divider '|' at the end of it and perform factorization mutations on it to obtain all possible two-part factorizations. By Theorem 4.3.4 all the reduced T-decompositions for a Coxeter element can be obtained by Hurwitz moves, therefore all two-part factorizations, being a subset of all the reduced T-decompositions, can also be obtained by Hurwitz moves.

It is not obvious that the resulting factorization of c is also a two-part factorization. The next lemma will address this concern.

Lemma 10.0.15. *For a Coxeter element c in a Coxeter group W let F_c denote a two part factorization of c then $\mu_k^{\text{fact}}(F_c) \in Fact_2(c)$, where k is an index on the left part of the two-part factorization F_c.*

Proof. Let $F_c = r_1 \cdots r_k \cdots r_i | r_{i+1} \cdots r_j \cdots r_l$ be a two-part factorization such that $r_{i+1} \leq_{\mathsf{c}} \cdots \leq_{\mathsf{c}} r_j \leq_{\mathsf{c}} r_k \leq_{\mathsf{c}} \cdots \leq_{\mathsf{c}} r_i$. Let

$$F_c' = r_1 \cdots r_{k-1} r_{k+1}^{r_k} \cdots r_i^{r_k} r_{i+1}^{r_k} \cdots r_j^{r_k} r_k r_{j+1} \cdots r_l$$

We will show that F_c' is a two-part factorization of c, i.e. $\mu_k^{\text{fact}}(F_c)$ is a two-part factorization.

Let $w_0(\mathsf{c}) = s_1 \cdots s_N$ then all the reflections in the group W may be arranged as follows in the order $\leq_{\mathsf{c}}$

$$s_1 \leq_{\mathsf{c}} s_1 s_2 s_1 \leq_{\mathsf{c}} \cdots \leq_{\mathsf{c}} s_1 s_2 \cdots s_N \cdots s_2 s_1$$

and they may be alternatively arranged as follows in the order $\leq_{\bar{\mathsf{s}}_1 \mathsf{c} \mathsf{s}_1}$

$$s_2 \leq_{\bar{\mathsf{s}}_1 \mathsf{c} \mathsf{s}_1} s_2 s_3 s_2 \leq_{\bar{\mathsf{s}}_1 \mathsf{c} \mathsf{s}_1} \cdots \leq_{\bar{\mathsf{s}}_1 \mathsf{c} \mathsf{s}_1} \underbrace{s_2 s_3 \cdots s_N s_1^{w^{-1}} s_N \cdots s_3 s_2}_{s_1}.$$

This follows from the result $w_0(\bar{\mathsf{s}}_1\mathsf{c}\mathsf{s}_1) \equiv \bar{\mathsf{s}}_1 w_0(\mathsf{c})\mathsf{s}_1^{w_0^{-1}}$ (Lemma 10.0.7). So, if we have a two-part factorization, say $\ell_1 \cdots \ell_q | r_1 r_2 \cdots r_{q'}$ that respects the order $\leq_{\mathsf{c}}$, then performing a left Cambrian rotation would give us a new two-part factorization involving modified reflections (t gets modified to t^{s_1}), that respects the order $\leq_{\bar{\mathsf{s}}_1\mathsf{c}\mathsf{s}_1}$, and the new two-part factorization would either be

$$\ell_1^{s_1} \cdots \ell_q^{s_1} | r_1^{s_1} r_2^{s_1} \cdots r_{q'}^{s_1}$$

or

$$\ell_1^{s_1} \cdots \ell_q^{s_1} r_1^{s_1} | r_2^{s_1} \cdots r_{q'}^{s_1}.$$

In summary, a Cambrian rotation, either doesn't affect the position of the '|' or just pushes it to the right by a position. Consequently, while undoing this Cambrian rotation, either the '|' stays put or moves left by a position, depending on whether it has moved earlier or not.

With this in mind now we observe that since $R^+ = \mathrm{inv}(w_0(\mathsf{c}))$ therefore there exists a simple reflection s_m, such that $w_0 = r_k s_1 \cdots \hat{s}_m \cdots s_N$, i.e. $s_1 \cdots s_m \cdots s_1^{-1} = r_k$. Set $w = s_1 \cdots s_{m-1}$ then $r_k = w s_m w^{-1}$. Conjugating c by $w^{-1} = s_{m-1} \cdots s_1$ (which is equivalent to $m-1$ Cambrian rotations) we obtain a new Coxeter element $w^{-1}cw$ (call it c') and $w_0(\mathsf{c}') \equiv s_m \cdots s_N s_1^{w^{-1}} \cdots s_{m-1}^{w^{-1}}$ (by repeated application of Lemma 10.0.7 and Lemma 10.0.11). These $m-1$ Cambrian rotations give us a new two-part factorization involving modified reflections (t gets modified to $t^{w^{-1}}$) that respects the order $\leq_{\mathsf{c}'}$ (up to commutation of commuting letters) in which s_m is the minimal element. We now choose the subword $r_k \cdots r_i | r_{i+1} \cdots r_j$ from F_c (the subword in F_c that gets affected due to the factorization mutation at r_k) and observe it as we conjugate each of the reflections in it by w^{-1} and thus obtain the following two-part factorization that respects the order $\leq_{\mathsf{c}'}$ (up to commutation of commuting letters)

$$s_m r_{k+1}^{w^{-1}} \cdots r_i^{w^{-1}} r_{i+1}^{w^{-1}} \cdots r_j^{w^{-1}} |. \tag{10.1}$$

This follows from our earlier discussion about how reflections move around the '|' due to Cambrian rotations and the fact that s_m is the minimal element in $\leq_{\mathsf{c}'}$. We also notice here that since s_m is S-initial in c' therefore by Lemma 10.0.3 the reflections $r_{k+1}^{w^{-1}}, \ldots, r_i^{w^{-1}}, r_{i+1}^{w^{-1}}, \cdots, r_j^{w^{-1}}$ are in $W_{\langle s_m \rangle}$. At this point we perform a factorization mutation on $s_m r_{k+1}^{w^{-1}} \cdots r_i^{w^{-1}} r_{i+1}^{w^{-1}} \cdots r_j^{w^{-1}} |$ at

s_m, which results in the following two-part factorization that still respects the order $\leq_{\mathsf{c}'}$ (upto commutation)

$$r_{k+1}^{(ws_m)^{-1}} \cdots r_i^{(ws_m)^{-1}} r_{i+1}^{(ws_m)^{-1}} \cdots r_j^{(ws_m)^{-1}} \Big| s_m. \tag{10.2}$$

Here the reflections $r_{k+1}^{(ws_m)^{-1}}, \ldots, r_i^{(ws_m)^{-1}}, r_{i+1}^{(ws_m)^{-1}}, \cdots, r_j^{(ws_m)^{-1}}$ are in $W_{\langle s_m \rangle}$[1]. For any root $v \in R^+_{\langle s_m \rangle}$, $s_m(v) = v + kv_{s_m}$, where v_{s_m} is the root corresponding to the reflection s_m and k is some non-negative number. Therefore,

$$\begin{aligned}
r_{k+1}^{(ws_m)^{-1}} &\leq_{\bar{\mathsf{s}}_m \mathsf{c}' \mathsf{s}_m} r_{k+1}^{w^{-1}} \\
\vdots \quad & \quad \vdots \quad \vdots \\
r_i^{(ws_m)^{-1}} &\leq_{\bar{\mathsf{s}}_m \mathsf{c}' \mathsf{s}_m} r_i^{w^{-1}} \\
r_{i+1}^{(ws_m)^{-1}} &\leq_{\bar{\mathsf{s}}_m \mathsf{c}' \mathsf{s}_m} r_{i+1}^{w^{-1}} \\
\vdots \quad & \quad \vdots \quad \vdots \\
r_j^{(ws_m)^{-1}} &\leq_{\bar{\mathsf{s}}_m \mathsf{c}' \mathsf{s}_m} r_j^{w^{-1}}.
\end{aligned}$$

Using the fact that $r_{k+1}^{(ws_m)^{-1}}, \ldots, r_i^{(ws_m)^{-1}}, r_{i+1}^{(ws_m)^{-1}}, \cdots, r_j^{(ws_m)^{-1}}, r_{k+1}^{w^{-1}}, \ldots, r_i^{w^{-1}}, r_{i+1}^{w^{-1}}, \cdots, r_j^{w^{-1}}$ are in $W_{\langle s_m \rangle}$ along with Lemma 10.0.9 we get

---

[1] Instead of performing the factorization mutation on Equation (10.1) if we performed another Cambrian rotation by conjugating c' by s_m, then we would have obtained the factorization

$$r_{k+1}^{(ws_m)^{-1}} \cdots r_i^{(ws_m)^{-1}} r_{i+1}^{(ws_m)^{-1}} \cdots r_j^{(ws_m)^{-1}} s_m \Big|.$$

which respects the order $\leq_{\bar{\mathsf{s}}_m \mathsf{c}' \mathsf{s}_m}$ (up to commutation of commuting letters) where s_m would have been the maximal simple reflection and using Lemma 10.0.4 we could say that the reflections $r_{k+1}^{(ws_m)^{-1}}, \ldots, r_i^{(ws_m)^{-1}}, r_{i+1}^{(ws_m)^{-1}}, \cdots, r_j^{(ws_m)^{-1}}$ are in $W_{\langle s_m \rangle}$.

$$
\begin{aligned}
r_{k+1}^{(ws_m)^{-1}} &\leq_{\mathsf{c}'} r_{k+1}^{w^{-1}} \\
\vdots \quad & \quad \vdots \quad \vdots \\
r_i^{(ws_m)^{-1}} &\leq_{\mathsf{c}'} r_i^{w^{-1}} \\
r_{i+1}^{(ws_m)^{-1}} &\leq_{\mathsf{c}'} r_{i+1}^{w^{-1}} \\
\vdots \quad & \quad \vdots \quad \vdots \\
r_j^{(ws_m)^{-1}} &\leq_{\mathsf{c}'} r_j^{w^{-1}}.
\end{aligned}
$$

Now we will undo the effect of the left Cambrian rotations that we performed earlier, so as to revert back to our original order, i.e. $\leq_{\mathsf{c}}$. This can be done by conjugating each of the reflections in the two-part factorization Equation (10.2) by w. The effect of this 'undoing' is quite simple on s_m, viz. $s_m^w = r_k$ but to see how it affects the other reflections we need to understand that if we conjugated each of the reflections in Equation (10.1) by w then we would end up with

$$
s_m^w r_{k+1}^{w^{-1}w} \cdots r_i^{w^{-1}w} | r_{i+1}^{w^{-1}w} \cdots r_j^{w^{-1}w}
$$

which is exactly what we started with, but since the reflections in Equation (10.2) have 'moved' earlier in the order $\leq_{\mathsf{c}'}$ than in Equation (10.1) therefore conjugating them by w would either give us the two-part factorization

$$
r_1 \cdots r_{k-1} r_{k+1}^{r_k} \cdots r_i^{r_k} \big| r_{i+1}^{r_k} \cdots r_j^{r_k} r_k r_{j+1} \cdots r_l
$$

or the same factorization with the '|' shifted to the right. In either case we obtain a two-part factorization.

