6.2. A sample of five measurements of the diameter of a sphere were recorded by a scientist as 6.33, 6.37, 6.36, 6.32, and 6.37 cm. Determine unbiased and efficient estimates of (*a*) the true mean, (*b*) the true variance. Assume that the measured diameter is normally distributed.

(*a*) An unbiased and efficient estimate of the true mean (i.e., the population mean) is

$$\bar{x} = \frac{\sum x}{n} = \frac{6.33 + 6.37 + 6.36 + 6.32 + 6.37}{5} = 6.35\,\text{cm}$$

(*b*) An unbiased and efficient estimate of the true variance (i.e., the population variance) is

$$\hat{s}^2 = \frac{n}{n-1}s^2 = \frac{\sum(x-\bar{x})^2}{n-1}$$
$$= \frac{(6.33-6.35)^2 + (6.37-6.35)^2 + (6.36-6.35)^2 + (6.32-6.35)^2 + (6.37-6.35)^2}{5-1}$$
$$= 0.00055\,\text{cm}^2$$

Note that $\hat{s} = \sqrt{0.00055} = 0.023$ is an estimate of the true standard deviation, but this estimate is neither unbiased nor efficient.

6.3. Suppose that the heights of 100 male students at XYZ University represent a random sample of the heights of all 1546 male students at the university. Determine unbiased and efficient estimates of (*a*) the true mean, (*b*) the true variance.

(*a*) From Problem 5.33:

$$\text{Unbiased and efficient estimate of true mean height} = \bar{x} = 67.45 \text{ inch}$$

(*b*) From Problem 5.38:

$$\text{Unbiased and efficient estimate of true variance} = \hat{s}^2 = \frac{n}{n-1}s^2 = \frac{100}{99}(8.5275) = 8.6136$$

Therefore, $\hat{s} = \sqrt{8.6136} = 2.93$. Note that since n is large there is essentially no difference between s^2 and $\hat{s}^2$ or between s and $\hat{s}$.

6.4. Give an unbiased and inefficient estimate of the true (mean) diameter of the sphere of Problem 6.2.

The median is one example of an unbiased and inefficient estimate of the population mean. For the five measurements arranged in order of magnitude, the median is 6.36 cm.

CONFIDENCE INTERVAL ESTIMATES FOR MEANS (LARGE SAMPLES)

6.5. Find (*a*) 95%, (*b*) 99% confidence intervals for estimating the mean height of the XYZ University students in Problem 6.3.

(*a*) The 95% confidence limits are $\bar{X} \pm 1.96\sigma/\sqrt{n}$.

Using $\bar{x} = 67.45$ inches and $\hat{s} = 2.93$ inches as an estimate of σ (see Problem 6.3), the confidence limits are $67.45 \pm 1.96(2.93/\sqrt{100})$, or 67.45 ± 0.57, inches. Then the 95% confidence interval for the population mean μ is 66.88 to 68.02 inches, which can be denoted by $66.88 < \mu < 68.02$.

We can therefore say that the probability that the population mean height lies between 66.88 and 68.02 inches is about 95%, or 0.95. In symbols we write $P(66.88 < \mu < 68.02) = 0.95$. This is equivalent to saying that we are 95% *confident* that the population mean (or true mean) lies between 66.88 and 68.02 inches.

(*b*) The 99% confidence limits are $\bar{X} \pm 2.58\sigma/\sqrt{n}$. For the given sample,

$$\bar{x} \pm 2.58\frac{\hat{s}}{\sqrt{n}} = 67.45 \pm 2.58\frac{2.93}{\sqrt{100}} = 67.45 \pm 0.76 \text{ inches}$$

Therefore, the 99% confidence interval for the population mean μ is 66.69 to 68.21 inches, which can be denoted by $66.69 < \mu < 68.21$.

In obtaining the above confidence intervals, we assumed that the population was infinite or so large that we could consider conditions to be the same as sampling with replacement. For finite populations where sampling is without replacement, we should use $\frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}}$ in place of $\frac{\sigma}{\sqrt{n}}$. However, we can consider the factor $\sqrt{\frac{N-n}{N-1}} = \sqrt{\frac{1546-100}{1546-1}} = 0.967$ as essentially 1.0, so that it need not be used. If it is used, the above confidence limits become 67.45 ± 0.56 and 67.45 ± 0.73 inches, respectively.

6.6. Measurements of the diameters of a random sample of 200 ball bearings made by a certain machine during one week showed a mean of 0.824 inch and a standard deviation of 0.042 inch. Find (*a*) 95%, (*b*) 99% confidence limits for the mean diameter of all the ball bearings.

Since $n = 200$ is large, we can assume that $\bar{X}$ is very nearly normal.

(*a*) The 95% confidence limits are

$$\bar{X} \pm 1.96\frac{\sigma}{\sqrt{n}} = \bar{x} \pm 1.96\frac{\hat{s}}{\sqrt{n}} = 0.824 \pm 1.96\frac{0.042}{\sqrt{200}} = 0.824 \pm 0.0058 \text{ inch}$$

or 0.824 ± 0.006 inch.

