EXAMINATION I

Show your work!

Problem 1

The following series is *not* a power series, but you can determine exactly where it converges:

$$\sum_{n=0}^{\infty} \frac{(n-1)\,3^n}{(n^2-1)(x+1)^n}$$

Of course we use the ratio test!:

$$\left| \frac{n\,3^{n+1}}{[(n+1)^2-1](x+1)^{n+1}} \cdot \frac{(n^2-1)(x+1)^n}{(n-1)\,3^n} \right| = \frac{n}{n-1} \cdot \frac{n^2-1}{n^2+2n} \cdot \frac{3}{|x+1|}$$

Taking the limit as $n \to \infty$, the n fractions all tend to one so the limit is: $\frac{3}{|x+1|}$. For convergence by this test we must have that ratio less than one, or $|x+1| > 3$. Since $|x+1| = |x-(-1)|$ is the distance from $x = -1$, we have convergence for $x > 2$ and for $x < -4$.

Finally, the end-points. At $x = 2$, the series is $\sum \frac{n-1}{n^2-1} \approx \sum \frac{1}{n}$ which diverges (p-series with $p = 1$). At $x = -4$, the series is $\sum (-1)^n \frac{n-1}{n^2-1} \approx \sum (-1)^n \frac{1}{n}$ which converges (alternating series test).

Thus the series **converges** exactly for $\boxed{x > 2}$ and $\boxed{x \le -4}$ and **diverges** everywhere else (*i.e.*, $\boxed{-4 < x \le 2}$).

Problem 2

Derive the Taylor series expansion for the function $f(x) = 1/x$ which has its interval of convergence centered at $x = 1$ (giving the general term and stating exactly where it does converge):

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n	$f^{(n)}(x)$	$f^{(n)}(1)$	$f^{(n)}(1)/n!$
0	$1/x$	1	1
1	$-1/x^2$	-1	-1
2	$2/x^3$	2	1
3	$-3 \cdot 2/x^4$	$-3 \cdot 2$	-1
4	$4 \cdot 3 \cdot 2/x^5$	$4 \cdot 3 \cdot 2$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$

By now it should be obvious that the coefficients are ± 1. In order to be "centered" at $x = 1$ the terms must be powers of $(x - 1)$. Thus

$$\frac{1}{x} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$$

BTW: Note that $\frac{1}{x} = \frac{1}{1+(x-1)} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$ from one of our most inportant well-known formulae!

Problem 3

Design a cylindrical can (with a lid and bottom) to contain 16π cubic units of liquid, using the minimum amount of metal.

The quantity we are trying to maximize is $M = 2\pi r^2 + 2\pi rh$ with the constraint that $V = \pi r^2 h = 16\pi$. Of course we use a Lagrange Multiplier:

$$\mathcal{L} = 2\pi r^2 + 2\pi rh + \lambda(\pi r^2 h)$$

Differentiating we get the two equations:

$$\partial\mathcal{L}/\partial r = 4\pi r + 2\pi h + \lambda(2\pi rh) = 0$$

$$\partial\mathcal{L}/\partial h = 2\pi r + \lambda(\pi r^2) = 0$$

These, together with the constraint, give us 3 equations in the 3 unknowns - as expected. Since r cannot be zero, the second equation may be solved for λ obtaining $\lambda = (-2\pi r)/(\pi r^2) = -2/r$. Substututing into the first equations yields:

$$4\pi r + 2\pi h - \frac{2}{r}(2\pi rh) = \pi(4r + 2h - 4h) = 0 \Rightarrow h = 2r$$

Now the contraint equations gives:

$$V = \pi r^2 h = \pi r^2 \cdot 2r = 2\pi r^3 = 16\pi \Rightarrow \boxed{\text{r=2}} \Rightarrow \boxed{\text{h=4}}$$

BTW: That means the height equals the diameter. It's the same problem as maximizing the volume of a can with a fixed amount of material!

Problem 4

Find the moment of inertia with respect to the origin (center) of a spherical shell whose inner radius is 1 and whose outer radius is 2, assuming constant density.