This entire episode of multiple conjugations didn't actually change the Coxeter element c. However, it gave rise to a new factorization for c and the '|' accordingly shifted to adjust for this change. This can be verified from the following steps

$$
\begin{aligned}
& r_k r_{k+1} \cdots r_i r_{i+1} \cdots r_j \\
=& (ww^{-1}) r_k (ww^{-1}) r_{k+1} (ww^{-1}) \cdots (ww^{-1}) r_i (ww^{-1}) r_{i+1} (ww^{-1}) \cdots (ww^{-1}) r_j (ww^{-1}) \\
=& w \left[(w^{-1} r_k w)(w^{-1} r_{k+1} w) \cdots (w^{-1} r_i w)(w^{-1} r_{i+1} w) \cdots (w^{-1} r_j w) \right] w^{-1} \\
=& w \left[s_m r_{k+1}^{w^{-1}} \cdots r_i^{w^{-1}} r_{i+1}^{w^{-1}} \cdots r_j^{w^{-1}} \right] w^{-1} \\
=& w \left[r_{k+1}^{(ws_m)^{-1}} \cdots r_i^{(ws_m)^{-1}} r_{i+1}^{(ws_m)^{-1}} \cdots r_j^{(ws_m)^{-1}} \right] s_m w^{-1} \\
=& w \left[(s_m w^{-1} r_{k+1} w s_m)(w^{-1} w) \cdots (w^{-1} w)(s_m w^{-1} r_j w s_m)(w^{-1} w) \right] s_m w^{-1} \\
=& (w s_m w^{-1}) r_{k+1} (w s_m w^{-1}) \cdots (w s_m w^{-1}) r_j (w s_m w^{-1})(w s_m w^{-1}) \\
=& r_{k+1}^{r_k} \cdots r_j^{r_k} r_k
\end{aligned}
$$

□

Definition 10.0.16. Given a two-part factorization of a Coxeter element c

$$F_c = \ell_1 \cdots \ell_i | r_{i+1} \cdots r_n,$$

we define an *associated quiver* $\mathcal{Q}_{F_c}$ with n vertices labelled $\ell_1, \cdots, \ell_i, r_{i+1}, \cdots, r_n$ and arrows

- from ℓ_b to ℓ_a if $\ell_a \leq_c \ell_b$ and ℓ_b, ℓ_a don't commute,
- from r_b to r_a if $r_a \leq_c r_b$ and r_b, r_a don't commute,
- from r_a to ℓ_b if $r_a \leq_c \ell_b$ and r_a, ℓ_b don't commute, and
- from ℓ_a to r_b if $\ell_a \leq_c r_b$ and r_b, ℓ_a don't commute.

The 'c' in F_c or $\mathcal{Q}_{F_c}$ may be dropped when the coxeter element c is clearly understood.

Theorem 10.0.17. *Let (W, S) be a Coxeter system of simply-laced type. Let c be a Coxeter element in W and let F_c be a two-part factorization of c, then*

$$\mu_k^{\text{quiv}}(\mathcal{Q}_{F_c}) = \mathcal{Q}_{\mu_k^{\text{fact}}(F_c)}.$$

where $\mathcal{Q}_{F_c}$ is the quiver associated to the two-part factorization F_c and $\mathcal{Q}_{\mu_k^{\text{fact}}(F_c)}$ is the quiver associated to the two-part factorization $\mu_k^{\text{fact}}(F_c)$.

Proof. It is easy to check for a Coxeter group of type A_3, see Figure 10.1. To see that this holds in general for Coxeter groups of simply-laced type, we consider the following facts from Chapter 8. Recall that a simply-laced Coxeter group is a Coxeter group whose Coxeter-Dynkin diagram corresponds to a simply-laced Dynkin diagrams i.e. Dynkin diagram of type A_n, D_n, E_6, E_7, E_8. In Lemma 8.2.10 and Lemma 8.2.11 we have shown that for any quiver, mutation equivalent to an orientation of a simply-laced Dynkin diagram all its edges have an assigned weight of 1 and all its 3-cycles are cyclic. Also, since quiver mutation at a node only affects the edges incident to the node and the edges opposite to the node, whenever the node is a part of a 3-cycle; therefore it suffices to show that for every subword w_c in a two-part factorization of a Coxeter element c, containing the reflection t_k, the quiver $\mu_k^{\text{quiv}}(\mathcal{Q}_{w_c})$ is the same as the quiver $\mathcal{Q}_{\mu_k^{\text{fact}}(w_c)}$, where the subword w_c consists of just the two reflections t_l and t_k such that in the associated quiver the nodes t_k and t_l share an edge; or consists of just the three reflections t_m, t_n and t_k such that in the associated quiver the nodes t_k, t_m and t_n form a cyclic 3-cycle. In Section 10.2 we have given a case by case proof by considering all such possible subwords arising in two-part factorizations of Coxeter elements. □

10.1 Properties of Factorization Mutation

This section lists a few of the properties of factorization mutation that we will be using in the next section for our case by case proof.

Lemma 10.1.1. *For a factorization mutation of the form*

$$\cdots t_k \cdots t_l \cdots \big| \cdots \xrightarrow[\mu_k^{\text{quiv}}]{} \cdots t_l^{t_k} \cdots \big| \cdots t_k \cdots$$

we have $t_k \leq_c t_l^{t_k} \leq_c t_l$.

Proof. Using the same technique as in Lemma 10.0.15 we perform left Cambrian rotations until t_k is modified to a simple reflection that is also the minimal element in the new Coxeter order (call it s_k). Now it is easy to see that in this new order the modified $t_l^{t_k}$ appears earlier than the modified t_l but appears later than s_k, therefore if we undo the Cambrian rotations we must have $t_k \leq_c t_l^{t_k} \leq_c t_l$. □

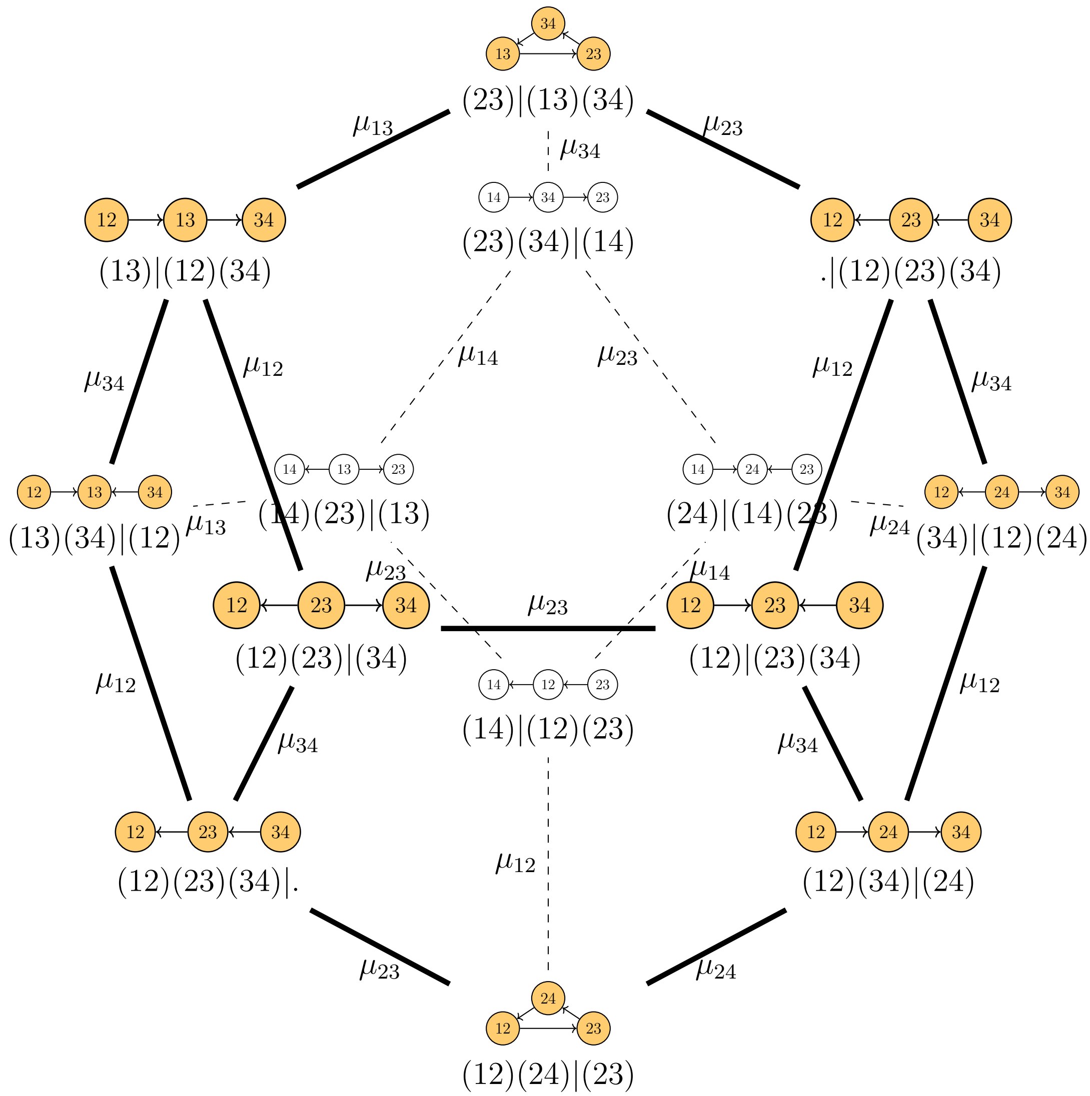


Figure 10.1. A correspondence between factorization mutation and quiver mutation for $\mathfrak{S}_3$.

Lemma 10.1.2. *For a factorization mutation of the form*

$$\cdots t_k \cdots \big| \cdots t_l \cdots \xrightarrow[\mu_k^{\text{quiv}}]{} \cdots \big| \cdots t_l^{t_k} \cdots t_k \cdots$$

we have $t_l^{t_k} \leq_c t_l \leq_c t_k$.

