Sliding bifurcations of limit cycles - derivation of canonical forms

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5 Summary
Consider Filippov systems (dynamical systems governed by discontinuous but piecewise smooth ordinary differential equations) of the form

\[
\dot{x} = \begin{cases} 
  f_+(x, \mu), & \text{if } h(x, \mu) \geq 0, \\
  f_-(x, \mu), & \text{if } h(x, \mu) < 0 
\end{cases}
\]

\[f_+, f_- : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}^N\] and \[h : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}\]
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The boundary \(\mathcal{H}_s := \{h(x, \mu) = 0\}\) is termed as the switching manifold.
On $\mathcal{H}_s$ we observe switching between $f_-$ and $f_+$ (or *vice versa*) or *sliding* (see Figure)
system evolution on $\mathcal{H}_s$ is termed as sliding; sliding motion is governed by the vector field $f_s = \alpha f_+ + (1 - \alpha)f_-$, $0 \leq \alpha(x) \leq 1$
Denote the boundaries of the region where sliding is possible by $\partial \hat{\Sigma}^{\pm}$.
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Suppose that there exists a limit cycle, say $L(x, \mu)$, built from one or distinct segments generated by vector fields $f_+$, and/or $f_-$, and/or $f_s$. 
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Suppose that under the variation of $\mu$ at some $\mu^*$ the limit cycle generated by $f_+$, or $f_-$, or $f_s$ has one point $x^* \in \partial \hat{\Sigma}^\pm$, or the limit cycle built from distinct segments switches between $f_+$ and $f_-$ (or $f_+$ and $f_-$), at one point $x^* \in \partial \hat{\Sigma}^\pm$. 
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We can distinguish between four different cases of codimension-one sliding bifurcations of limit cycles
Sliding bifurcations of limit cycles

**Sliding bifurcations – different case**

- **Crossing-sliding**
  - Flow transitions from one surface to another.
- **Switching-sliding**
  - Flow switches between surfaces.
- **Grazing-sliding**
  - Flow grazes along a boundary.
- **Adding-sliding**
  - Flow adds to a surface from another.
At grazing-sliding point $\mathbf{x}^*$ the following conditions have to hold

(i) $H(\mathbf{x}^*, \mu^*) = 0$, 

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(ii) $H_x F_1(x^*, \mu^*) = 0$, 

\[ x_f \quad PDM \quad x_s \]
Grazing-sliding, conditions

**Grazing-sliding**

At grazing-sliding point $\mathbf{x}^*$ the following conditions have to hold

(i) $H(\mathbf{x}^*, \mu^*) = 0$,
(ii) $H_x F_1(\mathbf{x}^*, \mu^*) = 0$,
(iii) $(H_x F_1)_x F_1(\mathbf{x}^*, \mu^*) > 0$, 

\[ \mathbf{x}^* \]
Grazing-sliding

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(i) $H(\mathbf{x}^*, \mu^*) = 0$,
(ii) $H_x F_1(\mathbf{x}^*, \mu^*) = 0$,
(iii) $(H_x F_1)_x F_1(\mathbf{x}^*, \mu^*) > 0$,
(iv) $\frac{\partial H(P(\mathbf{x},\mu))}{\partial \mu} < 0$, where $P : \Pi \hookrightarrow \Pi$ (ignoring the switching).
The PDM is computed as a map from $\Pi := \{H_x F_1(x) = 0\}$ back to itself. Define $H(x) + y^2 = 0$. 
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The PDM map is computed as the following composition of flows

$$\text{PDM}(x) = \phi_s(\phi_1(x, \delta), s)$$

where $\delta$ (negative) can be found by solving $H(\phi(x, \delta)) = 0$ and expressed as a power series in $y$. We then solve $H(\phi_s(\phi_1(x, \delta), s))F_1(\phi_s(\phi_1(x, \delta), s)) = 0$ for $s$ ($s$ is positive).
Map derivation