All these spheres tell us to use spherical coordinates and the square of the distance from the origin is r^2 immediately! We must remember that the element of volume is $r^2 \sin\theta \, dr \, d\theta \, d\phi$. The *hard* question is the limits of the integrals. Certainly, r goes from 1 to 2 is almost forced on us by the statement of the problem. Next, ϕ ("longitude") goes from 0 to 2π in order to cover the entire spheres. What about θ? Remember θ is the "co-latitude" or "polar latitude" so it goes from $\theta = 0$ at the "north pole" down only to $\theta = \pi$ at the "south pole", **not** to 2π. Therefore

$$I_{origin} = \int_0^{2\pi} \int_0^{\pi} \int_1^2 r^2 \cdot r^2 \, sin\theta \, dr \, d\theta \, d\phi$$

$$I_{origin} = 2\pi \cdot [-\cos\theta]_0^{\pi} \cdot \frac{1}{5} \cdot \left[r^5\right]_1^2 = 2\pi(1+1)\frac{1}{5}(32-1) = \underline{124\pi/5}$$

Problem 5

Find the volume of the solid bounded by the graphs of the equations $z = \sqrt{x^2 + y^2}$ and $x^2 + y^2 + z^2 = 9$.

When we draw the graphs, we see that $z = \sqrt{x^2 + y^2}$ is that part of the usual (ice cream) cone $z^2 = x^2 + y^2$ above the xy-plane. Most importantly, the angle it makes with the z-axis is $45^{\text{o}} = \pi/4$. The other equation is obviously a sphere of radius 3 around the origin. [Note: This is not a surface area problem so we don't have to figure out where the cone meets the sphere so we can tell what area in the xy-plane is covered.] The volume element in spherical coordinates is $r^2 \sin\theta \, dr \, d\theta \, d\phi$ so all we need is what are, in fact, the limits?

The "longitude" (ϕ) goes from 0 to 2π as usual to go all around the sphere. The distance from the origin (r) is clearly going from the center of the sphere ($r = 0$) to the outer edge ($r = 3$). The *hard* question (as in Problem 4) is what are the limits for θ? Remember θ is the "co-latitude" or "polar latitude" so it goes from $\theta = 0$ at the "north pole" down only to $\theta = \pi/4$ as mentioned above. Now we can set up the integrals:

$$V = \int_0^{2\pi} \int_0^{\pi/4} \int_0^3 r^2 \, \sin\theta \, dr \, d\theta \, d\phi = 2\pi \cdot [-\cos\theta]_0^{\pi/4} \cdot \left[r^3/3\right]_0^3 = 2\pi \cdot \left(1 - 1/\sqrt{2}\right) \cdot (9)$$

Thus

$$V = \underline{9\pi(2 - \sqrt{2})}$$

Problem 6

Find the surface area of that part of the paraboloid $z = 1 + x^2 + y^2$ which is inside the cylinder $x^2 + y^2 = 1$.

Immediately, $\phi = z - x^2 - y^2 - 1 \Rightarrow \phi_x = -2x,\ \phi_y = -2y,\ \phi_z = 1$ so

$$\sec\gamma = \sqrt{4x^2 + 4y^2 + 1}$$

The surface is over a circle so we will use polar coordinates:

$$S = \int\int \sec\gamma\, dA = \int\int \sqrt{4(x^2 + y^2) + 1}\, dx\, dy = \int_0^{2\pi} \int_0^1 \sqrt{4r^2 + 1}\, r\, dr\, d\theta$$

Integrating:

$$S = 2\pi \cdot \left[(4r^2 + 1)^{3/2} \cdot \frac{2}{3} \cdot \frac{1}{8} \right]_0^1 = \underline{\pi \left(5^{3/2} - 1 \right) / 6}$$

BTW: Notice that you cannot replace $x^2 + y^2$ by 1 because that is just the boundary of the cylinder in the xy-plane and you must use the area **inside** it. You could replace $x^2 + y^2$ by $z - 1$ from the surface of the paraboloid, but then you have to go back to x and y for the integral.

End of Test

Think of all the other material we couldn't find time for on this test! Change of variables with Jacobians, Max-Min problems on boundaries, convergence of lots of types of series, applications of power series, *etc.* You will need all this material in later courses.