Proof. Using the same technique as in Lemma 10.0.15 perform Cambrian rotations until t_k is modified to a simple reflection that is also the minimal element (call it s_k) in the new Coxeter order. Now it is easy to see that in this new order s_k appears earlier than the modified t_l and consequently the modified $t_l^{t_k}$ appears earlier than t_l and later than s_k therefore if we undo the Cambrian rotations and $t_l^{t_k}$ happens to end up on the right of | then we must have $t_l^{t_k} \leq_c t_l \leq_c t_k$. However if $t_l^{t_k}$ ends up on the left of | then we must have $t_l \leq_c t_k \leq_c t_l^{t_k}$, which gives us the next lemma. □

Lemma 10.1.3. *For a factorization mutation of the form*

$$\cdots t_k \cdots \big| \cdots t_l \cdots \xrightarrow[\mu_k^{\text{quiv}}]{} \cdots t_l^{t_k} \cdots \big| \cdots t_k \cdots$$

we have $t_l \leq_c t_k \leq_c t_l^{t_k}$.

Lemma 10.1.4. *For a factorization mutation of the form*

$$\cdots t_k \cdots t_l \cdots \big| \cdots \xrightarrow[\mu_k^{\text{quiv}}]{} \cdots t_l^{t_k} \cdots \big| \cdots t_k \cdots ,$$

if t_k *doesn't commute with* t_l *then* t_k *doesn't commute with* $t_l^{t_k}$.

Proof. It suffices to show that $(t_k t_l^{t_k})^3 = e$.

$(t_k t_l^{t_k})^3 = (t_l t_k)^3$ which is equal to identity since t_k doesn't commute with t_l. □

Lemma 10.1.5. *If* t_k *doesn't commute with* t_m *and* t_k *doesn't commute with* t_n *then* t_n *doesn't commute with* $t_m^{t_k}$ *and* t_m *doesn't commute with* $t_n^{t_k}$.

Proof. It suffices to show that $(t_n(t_m^{t_k}))^3 = e$,

$(t_n(t_m^{t_k}))^3 = t_n(t_m^{t_k})t_n(t_m^{t_k})t_n(t_m^{t_k})$

$= t_n(t_m t_k t_m)t_n(t_m t_k t_m)t_n(t_m t_k t_m) = t_n t_m t_k(t_m t_n)t_m t_k(t_m t_n)t_m t_k t_m$

$= t_n t_m t_k(t_m t_m)t_n t_k t_n(t_m t_m)t_k t_m = t_n t_m(t_n^{t_k})t_n t_k t_m$

$= t_n t_m t_n t_k(t_n t_n)t_k t_m = t_n t_m t_n(t_k t_k)t_m$

$= t_n(t_m t_n)t_m = (t_n t_n)(t_m t_m) = e.$

In a similar way it can also be shown that $t_m(t_n^{t_k})^3 = e$. □

Lemma 10.1.6. *If t_m doesn't commute with t_k, t_k doesn't commute with t_n and t_n doesn't commute with t_m then $t_m^{t_k}$ commutes with t_n and $t_n^{t_k}$ commutes with t_m.*

Proof. Since t_n doesn't commute with t_k and t_k doesn't commute with t_m and t_m doesn't commute with t_n therefore $[t_n, t_k t_m t_k] = e$, which implies $t_n t_m^{t_k} = t_m^{t_k} t_n$.

Thus $t_m^{t_k}$ commutes with t_n. Similarly it can also be shown that $t_m^{t_k}$ commutes with t_n. □

Lemma 10.1.7. *If t_m doesn't commute with t_k, t_k doesn't commute with t_n and t_n commutes with t_m then $t_m^{t_k}$ commutes with $t_n^{t_k}$.*

Proof.

$$\begin{aligned} t_n^{t_k} t_m^{t_k} &= t_k t_n (t_k t_k) t_m t_k \\ &= t_k t_n t_m t_k \\ &= t_k t_m t_n t_k \\ &= t_m^{t_k} t_n^{t_k} \end{aligned}$$

□

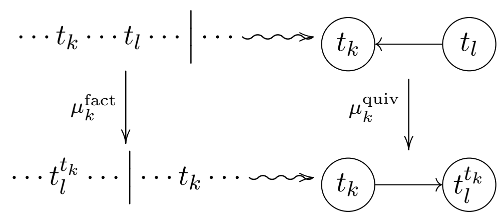
10.2 Case by Case Proof of the Correspondence between Factorization Mutation and Quiver Mutation

In all the following cases we perform a factorization mutation at k on a subword w_c of a two-part factorization of a Coxeter element c, and a quiver mutation on the quiver $\mathcal{Q}_{w_c}$(associated to the subword w_c) at k.

10.2.1 Subword w_c consisting of two reflections: t_k and t_l such that the corresponding nodes in $\mathcal{Q}_{w_c}$ are adjacent

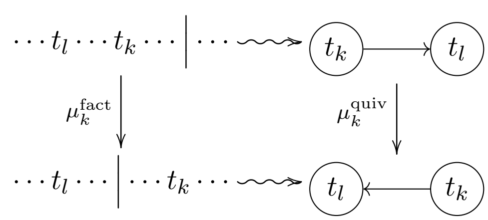
Let w_c be any subword of a two-part factorization of a Coxeter element c consisting of two reflections t_k and t_l and let the corresponding nodes in the associated quiver $\mathcal{Q}_{w_c}$ be adjacent i.e. the reflections t_k and t_l must not commute. Also, let t_k be on the left part of the two-part factorization, so that we can perform a factorization mutation at k. We will show that for all such subword w_c, $\mu_k^{\text{quiv}}(\mathcal{Q}_{w_c}) = \mathcal{Q}_{\mu_k^{\text{fact}}(w_c)}$.

Case I: Both t_k and t_l are are on the left part of the two-part factorization and $t_k \leq_c t_l$



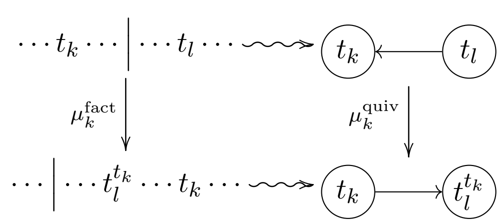
Where the nodes t_k and $t_l^{t_k}$ are adjacent by Lemma 10.1.4 and $t_k \leq_c t_l^{t_k}$ by Lemma 10.1.1.

Case II: Both t_k and t_l on the left part of the two-part factorization and $t_l \leq_c t_k$

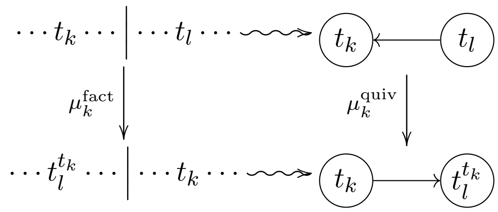


Nothing changes, so adjacency and order gets carried over!

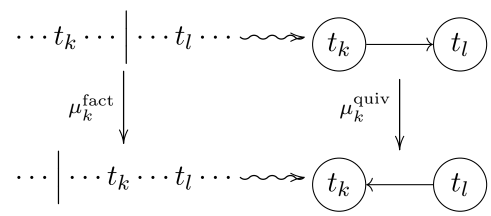
Case III: t_k on the left part and t_l on the right part of the two-part factorization and $t_l \leq_c t_k$



Where the nodes t_k and $t_l^{t_k}$ are adjacent by lemma 10.1.4 and $t_l^{t_k} \leq_c t_k$ by Lemma 10.1.2. Alternatively, we might also have the following case, where $t_k \leq_c t_l^{t_k}$ by Lemma 10.1.3.



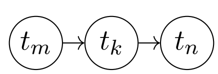
Case IV: t_k on the left part and t_l on the right part of the two-part factorization and $t_k \leq_c t_l$



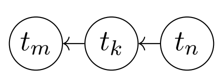
Nothing changes, so adjacency and order are carried forward!

10.2.2 Subword w_c consisting of three reflections: t_k, t_m and t_n such that the corresponding nodes in $\mathcal{Q}_{w_c}$ form a tree with the node t_k adjacent to the other two nodes.

Let w_c be any subword of a two-part factorization of a Coxeter element c consisting of three reflections t_m, t_n and t_k and let the corresponding nodes in the associated quiver $\mathcal{Q}_{w_c}$ be arranged as



or



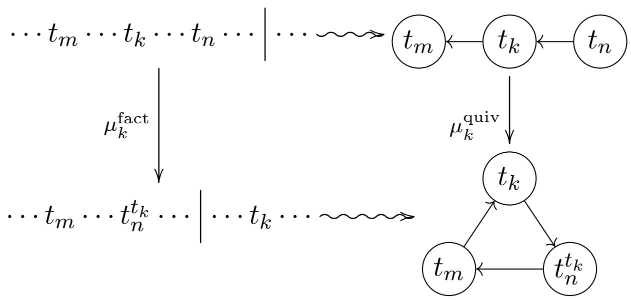
Consequently, the reflection t_k must not commute with the reflections t_n and t_m, and the reflection t_m must commute with the reflection t_n. Without loss of any generality we may assume that $t_m \leq_c t_n$. In Table 10.1 we have marked the cases that conform to these criteria with a ✓ beside them. We will now show that for all such subwords w_c listed in Table 10.1 $\mu_k^{\text{quiv}}(\mathcal{Q}_{w_c}) = \mathcal{Q}_{\mu_k^{\text{fact}}(w_c)}$,

Table 10.1. Counting all possible subwords (consisting of just 3 reflections) of two-part factorizations of a Coxeter element such that in the associated quiver the corresponding three nodes form a tree, with two fixed leaves.

