Hysteresis loops generated by perturbation of systems with Devil’s staircase nonlinearity

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Plan of the talk

Introduction

Feedback as a source of hysteresis

Hysteresis in slow-fast systems

The effect of positive feedback on Devil’s staircase
Systems with memory

Hysteresis operator (transducer)

\[
\begin{array}{c}
\text{Input} \\
\uparrow \\
x(t) \\
\downarrow \\
\text{Output} \\
\uparrow \\
y(t) \\
\downarrow \\
\text{Time} \\
\end{array}
\quad P
\quad \begin{array}{c}
\text{Input} \\
\uparrow \\
x(at+b) \\
\downarrow \\
\text{Output} \\
\uparrow \\
y(at+b) \\
\downarrow \\
\text{Time} \\
\end{array}
\quad \begin{array}{c}
\text{Input} \\
\uparrow \\
x(t) \\
\downarrow \\
\text{Output} \\
\uparrow \\
y(t) \\
\downarrow \\
\text{Time} \\
\end{array}
\]

with input \(u(t)\), output \(x(t)\).

**Rate-independence**, a characteristic property
Other natural properties of hysteresis operators:
Causality, Semi-group property of the input-state operator,...

Models with local memory:
Relay, Play, Stop, Multidimensional Play and Stop, Duhem model, Multidimensional relays, Armstrong-Frederick model,...

Models with non-local memory:
Models of Preisach, Prandtl-Ishlinskii, Mroz, Mayergoyz-Friedman, ...

Hysteresis operators are nonsmooth
Mathematical theory of hysteresis operators and applications
P. Krejčí, Hysteresis, Convexity and Dissipation in Hyperbolic Equations, 1996

Conferences
Limits to Rationality in Financial Markets, Glasgow, June 15 - July 3, 2009

Multi-Rate Processes and Hysteresis, Cork, March 2008
**Operator-differential equation** with a hysteresis operator $P$:

$$x' = f(t, u(t), x(t)) \quad OR \quad u' = f(t, u(t), x(t))$$

where $u(t)$ is an input, $x(t)$ is an output of the hysteresis operator:

$$x(t) = (Pu)(t).$$

In case of non-local memory, the system is characterised by a state of the hysteresis nonlinearity at each moment, in particular initial data involve an initial state.
Systems with hysteresis nonlinearities

Systems with relays, plays etc.
A.A. Andronov, A. A. Feldbaum, Ya. Z. Tsypkin, V. A. Yakubovich, G.A. Leonov ...

Bifurcations, oscillations, stability:


Numerical algorithms

http://euclid.ucc.ie/appliedmath/
Transducer

\[ x = f(u) \]  \hspace{1cm} (1)

with input \( u = u(t) \), output \( x = x(t) \) and a continuous input-output function \( f \).

Transducer with positive feedback:

\[ x = f(u + \varepsilon x) \]  \hspace{1cm} (2)

The coefficient \( \varepsilon > 0 \) measures the feedback strength.
Graphical interpretation of the implicit function $x = g(u)$ defined by $x = f(u + \varepsilon x)$.

Hence, if $f$ is globally Lipschitz continuous, $\varepsilon$ is sufficiently small, then $g$ is a single-valued continuous implicit function.
Multiple solutions of equation (2) at a point where $f'(u) = \infty$

The least and the largest (discontinuous) solutions:

$$g^-(u) = \min\{x : x = f(u + \varepsilon x)\}, \quad g^+(u) = \max\{x : x = f(u + \varepsilon x)\}$$
Positive feedback — a mechanism creating hysteresis in the input-output relationship of transducers with time-dependent input $u = u(t)$ and output $x = x(t)$.

(a) Ideal switch

$$x = \text{sign } u$$

(b) Ideal switch with positive feedback

$$x = \text{sign}(u + \varepsilon x)$$

Solutions $x = 1$ for $u > -\varepsilon$;

$$x = -1 \text{ for } u < \varepsilon.$$  

Choosing a solution $x(t)$ of this equation with varying $u = u(t)$ in a particular manner, which can be characterised as ‘the most inert’, leads to

(c) Input-output relationship of Non-ideal relay with thresholds $u = \pm \varepsilon$

$$u(t) \mapsto x(t) = (R_\varepsilon u)(t)$$
Feedback loop – a source of hysteresis

Hysteresis in slow-fast systems

The effect of positive feedback on Devil’s staircase

\[ x = (u - v)/\varepsilon \]

\[ x = g^-(u) = \text{sign}(u - \varepsilon); \]

Black: the upper solution \[ x = g^+(u) = \text{sign}(u + \varepsilon). \]

Rectangular hysteresis loop of the non-ideal relay.