(*b*) The 99% confidence limits are

$$\bar{X} \pm 2.58\frac{\sigma}{\sqrt{n}} = \bar{x} \pm 2.58\frac{\hat{s}}{\sqrt{n}} = 0.824 \pm 2.58\frac{0.042}{\sqrt{200}} = 0.824 \pm 0.0077 \text{ inch}$$

or 0.824 ± 0.008 inches.

Note that we have assumed the reported standard deviation to be the *modified* standard deviation $\hat{s}$. If the standard deviation had been s, we would have used $\hat{s} = \sqrt{n/(n-1)}\, s = \sqrt{200/199}\, s$ which can be taken as s for all practical purposes. In general, for $n \geqq 30$, we may take s and $\hat{s}$ as practically equal.

6.7. Find (*a*) 98%, (*b*) 90%, (*c*) 99.73% confidence limits for the mean diameter of the ball bearings in Problem 6.6.

(*a*) Let z_c be such that the area under the normal curve to the right of $z = z_c$ is 1%. Then by symmetry the area to the left of $z = -z_c$ is also 1%, so that the shaded area is 98% of the total area (Fig. 6-1).

Since the total area under the curve is one, the area from $z = 0$ is $z = z_c$ is 0.49; hence, $z_c = 2.33$. Therefore, 98% confidence limits are

$$\bar{x} \pm 2.33\frac{\sigma}{\sqrt{n}} = 0.824 \pm 2.33\frac{0.042}{\sqrt{200}} = 0.824 \pm 0.0069 \text{ inch}$$

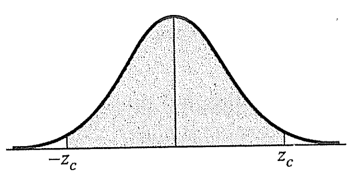

Fig. 6-1

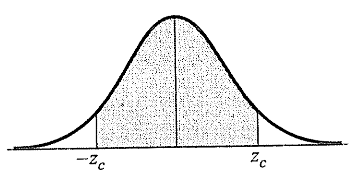

Fig. 6-2

(*b*) We require z_c such that the area from $z = 0$ to $z = z_c$ is 0.45; then $z_c = 1.645$ (Fig. 6-2).

Therefore, 90% confidence limits are

$$\bar{x} \pm 1.645\frac{\sigma}{\sqrt{n}} = 0.824 \pm 1.645\frac{0.042}{\sqrt{200}} = 0.824 \pm 0.0049 \text{ inch}$$

(c) The 99.73% confidence limits are

$$\bar{x} \pm 3\frac{\sigma}{\sqrt{n}} = 0.824 \pm 3\frac{0.042}{\sqrt{200}} = 0.824 \pm 0.0089 \text{ inch}$$

6.8. In measuring reaction time, a psychologist estimates that the standard deviation is 0.05 second. How large a sample of measurements must he take in order to be (a) 95%, (b) 99% confident that the error in his estimate of mean reaction time will not exceed 0.01 second?

(a) The 95% confidence limits are $\bar{X} \pm 1.96\sigma/\sqrt{n}$, the error of the estimate being $1.96\sigma/\sqrt{n}$. Taking $\sigma = s = 0.05$ second, we see that this error will be equal to 0.01 second if $(1.96)(0.05)/\sqrt{n} = 0.01$, i.e., $\sqrt{n} = (1.96)(0.05)/0.01 = 9.8$, or $n = 96.04$. Therefore, we can be 95% confident that the error in the estimate will be less than 0.01 if n is 97 or larger.

(b) The 99% confidence limits are $\bar{X} \pm 2.58\sigma/\sqrt{n}$. Then $(2.58)(0.05)/\sqrt{n} = 0.01$, or $n = 166.4$. Therefore, we can be 99% confident that the error in the estimate will be less than 0.01 only if n if 167 or larger.

Note that the above solution assumes a nearly normal distribution for $\bar{X}$, which is justified since the n obtained is large.

6.9. A random sample of 50 mathematics grades out of a total of 200 showed a mean of 75 and a standard deviation of 10. (a) What are the 95% confidence limits for the mean of the 200 grades? (b) With what degree of confidence could we say that the mean of all 200 grades is 75 ± 1?

(a) Since the population size is not very large compared with the sample size, we must adjust for *sampling without replacement*. Then the 95% confidence limits are

$$\bar{X} \pm 1.96\sigma_{\bar{X}} = \bar{X} \pm 1.96\frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}} = 75 \pm 1.96\frac{10}{\sqrt{50}}\sqrt{\frac{200-50}{200-1}} = 75 \pm 2.4$$

(b) The confidence limits can be represented by

$$\bar{X} \pm z_c\sigma_{\bar{X}} = \bar{X} \pm z_c\frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}} = 75 \pm z_c\frac{(10)}{\sqrt{50}}\sqrt{\frac{200-50}{200-1}} = 75 \pm 1.23z_c$$

Since this must equal 75 ± 1, we have $1.23z_c = 1$ or $z_c = 0.81$. The area under the normal curve from $z = 0$ to $z = 0.81$ is 0.2910; hence, the required degree of confidence is $2(0.2919) = 0.582$ or 58.2%.