Both t_m and t_n on the left			
$t_m \leq_c t_k \leq_c t_n$	$\cdots t_m \cdots t_k \cdots t_n \cdots \mid \cdots$	$t_m \leftarrow t_k \leftarrow t_n$	✓
$t_k \leq_c t_m \leq_c t_n$	$\cdots t_k \cdots t_m \cdots t_n \cdots \mid \cdots$	$t_m \rightarrow t_k \leftarrow t_n$	
$t_m \leq_c t_n \leq_c t_k$	$\cdots t_m \cdots t_n \cdots t_k \cdots \mid \cdots$	$t_m \leftarrow t_k \rightarrow t_n$	
Both t_m and t_n on the right			
$t_m \leq_c t_k \leq_c t_n$	$\cdots t_k \cdots \mid \cdots t_m \cdots t_n \cdots$	$t_m \rightarrow t_k \rightarrow t_n$	✓
$t_k \leq_c t_m \leq_c t_n$	$\cdots t_k \cdots \mid \cdots t_m \cdots t_n \cdots$	$t_m \leftarrow t_k \rightarrow t_n$	
$t_m \leq_c t_n \leq_c t_k$	$\cdots t_k \cdots \mid \cdots t_m \cdots t_n \cdots$	$t_m \rightarrow t_k \leftarrow t_n$	
t_m on the right and t_n on the left			
$t_m \leq_c t_k \leq_c t_n$	$\cdots t_k \cdots t_n \cdots \mid \cdots t_m \cdots$	$t_m \rightarrow t_k \leftarrow t_n$	
$t_k \leq_c t_m \leq_c t_n$	$\cdots t_k \cdots t_n \cdots \mid \cdots t_m \cdots$	$t_m \leftarrow t_k \leftarrow t_n$	✓
$t_m \leq_c t_n \leq_c t_k$	$\cdots t_n \cdots t_k \cdots \mid \cdots t_m \cdots$	$t_m \rightarrow t_k \rightarrow t_n$	✓
t_m on the left and t_n on the right			
$t_m \leq_c t_k \leq_c t_n$	$\cdots t_m \cdots t_k \cdots \mid \cdots t_n$	$t_m \leftarrow t_k \rightarrow t_n$	
$t_k \leq_c t_m \leq_c t_n$	$\cdots t_k \cdots t_m \cdots \mid \cdots t_n$	$t_m \rightarrow t_k \rightarrow t_n$	✓
$t_m \leq_c t_n \leq_c t_k$	$\cdots t_m \cdots t_k \cdots \mid \cdots t_n$	$t_m \leftarrow t_k \leftarrow t_n$	✓

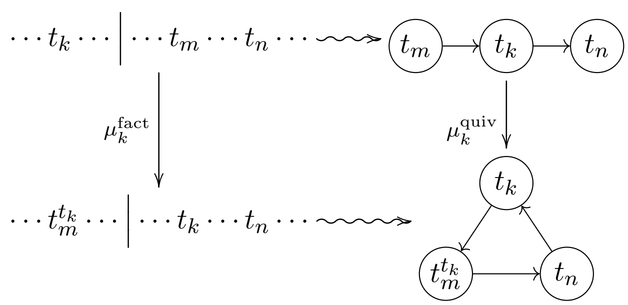
i.e. the quiver associated to the subword $\mu_k^{\text{fact}}(w_c)$ is a cyclic 3-cycle that is obtained by mutating $\mathcal{Q}_{w_c}$ at k.

Case I: μ_k^{fact} on the subword $\cdots t_m \cdots t_k \cdots t_n \cdots \Big| \cdots$ replaces t_n by $t_n^{t_k}$ where $t_m \leq_c t_k \leq_c t_n$. Thus there are three possible orderings, viz. $t_m \leq_c t_k \leq_c t_n^{t_k}$, $t_m \leq_c t_n^{t_k} \leq_c t_k$ and $t_n^{t_k} \leq_c t_m \leq_c t_k$. If $t_m \leq_c t_k \leq_c t_n^{t_k}$ then we have

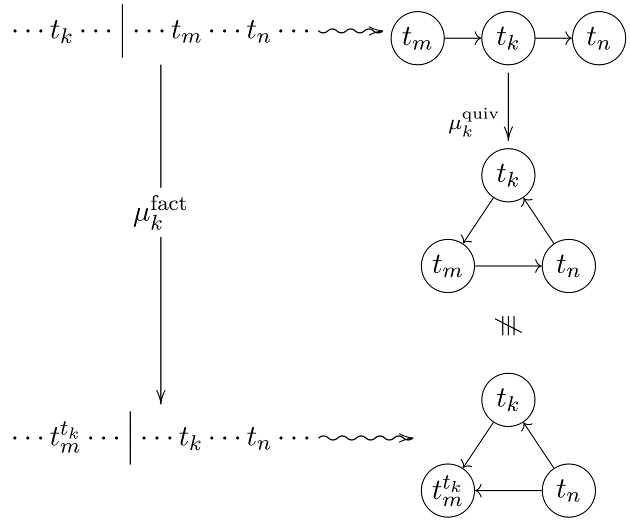


Here the node $t_n^{t_k}$ is adjacent to the nodes t_m and t_k by Lemma 10.1.5 and Lemma 10.1.4. Since Lemma 10.1.1 prevents the order $t_n^{t_k} \leq_c t_k$, thus this is the only possible outcome in this case.

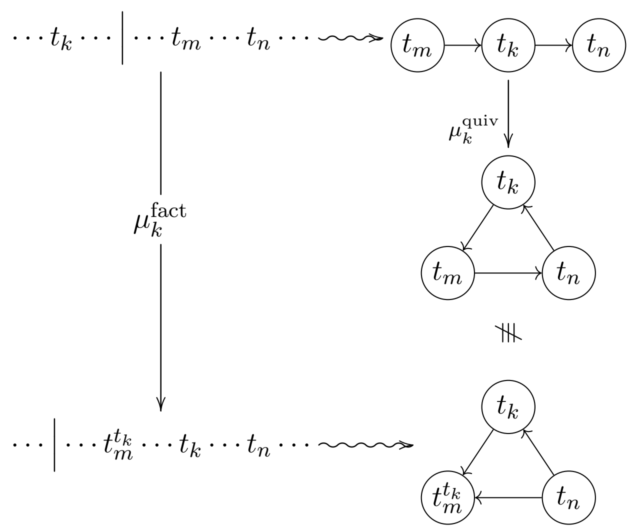
Case II: μ_k^{fact} on $\cdots t_k \cdots \Big| \cdots t_m \cdots t_n \cdots$ replaces t_m by $t_m^{t_k}$, where $t_m \leq_c t_k \leq_c t_n$. Thus there are three possible orderings, viz. $t_k \leq_c t_n \leq_c t_m^{t_k}$, $t_k \leq_c t_m^{t_k} \leq_c t_n$ and $t_m^{t_k} \leq_c t_k \leq_c t_n$. If $t_k \leq_c t_m^{t_k} \leq_c t_n$ then we have



Here the node $t_m^{t_k}$ is adjacent to the nodes t_n and t_k by Lemma 10.1.5 and lemma 10.1.4. However for the ordering $t_k \leq_c t_n \leq_c t_m^{t_k}$ we have



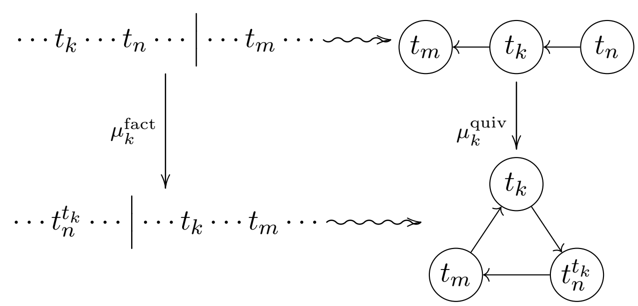
and for the ordering $t_m^{t_k} \leq_c t_k \leq_c t_n$ we have



However, it can be shown that the only possible ordering in this case is $t_k \leq_{\mathsf{c}} t_m^{t_k} \leq_{\mathsf{c}} t_n$. In a similar way as in the proof of Lemma 10.0.15, starting with the factorization $\cdots t_k \cdots \big| \cdots t_m \cdots t_n \cdots$ we perform right Cambrian rotations until t_n is modified to s_n—the maximal simple reflection in the new order $\leq_{\mathsf{c}'}$ (meanwhile c gets modified to c'). Let $\tilde{t}_k$ and $\tilde{t}_m$ denote the modified t_k and t_m respectively. At this point $\tilde{t}_k$ is still on the left and $\tilde{t}_m$ is still on the right part of '$\big|$' of the modified two-part factorization of c', this is because both t_m

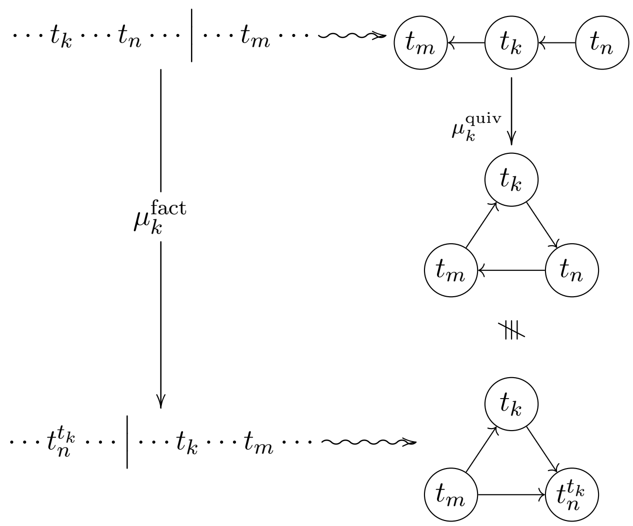
and t_k appeared earlier in the order $\leq_c$ than t_n therefore by Lemma 10.0.12 $\tilde{t}_m \leq_{\mathsf{c}'} \tilde{t}_k \leq_{\mathsf{c}'} s_n$. Now, by Lemma 10.0.4, $\tilde{t}_k$ and $\tilde{t}_m$ belong to the parabolic subgroup $W_{\langle s_n \rangle}$ and we can drop the last letter (i.e. s_n) from the factorization $\cdots \tilde{t}_k \cdots \big| \cdots \tilde{t}_m \cdots s_n = \mathsf{c}'$ and thus obtain a factorization of the parabolic Coxeter element $\mathsf{c}'_{\langle s_n \rangle}$, which is of the form $\cdots \tilde{t}_k \cdots | \cdots \tilde{t}_m \cdots$. Again, we replicate Lemma 10.0.15, and perform left Cambrian rotations, ($\mathsf{c}'_{\langle s_n \rangle}$ gets modified to $c''_{\langle s_n \rangle}$) until $\tilde{t}_m$ is modified to s_m—the minimal simple reflection in the order $\leq_{\mathsf{c}''_{\langle s_n \rangle}}$. Meanwhile $\tilde{t}_k$ gets modified to $\hat{t}_k$. In the modified factorization $\cdots \hat{t}_k \cdots \big| s_m \cdots = \mathsf{c}''_{\langle s_n \rangle}$ the subgroup generated by the reflections $\hat{t}_k$ and s_m is isomorphic to $\mathfrak{S}_3$. Since $s_m \leq_{\mathsf{c}''_{\langle s_n \rangle}} \hat{t}_k$ but $\hat{t}_k$ is to the left of s_m in the factorization $\cdots \hat{t}_k \cdots \big| s_m \cdots = \mathsf{c}''_{\langle s_n \rangle}$, therefore we must have $s_m \leq_{\mathsf{c}''_{\langle s_n \rangle}} \hat{t}_k \leq_{\mathsf{c}''_{\langle s_n \rangle}} s_m^{\hat{t}_k}$. Thus undoing all the Cambrian rotations we have $t_k \leq_{\mathsf{c}} t_m^{t_k} \leq_{\mathsf{c}} t_n$.