For example, if \( u = u(t) \) increases on the time interval \( t_0 \leq t \leq t_1 \) from a value \( u(t_0) = \alpha < -\varepsilon \) to a value \( u(t_1) = \beta > \varepsilon \) and then decreases on the time interval \( t_1 \leq t \leq t_2 \) back from \( \beta \) to \( \alpha \), then

\[ x(t) = \begin{cases} 
  g^-(u(t)), & t_0 \leq t \leq t_1 \\
  g^+(u(t)), & t_1 < t \leq t_2 
\end{cases} \]
A function with $f' = \infty$ at some point and a corresponding hysteresis loop
This picture suggests that for systems with complex continuous input-output functions $f$, such as Devil's staircase, an introduction of a small positive feedback generates an input-output relationship with many jumps. Taking the Cantor function as a model, we ask questions like

(a) Is the number of jumps finite or infinite?

(b) What is the distribution/size of the jumps?
Hysteresis in slow-fast systems

Van der Pol oscillator

\[
\begin{align*}
\dot{u} &= -x \\
\delta \dot{x} &= u + \epsilon x - x^3, \quad \delta \ll 1
\end{align*}
\]

Using \( \epsilon \) as a bifurcation parameter, Hopf bifurcation at \( \epsilon = 0 \); relaxation oscillations for \( \epsilon > 0 \)

Slow manifold (black): \( 0 = u + \epsilon x - x^3 \), i.e. \( x = \frac{3}{\delta} \sqrt{u + \epsilon x} \)
Slow manifold $x = \sqrt[3]{u}$:

\begin{align*}
\dot{u} &= -x \\
\delta \dot{x} &= u - x^3
\end{align*}

Slow manifold $x = \sqrt[3]{u + \varepsilon x}$:

\begin{align*}
\dot{u} &= -x \\
\delta \dot{x} &= u + \varepsilon x - x^3
\end{align*}

Slow manifold $x = f(u + \varepsilon x)$:

\begin{align*}
\dot{u} &= -x \\
\delta \dot{x} &= u + \varepsilon x - f^{-1}(x)
\end{align*}
\[ f(u) = \text{sign } u \text{ (or } f(u) = \text{sign } u + au \text{ with small } a) \]

Cycle of the system

\[ \dot{u} = -x \]
\[ \delta \dot{x} = u + \varepsilon x - f^{-1}(x) \]

Transmission channel: input $x(t)$, output $y(t)$ related by

$$\delta z' = Az + bx(t), \quad z \in \mathbb{R}^N, \delta \ll 1$$

$$y = (c, z)$$

where $b, c \in \mathbb{R}^N$, $A$ is a Hurwitzian matrix, and $(c, A^{-1}b) = -1$. Hence,

$$x = f(u + \varepsilon y).$$

The degenerate equation is

$$x = f(u + \varepsilon x).$$
**System with a small delay in the feedback loop**

\[ x(t) = f(u(t) + \varepsilon x(t - h)) \]

Expanding in \( h \) gives

\[ f^{-1}(x) = u + \varepsilon x - \varepsilon x' h + \cdots \]

i.e.,

\[ (\varepsilon h)x' = u + \varepsilon x - f^{-1}(x) \]

**Degenerate equation**

\[ x = f(u + \varepsilon x) \]
Devil’s staircase $x = f(u)$

Solution of $x = f(u + \varepsilon x)$?

Notation

Partitions

\[ \mathcal{P}_k : \quad 0 < v_1 < u_1 < v_2 < u_2 < \cdots < v_{2^k-1} < u_{2^k-1} < 1 \]

of \([0,1]\) generating the Cantor Middle Third set

\[ \mathcal{C} = \bigcap_{k=1}^{\infty} \left( \left[0, 1\right] \setminus \bigcup_{u_i, v_i \in \mathcal{P}_k} (v_i, u_i) \right) \]
**Theorem**

For any $\varepsilon > 0$ the range of the function $g^-$ is a finite subset of the set of all the fractions $0 \leq \frac{m}{2k} \leq 1$ with integer $m, k$.

