EXAMINATION II

Show your work!

$$\oint_{\partial(\sigma)} \mathbf{V} \bullet d\mathbf{r} = \int\int_{\sigma} (\nabla \times \mathbf{V}) \bullet \mathbf{n}\, d\sigma$$

$$\int\int\int_{\tau} \nabla \bullet \mathbf{V}\, d\tau = \int\int_{\sigma(\tau)} \mathbf{V} \bullet \mathbf{n}\, d\sigma$$

Problem 1 [6-6.6]

Find a vector normal to the surface $x^2 + y^2 - z = 0$ at the point $(3, 4, 25)$. Find the equations of the tangent plane and normal line to the surface at that point.

$\phi = x^2 + y^2 - z \Rightarrow \nabla\phi = 2x\,\mathbf{i} + 2y\,\mathbf{j} - \mathbf{k} \Rightarrow \nabla\phi(3, 4, 25) = \underline{6\,\mathbf{i} + 8\,\mathbf{j} - \mathbf{k}}$

Tangent Plane: $6(x - 3) + 8(y - 4) - (z - 25) = 0 \Rightarrow \underline{6x + 8y - z = 25}$

Normal Line: $\underline{\frac{x-3}{6} = \frac{y-4}{8} = \frac{z-25}{-1}}$

Problem 2 [6-8.2]

Evaluate the line integral $\oint (x+2y)\,dx \,-\, 2x\,dy$ along each of the following closed paths, taken counterclockwise:

(a) the circle $x^2+y^2=1$;

(b) the square with corners at $(1,1),(-1,1),(-1,-1),(1,-1)$;

(c) the square with corners $(0,1),(-1,0),(0,-1),(1,0)$.

We may use Green's (Stokes') Theorem with $P=(x+2y)$ and $Q=-2x$:

$\oint (x+2y)\,dx \,-\, 2x\,dy = \int\int_\sigma \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,d\sigma = \int\int_\sigma (-4)\,d\sigma = -4 \,\times$ Area

Therefore, (a) $-4\times\pi\,1^2 = \underline{-4\pi}$

(b) $-4\times 2^2 = \underline{-16}$

(c) $-4\times(\sqrt{2})^2 = \underline{-8}$

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Note: Of course you can parameterize each of the curves and do the integrations and get the same answers, but it's much more work!

Problem 3 [6-9.4]

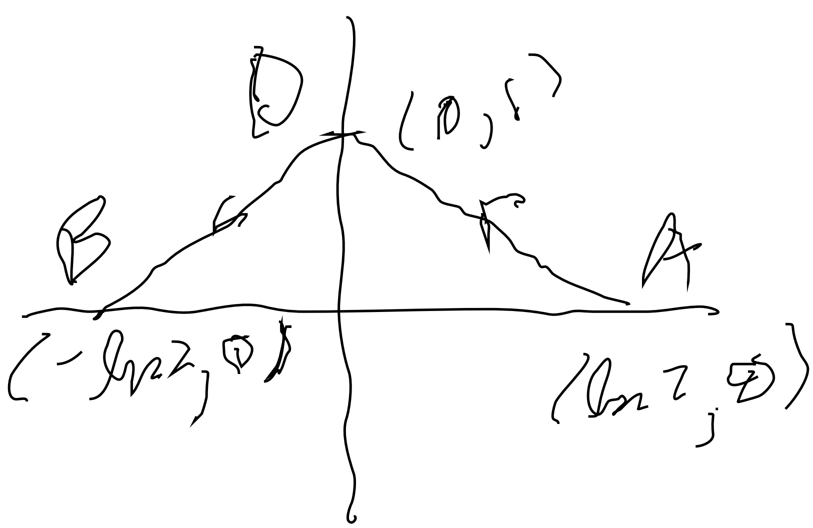
Evaluate the integral $\int_{\mathcal{C}} e^x \cos y \, dx \; - \; e^x \sin y \, dy$, where $\mathcal{C}$ is the broken line from $A = (\ln 2, 0)$ to $D = (0, 1)$ and then from D to $B = (-\ln 2, 0)$. *Hint:* Consider the additional line from B to A.