Case III: μ_k^{fact} on $\cdots t_k \cdots t_n \cdots \Big| \cdots t_m \cdots$ replaces t_n by $t_n^{t_k}$, where $t_k \leq_c t_m \leq_c t_n$. Thus there are three possible orderings, viz. $t_k \leq_c t_m \leq_c t_n^{t_k}$, $t_k <_c t_n^{t_k} \leq_c t_m$ and $t_n^{t_k} \leq_c t_k <_c t_m$. For the ordering $t_k \leq_c t_n^{t_k} \leq_c t_m$ we have

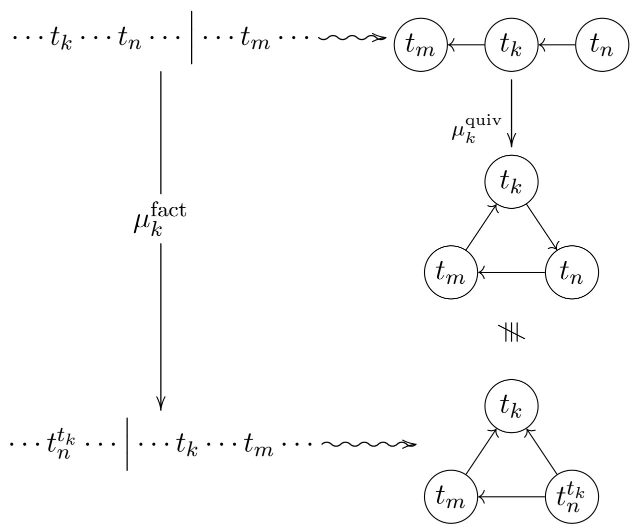


where the node $t_n^{t_k}$ is adjacent to the nodes t_k and t_m due to Lemma 10.1.4 and Lemma 10.1.5 respectively.

For the ordering $t_k \leq_c t_m \leq_c t_n^{t_k}$ we have

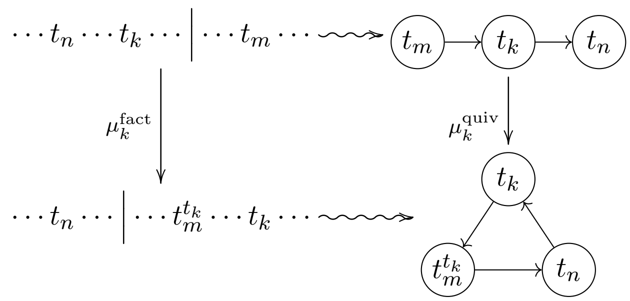


and for the ordering $t_n^{t_k} \leq_c t_k \leq_c t_m$ we have

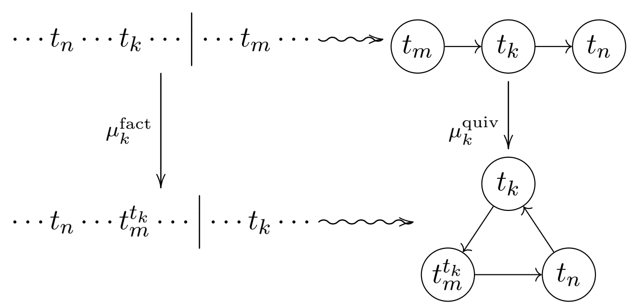


But the case $t_n^{t_k} \leq_c t_k \leq_c t_m$ can be invalidated on the grounds that $t_n^{t_k} \leq_c t_k$ by Lemma 10.1.1. In fact we can show that the only case that will occur is $t_k \leq_c t_n^{t_k} \leq_c t_m$ using a similar technique as used in Case II:. Starting with the factorization $\cdots t_k \cdots t_n \cdots \big| \cdots t_m \cdots$ we perform right Cambrian rotations until t_n is modified to s_n—the minimal simple reflection in the new order $\leq_{\mathsf{c}'}$ (this is accomplished by first performing right Cambrian rotations until t_n becomes the maximal simple reflection and then we perform one more right Cambrian rotation). Meanwhile t_k and t_m got modified to $\tilde{t}_k$ and $\tilde{t}_m$ respectively. Thus we have the factorization $\cdots \tilde{t}_k \cdots \big| s_n \cdots \tilde{t}_m \cdots$ for the modified Coxeter element c', where $s_n \leq_{\mathsf{c}'} \tilde{t}_k \leq_{\mathsf{c}'} \tilde{t}_m$. This is exactly the case we dealt with in Case II: so we must have $\tilde{t}_k \leq_{\mathsf{c}'} s_n^{\tilde{t}_k} \leq_{\mathsf{c}'} \tilde{t}_m$. Therefore undoing the Cambrian rotations we must have $t_k \leq_c t_n^{t_k} \leq_c t_m$.

Case IV: μ_k^{fact} on $\cdots t_n \cdots t_k \cdots \Big| \cdots t_m \cdots$ replaces t_m by $t_m^{t_k}$, where $t_m \leq_c t_n \leq_c t_k$. Thus there are three possible orderings, viz. $t_m^{t_k} \leq_c t_n \leq_c t_k$, $t_n \leq_c t_m^{t_k} \leq_c t_k$ and $t_n \leq_c t_k \leq_c t_m^{t_k}$. For the ordering $t_m^{t_k} \leq_c t_n \leq_c t_k$ we have

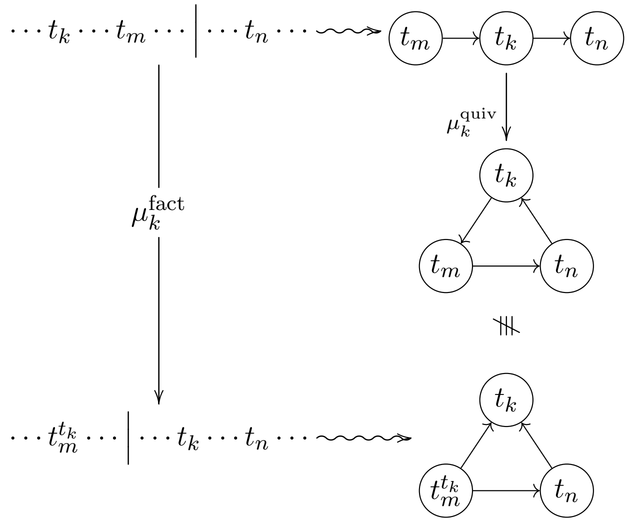


The node $t_m^{t_k}$ is adjacent to t_k and t_n due to Lemma 10.1.4 and Lemma 10.1.5 respectively. For the ordering $t_n \leq_c t_k \leq_c t_m^{t_k}$ we have

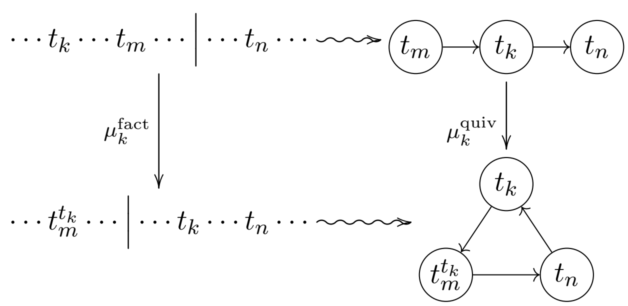


The case $t_n \leq_c t_m^{t_k} \leq_c t_k$ is not possible since if $t_m^{t_k} \leq_c t_k$, then by Lemma 10.1.2, we must have $t_m^{t_k} \leq_c t_m$, but $t_m \leq_c t_n$ which implies $t_m^{t_k} \leq_c t_n$, a contradiction!

Case V: μ_k^{fact} on $\cdots t_k \cdots t_m \cdots \Big| \cdots t_n \cdots$ replaces t_m by $t_m^{t_k}$, where $t_k \leq_c t_m \leq_c t_n$. Thus there are three possible orderings, viz. $t_m^{t_k} \leq_c t_k \leq_c t_n$, $t_k \leq_c t_m^{t_k} \leq_c t_n$ and $t_k \leq_c t_n \leq_c t_m^{t_k}$. For the ordering $t_m^{t_k} \leq_c t_k \leq_c t_n$ we have

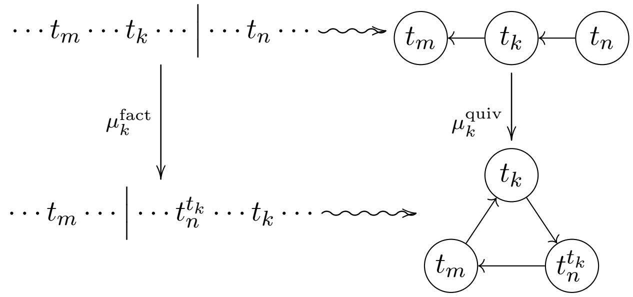


Here the node $t_m^{t_k}$ is adjacent to the nodes t_k and t_n due to Lemma 10.1.4 and Lemma 10.1.5 respectively. But this is not possible since $t_k \leq_c t_m^{t_k}$ by Lemma 10.1.1. Now for the ordering $t_k \leq_c t_m^{t_k} \leq_c t_n$ we have



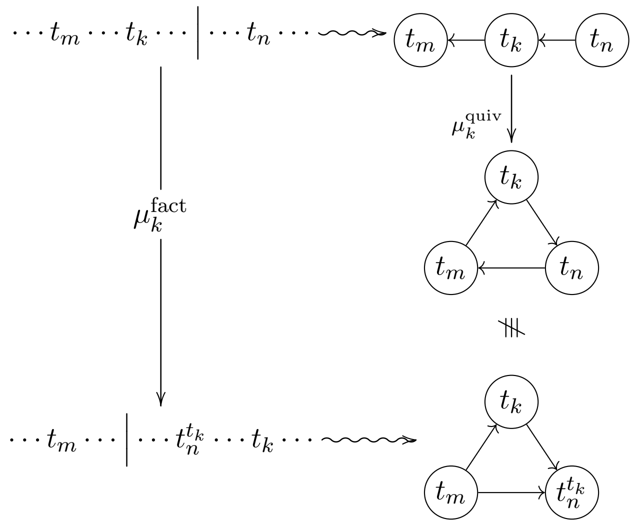
The ordering $t_k \leq_c t_n \leq_c t_m^{t_k}$ is not possible because by Lemma 10.1.1 $t_m^{t_k} \leq_c t_m$ and $t_m \leq_c t_n$, therefore we must always have $t_m^{t_k} \leq_c t_n$.

Case VI: μ_k^{fact} on $\cdots t_m \cdots t_k \cdots \Big| \cdots t_n \cdots$ replaces t_n by $t_n^{t_k}$, where $t_m \leq_c t_n \leq_c t_k$. Thus there are three possible orderings, viz. $t_n^{t_k} \leq_c t_m \leq_c t_k$, $t_m \leq_c t_n^{t_k} \leq_c t_k$ and $t_m \leq_c t_k \leq_c t_n^{t_k}$. For the ordering $t_n^{t_k} \leq_c t_m \leq_c t_k$ we have



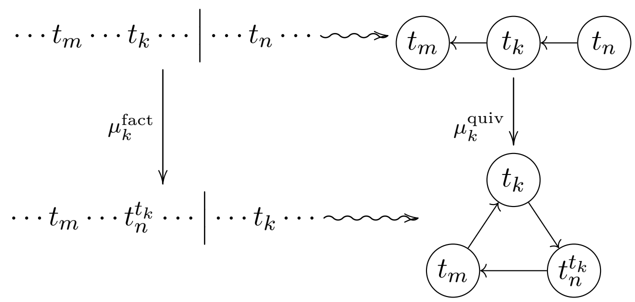
Here, the node $t_n^{t_k}$ is adjacent to the nodes t_k and t_m due to Lemma 10.1.4 and Lemma 10.1.5 respectively.