More precisely, for $2 \left( \frac{2}{3} \right)^{k+1} \leq \varepsilon < 2 \left( \frac{2}{3} \right)^k$ with $k \geq 1$, the increasing piecewise constant left-continuous function $g^-$ is defined by

$$g^-(u) = \begin{cases} 
0, & u \leq u_0 = 0, \\
\frac{m}{2k}, & u_{m-1} - \frac{\varepsilon(m-1)}{2^k} < u \leq u_m - \frac{\varepsilon m}{2^k}, \quad m = 1, \ldots, 2^k - 1, \\
1, & u > u_{2^k-1} - \frac{\varepsilon(2^k-1)}{2^k},
\end{cases}$$

The proof is based on the self-similarity of the Cantor function.

If $\varepsilon \geq 4/3$, then $g^-(u) = 0$ for $u \leq 0$ and $g^-(u) = 1$ for $u > 0$. 
Theorem

For the Cantor function $f$, $g^+$ is the increasing piecewise constant right-continuous function defined by

$$g^+(u) = \begin{cases} 
0, & u < v_1 - \frac{\varepsilon}{2^k}, \\
\frac{m}{2^k}, & v_m - \frac{\varepsilon m}{2^k} \leq u < v_{m+1} - \frac{\varepsilon(m+1)}{2^k}, \quad m = 1, \ldots, 2^k - 1, \\
1, & u \geq v_{2^k} - \varepsilon = 1 - \varepsilon
\end{cases}$$

for $2 \left(\frac{2}{3}\right)^{k+1} \leq \varepsilon < 2 \left(\frac{2}{3}\right)^k$. 
Suppose an input \( u = u(t) \) increases on the time interval \( t_0 \leq t \leq t_1 \) from a value \( u(t_0) = \alpha < -\epsilon \) to a value \( u(t^1) = \beta > \epsilon \) and then decreases on the time interval \( t_1 \leq t \leq t_2 \) back from the value \( \beta \) to the value \( \alpha = u(t_2) \).

Hysteresis loops of the input-output relationship for the output

\[
x(t) = \begin{cases} 
g^-(u(t)), & t_0 \leq t \leq t_1, 
g^+(u(t)), & t_1 < t \leq t_2, \end{cases}
\]

where \( g^- \) is the min solution, and \( g^+ \) is the max solution, of \( x = f(u + \epsilon x) \):

(a) The input \( u \) increases from the value \( \alpha \) to the value \( \beta \), the point \((u(t), x(t))\) follows the red path moving right and upwards; the input \( u \) decreases from \( \beta \) to \( \alpha \), the point \((u, x)\) follows the black path moving left and downwards.

(b) The input \( u = u(t) \) varies between the values \( \gamma \) and \( \beta \).
Theorem

The input-output relationship

\[ x(t) = \begin{cases} 
  g^-(u(t)), & \text{if } t_0 \leq t \leq t_1, \\
  g^+(u(t)), & \text{if } t_1 < t \leq t_2,
\end{cases} \]

created by the positive feedback with \(2\left(\frac{2}{3}\right)^{k+1} \leq \varepsilon < 2\left(\frac{2}{3}\right)^k\), and the hysteresis loops, are the same as those of the Preisach operator

\[ x(t) = \sum_{m=1}^{M} \mu_m(\mathcal{R}_{s_m, \delta_m} u)(t) \]

with \(M = 2^k\) relays, which have equal weight and equal width

\[ \mu_1 = \cdots = \mu_k = 2^{-k}, \quad \delta_1 = \cdots = \delta_{2^k} = 2^{-k-1}\varepsilon - 0.5 \cdot 3^{-k}, \]

and \(s_m = u_{m-1} - \varepsilon (2m - 1)2^{-k-1} + 0.5 \cdot 3^{-k}, \quad m = 1, \ldots, 2^k.\)
The area $\sigma_\varepsilon$ of the outer hysteresis loop (the some of the areas of all the relays) on the input-output plane satisfies

$$3^{-k-1} \leq \sigma_\varepsilon = 2^{-k} \varepsilon - 3^{-k} < 3^{-k}.$$ 

Hence the area $\sigma_\varepsilon$ tends to zero as $\varepsilon \to 0$, i.e. when the feedback decreases:

$$\sigma_\varepsilon \asymp \varepsilon^{\log_2 3 \log_2 3^{-1}}.$$
Thank you!