Green's Theorem only applies to a closed curve but (even without the hint) we note that with the additional line L from B to A we have enclosed an area. The given integrand taken around the area on $\mathcal{C} + L$ yields zero by Green's Theorem so the value on L must cancel out the value on $\mathcal{C}$. Thus,

$$\int_{\mathcal{C}} e^x \cos y \, dx \; - \; e^x \sin y \, dy = -\int_L e^x \cos y \, dx \; - \; e^x \sin y \, dy =$$

But $y = 0$ on L so

$$= -\int_{-\ln 2}^{\ln 2} e^x \, dx = -\left[e^x\right]_{-\ln 2}^{\ln 2} = -(2 - 1/2) = \underline{-3/2}$$



Problem 4 [6-10.5]

Evaluate $\int\int\int(\nabla \bullet \mathbf{F})\, d\tau$ over the region $x^2 + y^2 + z^2 \leq 25$, where

$$\mathbf{F} = (x^2 + y^2 + z^2)(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}).$$

Calculating the gradient: $\mathbf{n} = \frac{x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}}{\sqrt{x^2+y^2+z^2}} \Rightarrow \mathbf{F} \bullet \mathbf{n} = (x^2 + y^2 + z^2)^{3/2}$.

By the Divergence Theorem, where the surface is $x^2 + y^2 + z^2 = 25$ so $\mathbf{F} \bullet \mathbf{n} = (25)^{3/2}$:

$$\int\int\int(\nabla \bullet \mathbf{F})\, d\tau = (25)^{3/2} \times \text{Area} = 5^3 \cdot 4\pi \cdot 5^2 = \underline{4\pi \cdot 5^5}$$

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BTW: If the Divergence Theorem is not obvious immediately (a volume, the divergence of a field, the left hand side of the Divergence Theorem, ENGR 3300), you can do it directly. Work out carefully that

$$\nabla \bullet \mathbf{F} = 5(x^2 + y^2 + z^2) = 5r^2$$

in spherical coordinates (naturally, since it's a sphere). Integrate

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{5}(5r^2)(r^2 \sin\theta)\, dr\, d\theta\, d\phi = 2\pi \cdot 2 \cdot 5^5 = \underline{4\pi \cdot 5^5}$$

Problem 5 [6-11.12]

Evaluate $\oint \mathbf{V} \bullet d\mathbf{r}$ around the circle $(x-2)^2 + (y-3)^2 = \mathbf{4}$, $z = 0$, where

$$\mathbf{V} = (x^2 + yz^2)\,\mathbf{i} + (2x - y^3)\,\mathbf{j}.$$

For Stokes' Theorem, we calculate

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2 + yz^2) & (2x - y^3) & 0 \end{vmatrix} = 2yz\mathbf{j} + (2 - z^2)\mathbf{k} = 2\mathbf{k}$$

on the (x, y)-plane with $z = 0$.

Since $\mathbf{n} = \mathbf{k}$ (see Example 1 on page 328),

$$\int\int_\sigma (2\mathbf{k}) \bullet \mathbf{k}\, d\sigma = 2 \int\int_C d\sigma = 2 \cdot \pi\, \mathbf{2}^2 = \underline{\mathbf{8}\pi}$$

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Note: Of course Green's Theorem yields the same result.

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BTW: If you don't immediately think of Green's (or Stokes') Theorem, it can be done directly!

$$\oint \mathbf{V} \bullet\ d\mathbf{r} = \oint (x^2 + yz^2)\,dx + (2x - y^2)\,dy = \oint x^2\,dx + (2x - y^3)\,dy$$

on $z = 0$. But

$$= \oint x^2\,dx - \oint y^3\,dy + 2\oint x\,dy = 2\oint x\,dy$$

because the first two terms are exact differentials and so give zero around a circle. To parameterize the circle with center at $(2, 3)$, let $x = 2 + 2\cos\theta$ and $y = 3 + 2\sin\theta$; so $dx = -2\sin\theta\,d\theta$ and $dy = 2\cos\theta\,d\theta$.

$$= 2\int_0^{2\pi} (2 + 2\cos\theta)(2\cos\theta)\,d\theta = 8\int_0^{2\pi} \cos^2\theta = \underline{8\pi}$$

Problem 6 [6-10.7]

Evaluate $\int\int \mathbf{r} \bullet \mathbf{n}\, d\sigma$ over the entire surface of the cone with base $x^2+y^2 \leq 16,\ z=0$, and vertex at $(0,0,3)$, where $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$.

Divergence Theorem: $= \int\int\int_\tau \nabla \bullet \mathbf{r}\, d\tau = 3 \times \text{Volume} = 3 \cdot \frac{1}{3}\pi \cdot 4^2 \cdot 3 = \underline{48\pi}$

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Note: If you didn't remember the 1/3, this test was so short that you could integrate and find the volume of a cone using cylindrical coordinates with $z = -\frac{H}{R}r + H$ for a general cone of height H and radius R!

End of Test