For the case $t_m \leq_c t_n^{t_k} \leq_c t_k$ we have

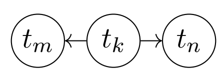


Using a similar technique as used in Case II: we can show that this case will never occur. Starting with the factorization $\cdots t_m \cdots t_k \cdots \big| \cdots t_n \cdots$ we perform left Cambrian rotations until t_m is modified to s_m—the minimal simple reflection in the new order $\leq_{\mathsf{c}'}$. Let $\tilde{t}_n$ and $\tilde{t}_k$ denote the modified t_n and t_k respectively. Since both t_n and t_k appeared later than t_n in the order $\leq_{\mathsf{c}}$ therefore by Lemma 10.0.12 we will have a factorization of the form $s_m \cdots \tilde{t}_k \cdots \big| \cdots \tilde{t}_n = \mathsf{c}'$. Now, with the help of Lemma 10.0.3 we can drop the letter s_m from the factorization $s_m \cdots \tilde{t}_k \cdots \big| \cdots \tilde{t}_n$ and thus obtain a the factorization $\cdots \tilde{t}_k \cdots \big| \cdots \tilde{t}_n \cdots$ of the parabolic Coxeter element $\mathsf{c}'_{\langle s_m \rangle}$ with $\tilde{t}_n \leq_{\mathsf{c}'_{\langle s_m \rangle}} \tilde{t}_k$. Now we again perform left Cambrian rotations until $\tilde{t}_n$ is modified to the minimal simple reflection, call it s_n, in the new order $\leq_{\mathsf{c}''_{\langle s_m \rangle}}$. Meanwhile t_k gets modified to $\hat{t}_k$, thus giving us the factorization $\cdots \hat{t}_k \cdots \big| s_n \cdots = \mathsf{c}''_{\langle s_m \rangle}$. Notice that $\hat{t}_k$ and s_n generates a subgroup isomorphic to $\mathfrak{S}_3$. Now because $s_n \leq_{\mathsf{c}''_{\langle s_m \rangle}} \hat{t}_k$ but $\hat{t}_k$ appears to the left of s_n in the factorization $\cdots \hat{t}_k \cdots \big| s_n \cdots = \mathsf{c}''_{\langle s_m \rangle}$, therefore $s_n \leq_{\mathsf{c}''_{\langle s_m \rangle}} \hat{t}_k \leq_{\mathsf{c}''_{\langle s_m \rangle}} \hat{t}_k^{s_n}$, consequently $t_k \leq_c t_n^{t_k}$.

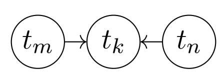
And finally for the case $t_m \leq_c t_k \leq_c t_n^{t_k}$, we have



Now, let us consider the case where w_c is any subword of a two-part factorization of a Coxeter element c consisting of three reflections t_m, t_n and t_k such that the corresponding nodes in the associated quiver $\mathcal{Q}_{w_c}$ are arranged as

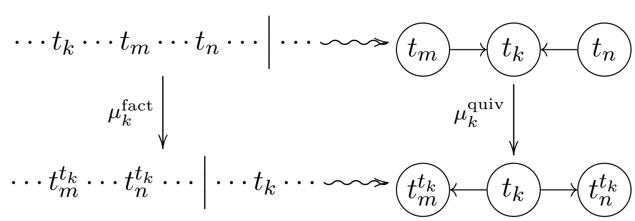


or



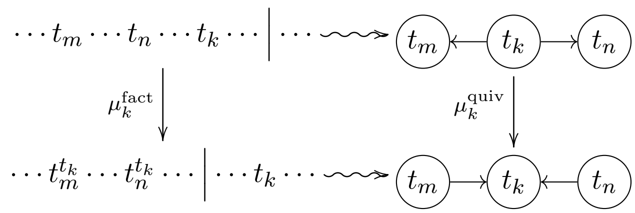
It is required to show that for all such subwords w_c, the quiver $\mathcal{Q}_{\mu_k^{\text{fact}}(w_c)}$ is not a cycle. The orientations of the edges of $\mathcal{Q}_{\mu_k^{\text{fact}}(w_c)}$ are already taken care of by the first part of the proof where we dealt with subwords consisting of just two reflections. So, now we illustrate the remaining 6 cases from Table 10.1.

Case I: $w_c = \cdots t_k \cdots t_m \cdots t_n \cdots \big| \cdots$ and $t_k \leq_c t_m \leq_c t_n$

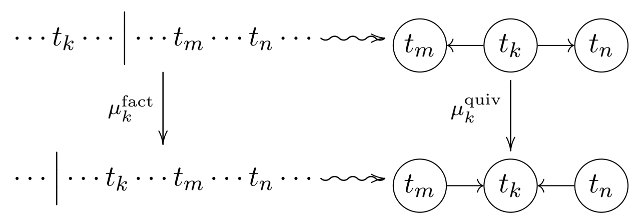


Here the nodes $t_m^{t_k}$ and $t_n^{t_k}$ are non-adjacent by Lemma 10.1.7.

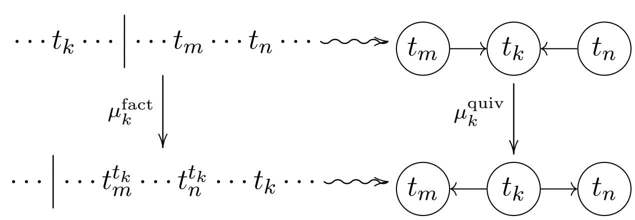
Case II: $w_c = \cdots t_m \cdots t_n \cdots t_k \cdots \Big| \cdots$ and $t_m \leq_c t_n \leq_c t_k$



Case III: $w_c = \cdots t_k \cdots \Big| \cdots t_m \cdots t_n \cdots$ and $t_k \leq_c t_m \leq_c t_n$

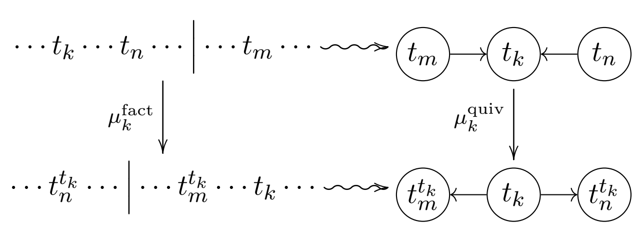


Case IV: $w_c = \cdots t_k \cdots \Big| \cdots t_m \cdots t_n \cdots$ and $t_m \leq_c t_n \leq_c t_k$



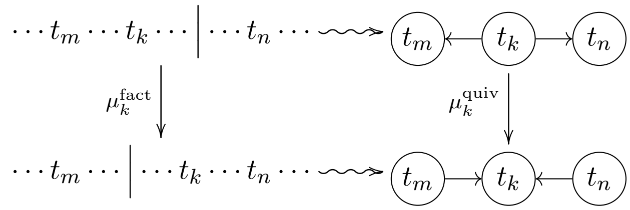
Here the nodes $t_m^{t_k}$ and $t_n^{t_k}$ are non-adjacent by Lemma 10.1.7.

Case V: $w_c = \cdots t_k \cdots t_n \cdots \Big| \cdots t_m \cdots$ and $t_m \leq_c t_k \leq_c t_n$



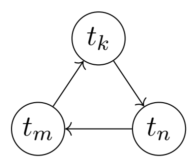
Here the nodes $t_m^{t_k}$ and $t_n^{t_k}$ are non-adjacent by Lemma 10.1.7.

Case VI: $w_c = \cdots t_m \cdots t_k \cdots \Big| \cdots t_n \cdots$ and $t_m \leq_c t_k \leq_c t_n$

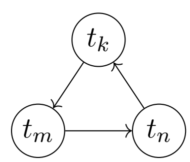


10.2.3 Subword w_c consisting of three reflections: t_k, t_m and t_n such that the corresponding nodes in $\mathcal{Q}_{w_c}$ form a cyclic 3-cycle.

Let w_c be any subword of a two-part factorization of a Coxeter element c consisting of three reflections t_m, t_n and t_k and let the corresponding nodes in the associated quiver $\mathcal{Q}_{w_c}$ be arranged as



or



Therefore, the reflection t_k must not commute with the reflections t_m and t_n and the reflections t_m and t_n must not commute with each other. Without loss of any generality we may assume that $t_m \leq_c t_n$. In Table 10.2 we have marked the cases that conform to these criteria with a ✓ beside them.

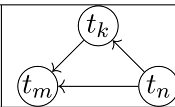
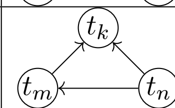
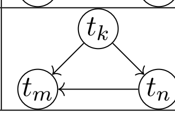
Now we shall show that in those 3 cases we identified, $\mu_k^{\text{quiv}}(\mathcal{Q}_{w_c}) = \mathcal{Q}_{\mu_k^{\text{fact}}(w_c)}$.

Case I: For $w_c = \cdots t_k \cdots \Big| \cdots t_m \cdots t_n \cdots$ and $t_m \leq_c t_k \leq_c t_n$, t_m is replaced by $t_m^{t_k}$ in $\mu_k^{\text{fact}}(w_c)$.

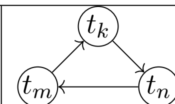
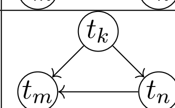
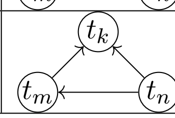
Therefore we consider the three possible orderings, viz. $t_m^{t_k} \leq_c t_k \leq_c t_n$, $t_k \leq_c t_m^{t_k} \leq_c t_n$ and $t_k \leq_c t_n \leq_c t_m^{t_k}$. If $t_m^{t_k} \leq_c t_k \leq_c t_n$ then we have

Table 10.2. Counting all possible subwords (consisting of just 3 reflections) of two-part factorizations of a Coxeter element such that the associated quiver forms a cyclic 3-cycle.