CONFIDENCE INTERVAL ESTIMATES FOR MEANS (SMALL SAMPLES)

6.10. The 95% critical values (two-tailed) for the normal distribution are given by ± 1.96. What are the corresponding values for the t distribution if the number of degrees of freedom is (a) $\nu = 9$, (b) $\nu = 20$, (c) $\nu = 30$, (d) $\nu = 60$?

For 95% critical values (two-tailed) the total shaded area in Fig. 6-3 must be 0.05. Therefore, the shaded area in the right tail is 0.025, and the corresponding critical value is $t_{0.975}$. Then the required critical values are $\pm t_{0.975}$. For the given values of ν these are (a) ± 2.26, (b) ± 2.09, (c) ± 2.04, (d) ± 2.00.

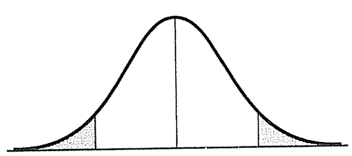

Fig. 6-3

6.11. A sample of 10 measurements of the diameter of a sphere gave a mean $\bar{x} = 4.38$ inches and a standard deviation $s = 0.06$ inch. Find (*a*) 95%, (*b*) 99% confidence limits for the actual diameter.

(*a*) The 95% confidence limits are given by $\bar{X} \pm t_{0.975}(S/\sqrt{n-1})$.

Since $\nu = n - 1 = 10 - 1 = 9$, we find $t_{0.975} = 2.26$ [see also Problem 6.10(*a*)]. Then using $\bar{x} = 4.38$ and $s = 0.06$, the required 95% confidence limits are

$$4.38 \pm 2.26 \frac{0.06}{\sqrt{10-1}} = 4.38 \pm 0.0452 \text{ inch}$$

Therefore, we can be 95% confident that the true mean lies between $4.38 - 0.045 = 4.335$ inches and $4.38 + 0.045 = 4.425$ inches.

(*b*) For $\nu = 9$, $t_{0.995} = 3.25$. Then the 99% confidence limits are

$$\bar{X} \pm t_{0.995}(S/\sqrt{n-1}) = 4.38 \pm 3.25(0.06/\sqrt{10-1}) = 4.38 \pm 0.0650 \text{ inch}$$

and the 99% confidence interval is 4.315 to 4.445 inches.

6.12. (*a*) Work Problem 6.11 assuming that the methods of large sampling theory are valid.
(*b*) Compare the results of the two methods.

(*a*) Using large sampling theory, the 95% confidence limits are

$$\bar{X} \pm 1.96 \frac{\sigma}{\sqrt{n}} = 4.38 \pm 1.96 \frac{0.06}{\sqrt{10}} = 4.38 \pm 0.037 \text{ inch}$$

where we have used the sample standard deviation 0.06 as estimate of σ. Similarly, the 99% confidence limits are $4.38 \pm (2.58)(0.06)/\sqrt{10} = 4.38 \pm 0.049$ inch.

(*b*) In each case the confidence intervals using the small or exact sampling methods are wider than those obtained by using large sampling methods. This is to be expected since less precision is available with small samples than with large samples.

CONFIDENCE INTERVAL ESTIMATES FOR PROPORTIONS

6.13. A sample poll of 100 voters chosen at random from all voters in a given district indicated that 55% of them were in favor of a particular candidate. Find (*a*) 95%, (*b*) 99%, (*c*) 99.73% confidence limits for the proportion of all the voters in favor of this candidate.

(*a*) The 95% confidence limits for the population p are

$$P \pm 1.96\sigma_P = P \pm 1.96\sqrt{\frac{p(1-p)}{n}} = 0.55 \pm 1.96\sqrt{\frac{(0.55)(0.45)}{100}} = 0.55 \pm 0.10$$

where we have used the sample proportion 0.55 to estimate p.

(*b*) The 99% confidence limits for p are $0.55 \pm 2.58\sqrt{(0.55)(0.45)/100} = 0.55 \pm 0.13$.

(*c*) The 99.73% confidence limits for p are $0.55 \pm 3\sqrt{(0.55)(0.45)/100} = 0.55 \pm 0.15$.

For a more exact method of working this problem, see Problem 6.27.

6.14. How large a sample of voters should we take in Problem 6.13 in order to be 95% confident that the candidate will be elected?

The candidate is elected if $p > 0.50$, and to be 95% confident of his being elected, we require that Prob.$(p > 0.50) = 0.95$. Since $(P-p)/\sqrt{p(1-p)/n}$ is asymptotically normal,

$$\text{Prob.}\left(\frac{P-p}{\sqrt{p(1-p)/n}} < \beta\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\beta} e^{-u^2/2}\, du$$

or

$$\text{Prob.}(p > P - \beta\sqrt{p(1-p)/n}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\beta} e^{-u^2/2}\, du$$