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- To leading order we have

\[
PDM(x) = \begin{cases} 
  x & \text{for } H(x) \geq 0, \\
  x - H(x)(C_0F_2(x^*) - C_1F_1(x^*)) & \text{for } H(x) < 0.
\end{cases}
\]
The map $P(x, \mu)$ can be approximated by an affine map $P_A$

$$P_A = Ax + B\mu,$$

where $A$ is an $n \times n$ matrix, and it is characterized by the full rank, and $B$ is a $1 \times n$ column vector (assume $(x^*, \mu^*) = (0, 0)$)
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$$P_F(x, \mu) = \begin{cases} 
Ax + B\mu & \text{for } H(x) \geq 0, \\
A(I - F_C H_x)x + B\mu & \text{for } H(x) < 0
\end{cases}$$

$F_C = F_2(x^*)C_0 - C_1F_1(x^*)$ or $F_C = F_2(x^*)C_0$ (for non-autonomous systems)
Consider 3-dimensional Filippov type system where a limit cycle undergoes a grazing-sliding bifurcation at \((x^*, \mu^*) = (0, 0)\), and the grazing cycle is generated by \(F_1\).
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Choose the coordinate system \((x, y, z)\) such that \(H(x) = x\) and the Poincaré section \(\Pi\) is given by \(H_x F_1(x) = 0 = y\).
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Choose the coordinate system $(x, y, z)$ such that $H(x) = x$ and the Poincaré section $\Pi$ is given by $H_x F_1(x) = 0 = y$.

Then using the formula for the PDM it can be expressed as

$$PDM(x, z) = \begin{cases} 
\begin{pmatrix} x \\ 0 \\ Cx + z \end{pmatrix} & \text{for } x \geq 0, \\
0 & \text{for } x < 0.
\end{cases}$$
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Then using the formula for the PDM it can be expressed as

\[
PDM(x, z) = \begin{cases} 
  x & \text{for } x \geq 0, \\
  0 & \text{for } x < 0.
\end{cases}
\]

The affine map \(P_A : \Pi \mapsto \Pi\) can be written as

\[
P_A(x, z; \mu) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} + \mu \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}
\]

where \(a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2\) are arbitrary but fixed constants and the coefficient matrix is full rank and \(b_1\) is non-zero.
Composing \( PDM \) and \( P_A \) and scaling \( \mu \) gives

\[
G(x, z) = \begin{cases} 
\left( \begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right) \left( \begin{array}{c} x \\ z \end{array} \right) + \mu \left( \begin{array}{c} 1 \\ 0 \end{array} \right) & \text{for } x \geq 0, \\
\left( \begin{array}{cc} a_{21} C & a_{21} \\ a_{22} C & a_{22} \end{array} \right) \left( \begin{array}{c} x \\ z \end{array} \right) + \mu \left( \begin{array}{c} 1 \\ 0 \end{array} \right) & \text{for } x < 0.
\end{cases}
\]
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\[
G(x, z) = \begin{cases} 
\begin{pmatrix}
    a_{11} & a_{12} \\
    a_{21} & a_{22}
\end{pmatrix}
\begin{pmatrix}
    x \\
    z
\end{pmatrix}
+ \mu \begin{pmatrix}
    1 \\
    0
\end{pmatrix}
& \text{for } x \geq 0, \\
\begin{pmatrix}
    a_{21} C & a_{21} \\
    a_{22} C & a_{22}
\end{pmatrix}
\begin{pmatrix}
    x \\
    z
\end{pmatrix}
+ \mu \begin{pmatrix}
    1 \\
    0
\end{pmatrix}
& \text{for } x < 0.
\end{cases}
\]

Alternative form of the full map is given by

\[
g(\tilde{x}, \tilde{z}) = \begin{cases} 
\begin{pmatrix}
    \tilde{a}_{11} & 1 \\
    \tilde{a}_{21} & 0
\end{pmatrix}
\begin{pmatrix}
    \tilde{x} \\
    \tilde{z}
\end{pmatrix}
+ \mu \begin{pmatrix}
    1 \\
    0
\end{pmatrix}
& \text{for } \tilde{x} \geq 0, \\
\begin{pmatrix}
    \tilde{a}_{12}' & 1 \\
    0 & 0
\end{pmatrix}
\begin{pmatrix}
    \tilde{x} \\
    \tilde{z}
\end{pmatrix}
+ \mu \begin{pmatrix}
    1 \\
    0
\end{pmatrix}
& \text{for } \tilde{x} < 0.
\end{cases}
\]

in this case $\{\tilde{x} = 0\}$ does not correspond to the switching surface.
Consider a Filippov type system of the form

\[ \ddot{x} + x = \alpha_4 \cos(\omega t) + \alpha_1 \text{sgn}(1 - \dot{x}) + \alpha_2 (1 - \dot{x}) - \alpha_3 (1 - \dot{x})^3. \]