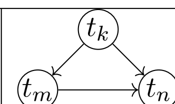
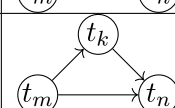
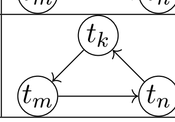
Both t_m and t_n on the left

$t_m \leq_c t_k \leq_c t_n$	$\cdots t_m \cdots t_k \cdots t_n \cdots \mid \cdots$		
$t_k \leq_c t_m \leq_c t_n$	$\cdots t_k \cdots t_m \cdots t_n \cdots \mid \cdots$		
$t_m \leq_c t_n \leq_c t_k$	$\cdots t_m \cdots t_n \cdots t_k \cdots \mid \cdots$		

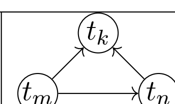
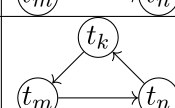
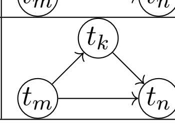
Both t_m and t_n on the right

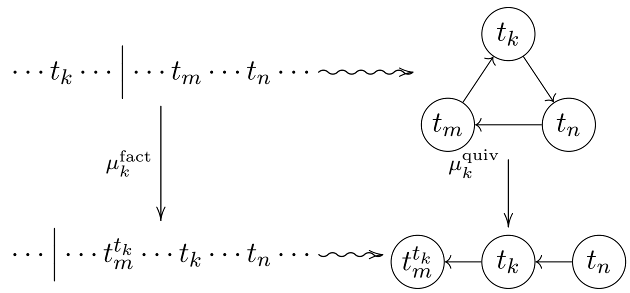
$t_m \leq_c t_k \leq_c t_n$	$\cdots t_k \cdots \mid \cdots t_m \cdots t_n \cdots$		✓
$t_k \leq_c t_m \leq_c t_n$	$\cdots t_k \cdots \mid \cdots t_m \cdots t_n \cdots$		
$t_m \leq_c t_n \leq_c t_k$	$\cdots t_k \cdots \mid \cdots t_m \cdots t_n \cdots$		

t_m on the right and t_n on the left

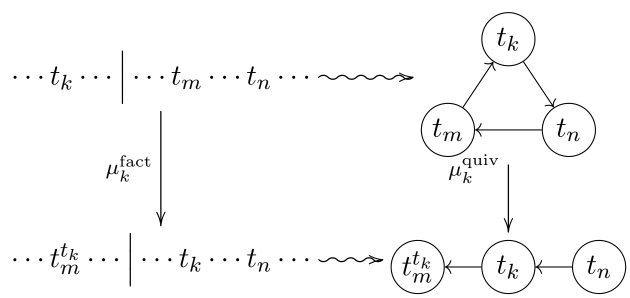
$t_m \leq_c t_k \leq_c t_n$	$\cdots t_k \cdots t_n \cdots \mid \cdots t_m \cdots$		
$t_k \leq_c t_m \leq_c t_n$	$\cdots t_k \cdots t_n \cdots \mid \cdots t_m \cdots$		
$t_m \leq_c t_n \leq_c t_k$	$\cdots t_n \cdots t_k \cdots \mid \cdots t_m \cdots$		✓

t_m on the left and t_n on the right

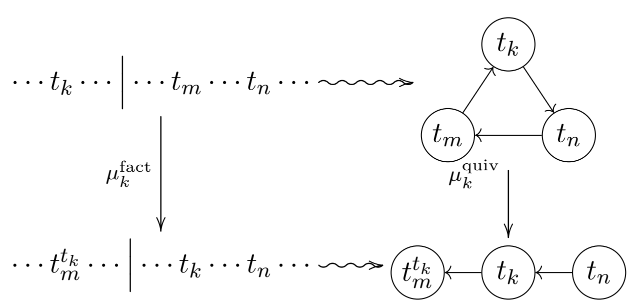
$t_m \leq_c t_k \leq_c t_n$	$\cdots t_m \cdots t_k \cdots \mid \cdots t_n$		
$t_k \leq_c t_m \leq_c t_n$	$\cdots t_k \cdots t_m \cdots \mid \cdots t_n$		✓
$t_m \leq_c t_n \leq_c t_k$	$\cdots t_m \cdots t_k \cdots \mid \cdots t_n$		



The adjacency between the nodes t_k and $t_m^{t_k}$ can be explained by Lemma 10.1.4 and the non-adjacency between the nodes t_n and $t_m^{t_k}$ can be explained by Lemma 10.1.6. If $t_k \leq_c t_m^{t_k} \leq_c t_n$ then we have

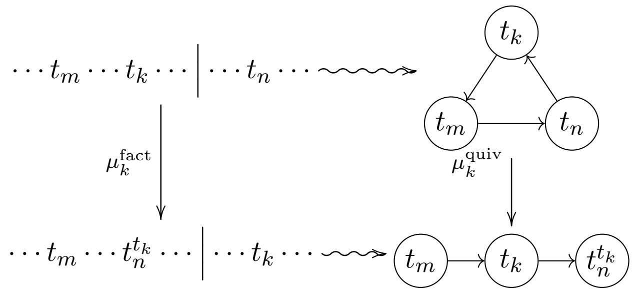


and finally for $t_k \leq_c t_n \leq_c t_m^{t_k}$ we have

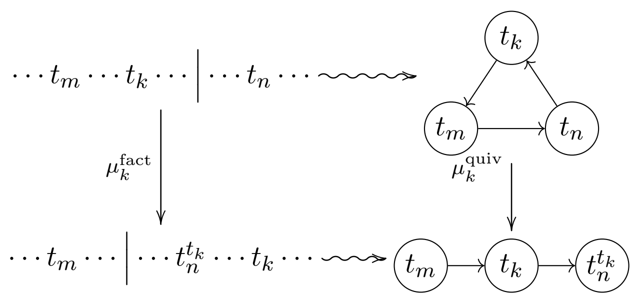


Case II: For $w_c = \cdots t_m \cdots t_k \cdots \big| \cdots t_n \cdots$ and $t_m \leq_c t_n \leq_c t_k$, t_n is replaced by $t_n^{t_k}$ in $\mu_k^{\text{fact}}(w_c)$. Therefore we consider the three possible orderings, viz. $t_m \leq_c t_k \leq_c t_n^{t_k}$, $t_m \leq_c t_n^{t_k} \leq_c t_k$ and

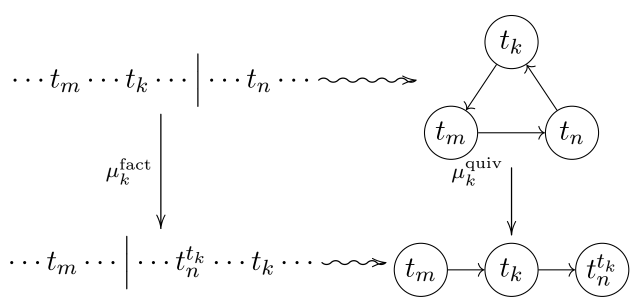
$t_n^{t_k} \leq_c t_m \leq_c t_k$ If $t_m \leq_c t_k \leq_c t_n^{t_k}$ then we have



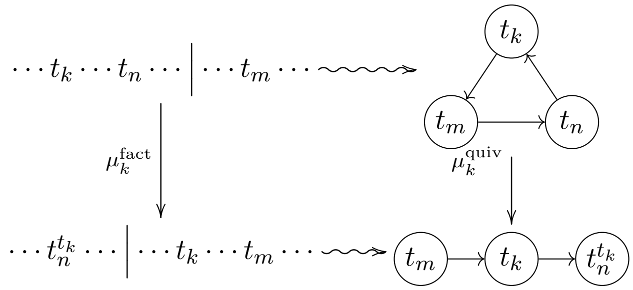
where the adjacency between the nodes t_k and $t_n^{t_k}$ is due to Lemma 10.1.4 and the non-adjacency between the nodes t_m and $t_n^{t_k}$ is due to Lemma 10.1.6. If $t_m \leq_c t_n^{t_k} \leq_c t_k$ then we have



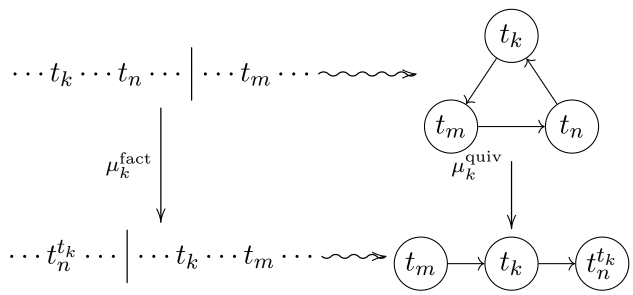
and finally for the case $t_n^{t_k} \leq_c t_m \leq_c t_k$ we have



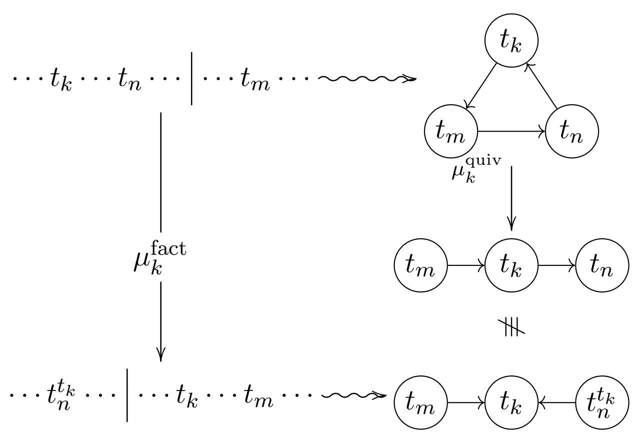
Case III: For $w_c = \cdots t_k \cdots t_n \cdots \Big| \cdots t_m \cdots$ and $t_k \leq_c t_m \leq_c t_n$, t_n is replaced by $t_n^{t_k}$ in $\mu_k^{\text{fact}}(w_c)$. So we have three possible orderings, viz. $t_k \leq_c t_m \leq_c t_n^{t_k}$, $t_k \leq_c t_n^{t_k} \leq_c t_m$ and $t_n^{t_k} \leq_c t_k \leq_c t_m$. If $t_k \leq_c t_m \leq_c t_n^{t_k}$ then we have



Similar to the last two cases, the node t_k is adjacent to the node $t_n^{t_k}$ by Lemma 10.1.4 and the node t_m is not adjacent to the node $t_n^{t_k}$ by Lemma 10.1.6. Now, for the case $t_k \leq_c t_n^{t_k} \leq_c t_m$ we have,



and finally we have the following for the case $t_n^{t_k} \leq_c t_k \leq_c t_m$



But this is not possible due to lemma 10.1.1.

It is worth mentioning here that quivers associated to two-part factorizations of a Coxeter element of simply laced type never contain any acyclic 3-cycles. This is because quivers associated to the initial two-part factorization (involving the defining simple reflections of the Coxeter element only on the left part of the two-part factorization) of a Coxeter element of simply-laced type don't accommodate any cycles, and by Section 10.2.2 we know that any quiver associated to a two-part factorization obtained from the initial two-part factorization by factorization mutation doesn't contain any acyclic 3-cycles. Therefore in our proof for Section 10.2 we need not consider sub-quivers which are associated to acyclic 3-cycles.

CHAPTER 11

RECOVERING BAROT-GRANT-MARSH PRESENTATION

In (Barot and Marsh, 2015), Barot and Marsh have constructed groups from diagrams associated to a seed in a cluster algebra of finite type, that are preserved under seed mutation. In (Grant and Marsh, 2017), Grant and Marsh generalized it for braid groups.

Theorem 11.0.1 ((Grant and Marsh, 2017))**.** *Let $\mathcal{Q}$ be a quiver, mutation equivalent to an orientation of a simply-laced Dynkin diagram (also known as Dynkin quiver) with vertices $v_1, v_2, \ldots, v_n$. Let $\boldsymbol{B}(\mathcal{Q})$ denote the group generated by the generators $\boldsymbol{s}_1, \boldsymbol{s}_2, \cdots, \boldsymbol{s}_n$ and the following 3 relations*

1. $\boldsymbol{s}_i\boldsymbol{s}_j = \boldsymbol{s}_j\boldsymbol{s}_i$ *if v_i and v_j are non-adjacent in $\mathcal{Q}$.*
2. $\boldsymbol{s}_i\boldsymbol{s}_j\boldsymbol{s}_i = \boldsymbol{s}_j\boldsymbol{s}_i\boldsymbol{s}_j$ *if v_i and v_j are adjacent in $\mathcal{Q}$.*
3. $\boldsymbol{s}_{i_1}\boldsymbol{s}_{i_2}\cdots\boldsymbol{s}_{i_m}\boldsymbol{s}_{i_1}\cdots\boldsymbol{s}_{i_{m-2}} = \boldsymbol{s}_{i_2}\boldsymbol{s}_{i_3}\cdots\boldsymbol{s}_{i_m}\boldsymbol{s}_{i_1}\boldsymbol{s}_{i_2}\cdots\boldsymbol{s}_{i_{m-1}}$ *for every chordless cycle $v_{i_1} \to v_{i_2} \to \cdots \to v_{i_{m-1}} \to v_{i_m} \to v_{i_1}$ in $\mathcal{Q}$.*

then $\boldsymbol{B}(\mathcal{Q})$ is isomorphic to the Artin group of the same type as the underlying simply-laced Dynkin diagram, whose orientation is mutation-equivalent to $\mathcal{Q}$.

In other words let $\mathcal{Q}$ be an orientation of a simply-laced Dynkin diagram. Let $\mathcal{Q}'$ be a quiver obtained by performing a finite number of quiver mutations on $\mathcal{Q}$ then $\boldsymbol{B}(\mathcal{Q}) \cong \boldsymbol{B}(\mathcal{Q}')$, and are of the same simply-laced type as $\mathcal{Q}$.

In Theorem 5.1.2 we have shown that each reduced T-decomposition of a Coxeter element c of a simply-laced type Coxeter group W encode a presentation of the Artin group $\boldsymbol{B}(W)$ corresponding to the Coxeter group W. And since the set of reduced T-decompositions of the Coxeter element c is transitive under the action of Hurwitz moves (Theorem 4.3.4), therefore reduced T-decompositions obtained from these Hurwitz moves also encode presentations of the same Artin braid group.

In Chapter 10 we introduced a special class of factorizations of the Coxeter elements called the two-part factorizations (definition 10.0.13) and a special sequence of successive Hurwitz moves called factorization mutations (definition 10.0.14). Using a simple rule Definition 10.0.16 to associate these two-part factorizations with quivers we have shown that factorization mutations on a two-part factorization are exactly the quiver mutations on the associated quiver.

Now, since two-part factorizations of a Coxeter element form a subset of the reduced T-decompositions of the Coxeter element and factorization mutations are just a sequence of successive Hurwitz moves, furthermore since we can rewrite the relation

$$\boldsymbol{s}_{i_1}\boldsymbol{s}_{i_2}\cdots\boldsymbol{s}_{i_m}\boldsymbol{s}_{i_1}\cdots\boldsymbol{s}_{i_{m-2}} = \boldsymbol{s}_{i_2}\boldsymbol{s}_{i_3}\cdots\boldsymbol{s}_{i_m}\boldsymbol{s}_{i_1}\boldsymbol{s}_{i_2}\cdots\boldsymbol{s}_{i_{m-1}}$$

in Theorem 11.0.1 as

$$[\boldsymbol{s}_{i_1}, \boldsymbol{s}_{i_2}\cdots\boldsymbol{s}_{i_{m-1}}\boldsymbol{s}_{i_m}\boldsymbol{s}_{i_{m-1}}^{-1}\cdots\boldsymbol{s}_{i_2}^{-1}] = e$$

therefore the relations in our presentation (Theorem 5.1.2) are exactly the relations in the Grant-Marsh's presentation, thus it follows:

Theorem 11.0.2. *Let $s_1\cdots s_n$ be a reduced S-decomposition of a Coxeter element c. Let $Fact_2(c)$ denote the set of all two-part factorizations of c and $\mathcal{Q}$ denote the quiver associated to the two-part factorization $s_1\cdots s_n|\cdot$, then the Artin group presentations arising from the reduced T-decompositions in $Fact_2(c)$ using Theorem 5.1.2 are precisely the presentations arising from the quivers in the mutation class of $\mathcal{Q}$ using Theorem 11.0.1.*

Therefore our presentation recovers Grant and Marsh's presentation as a special case. Starting with a reduced T-decomposition of a Coxeter element c using Hurwitz moves we can produce all possible reduced T-decompositions of c. However, only a few of these reduced T-decompositions will qualify as two-part factorizations. It is only at these points that our presentations coincides with Grant and Marsh's presentations. Figure 11.1 demonstrates an example.

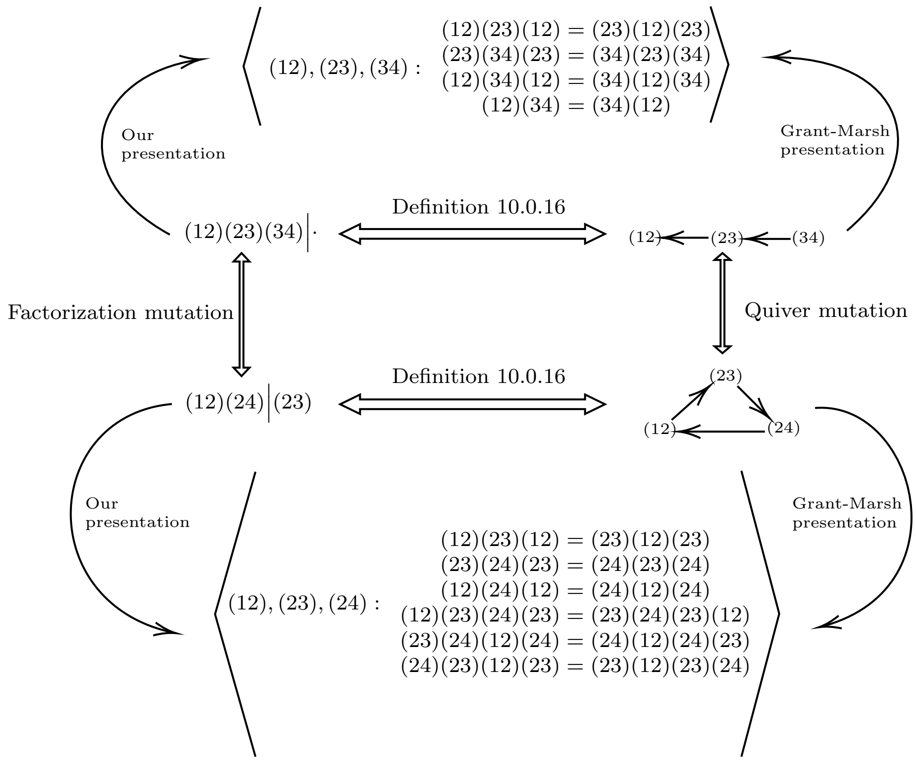


Figure 11.1. Our presentation vs Grant-Marsh's presentation.

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BIOGRAPHICAL SKETCH

Priyojit Palit is currently in his 5$^{\text{th}}$ year of study in the mathematics graduate program at The University of Texas at Dallas. In 2013, he entered the Institute of Mathematics and Applications in India and received a Bachelor of Science (Honors) in April 2016, with a major in Mathematics and Computer Science.

In August 2016 Priyojit entered the applied mathematics graduate program at The South Asian University in India. After completing 30 credit hours, he entered the mathematics graduate program at The University of Texas at Dallas in August 2017.

During these last 5 years, Priyojit has taught several undergraduate courses as a graduate teaching assistant and carried out research in algebraic combinatorics under the direction of Dr. Nathan Williams as a graduate research assistant. His research interest is in algebraic combinatorics within the framework of reflection groups with a broad interest in motivation from other areas of mathematics such as Lie theory, cluster algebras and Artin/braid groups.

CURRICULUM VITAE

Priyoijt Palit

June 6, 2022

Contact Information:

Department of Mathematics
The University of Texas at Dallas
800 W. Campbell Rd.
Richardson, TX 75080-3021, U.S.A.

Educational History:

BSc (Hons), Mathematics and Computing, Institute of Mathematics and Applications, India, 2013 – 2016
MSc, (30 credit hours completed), Applied Mathematics, South Asian University, 2016 – 2017
MS & PhD, Mathematics, University of Texas at Dallas, 2017 – 2022

Dual Braid Presentations and Cluster Algebras
PhD Dissertation
Mathematical Sciences, University of Texas at Dallas
Advisor: Dr. Nathan Williams

Employment History:

Graduate Teaching Assistant, The University of Texas at Dallas, January 2022 – August 2022
Graduate Research Assistant, The University of Texas at Dallas, May 2021 – December 2021
Graduate Teaching Assistant, The University of Texas at Dallas, August 2017 – April 2021

Conference Talks and Poster presentations:

Dynkin diagrams and dual braid presentations, 17^{th} Graduate Student Combinatorics Conference (GSCC), University of California San Diego 2022.
A Correspondence between Quiver Mutation and Hurwitz Action, DFW Virtual Poster Competition in Mathematics, University of Texas at Dallas, May 2021.
Coxeter-like presentation from factorizations of Coxeter elements, 16^{th} Graduate Student Combinatorics Conference (GSCC), University of Minnesota, April 2021.
Cluster Algebras and Artin braid presentations, Graduate Algebra Symposium, University of North Texas, February 2020.

Awards and Honors:

Best Graduate Blitz, DFW Virtual Poster Competition in Mathematics, May 2021.

Organizing:

Texas Geometry and Topology Conference (TGTC), 2022 – Co-organizer of an annual 3-day conference for the community of geometers and topologists in Texas and surrounding states.