Setting the variables \( x_1 = x, \ x_2 = \dot{x}, \) and \( x_3 = \omega t \mod 2\pi \) we obtain a Filippov system

\[
F_{1,2} = \begin{cases} 
-\dot{x}_1 & \pm \alpha_1 - \alpha_2 (1 - x_2) + \alpha_3 (1 - x_2)^3 + \alpha_4 \cos(x_3) \\
\omega & \end{cases}
\]

with + sign for \( F_1, \) and \( H(x) = 1 - x_2. \)
We set the parameters to $\alpha_1 = \alpha_2 = 1.5$, $\alpha_3 = 0.45$, $\alpha_4 = 0.7$
To prove our numerical finding we will study the normal form map \( g(\tilde{x}_1, \tilde{x}_2) \)
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the non-trivial Floquet multipliers of the grazing cycle viewed as not interacting with the boundary of the sliding region and as a cycle with a zero-length sliding segment determine the traces and determinants of the matrix coefficients and hence the structure of the map
To prove our numerical finding we will study the normal form map
\[ g(\tilde{x}_1, \tilde{x}_2) \]
the non-trivial Floquet multipliers of the grazing cycle viewed as not interacting with the boundary of the sliding region and as a cycle with a zero-length sliding segment determine the traces and determinants of the matrix coefficients and hence the structure of the map.

We found that the Floquet multipliers of the non-sliding cycle are \( \lambda_{1s} = -5.732288 \) and \( \lambda_{2s} = -0.016799 \) and of the sliding one are \( \lambda_{1ns} = 0.619648 \) and \( \lambda_{2ns} = 0 \).
\[
\Sigma(\ddot{x}_1, \ddot{x}_2, \tilde{\mu}) = \begin{cases}
\begin{pmatrix}
-5.7490869 & 1 \\
0.0965801 & 0 \\
0.6196484 & 1
\end{pmatrix}
\begin{pmatrix}
\ddot{x}_1 \\
\ddot{x}_2
\end{pmatrix}
+ \tilde{\mu}
\begin{pmatrix}
1 \\
0
\end{pmatrix}
& \ddot{x}_1 > 0 \\
\begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
\ddot{x}_1 \\
\ddot{x}_2
\end{pmatrix}
+ \tilde{\mu}
\begin{pmatrix}
1 \\
0
\end{pmatrix}
& \ddot{x}_1 < 0,
\end{cases}
\]
Grazing-sliding in 3-dimensional Filippov type flows

\[
\Sigma(\tilde{x}_1, \tilde{x}_2, \tilde{\mu}) = \begin{cases} 
\begin{pmatrix} -5.7490869 & 1 \\ 0.0965801 & 0 \\ 0.6196484 & 1 \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} + \tilde{\mu} \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \tilde{x}_1 > 0 \\
\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} + \tilde{\mu} \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \tilde{x}_1 < 0,
\end{cases}
\]

We prove the existence of robust chaos in the map for $\tilde{\mu} > 0$ by reducing the map to a 1-dimensional map.
Using local theory we can analyze sliding bifurcations of limit cycles in $n$-dimensional Filippov type flows
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Using local theory we can analyze sliding bifurcations of limit cycles in \( n \)-dimensional Filippov type flows.

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In the case of grazing-sliding bifurcations that canonical map that captures the systems dynamics is piecewise affine to leading order.

In the case of 3-dimensional Filippov type flows the analysis of the system around grazing-sliding bifurcations is reduced to the analysis of 1-dimensional map.