would suggest a sampling rate $F_s \geq 2F_H$; however, as we show in this section, there are sampling techniques that allow sampling rates consistent with the bandwidth B, rather than the highest frequency, F_H, of the signal spectrum. Sampling of bandpass signals is of great interest in the areas of digital communications, radar, and sonar systems.

6.4.1 Uniform or First-Order Sampling

Uniform or first-order sampling is the typical periodic sampling introduced in Section 6.1. Sampling the bandpass signal in Figure 6.4.1(a) at a rate $F_s = 1/T$ produces a sequence $x(n) = x_a(nT)$ with spectrum

$$X(F) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_a(F - kF_s) \tag{6.4.1}$$

The positioning of the shifted replicas $X(F - kF_s)$ is controlled by a single parameter, the sampling frequency F_s. Since bandpass signals have two spectral bands, in general, it is more complicated to control their positioning, in order to avoid aliasing, with the single parameter F_s.

Integer Band Positioning. We initially restrict the higher frequency of the band to be an integer multiple of the bandwidth, that is, $F_H = mB$ (*integer band positioning*). The number $m = F_H/B$, which is in general fractional, is known as the *band position*. Figures 6.4.1(a) and 6.4.1(d) show two bandpass signals with even ($m = 4$) and odd ($m = 3$) band positioning. It can be easily seen from Figure 6.4.1(b) that, for integer-positioned bandpass signals, choosing $F_s = 2B$ results in a sequence with a spectrum without aliasing. From Figure 6.4.1(c), we see that the original bandpass signal can be reconstructed using the reconstruction formula

$$x_a(t) = \sum_{n=-\infty}^{\infty} x_a(nT) g_a(t - nT) \tag{6.4.2}$$

where

$$g_a(t) = \frac{\sin \pi Bt}{\pi Bt} \cos 2\pi F_c t \tag{6.4.3}$$

is the inverse Fourier transform of the bandpass frequency gating function shown in Figure 6.4.1(c). We note that $g_a(t)$ is equal to the ideal interpolation function for lowpass signals [see (6.1.21)], modulated by a carrier with frequency F_c.

It is worth noticing that, by properly choosing the center frequency F_c of $G_a(F)$, we can reconstruct a continuous-time bandpass signal with spectral bands centered at $F_c = \pm(kB + B/2)$, $k = 0, 1, \ldots$. For $k = 0$ we obtain the equivalent baseband signal, a process known as *down-conversion*. A simple inspection of Figure 6.4.1 demonstrates that the baseband spectrum for $m = 3$ has the same spectral structure as the original spectrum; however, the baseband spectrum for $m = 4$ has been "inverted." In general, when the band position is an *even* integer the baseband spectral images are inverted versions of the original ones. Distinguishing between these two cases is important in communications applications.

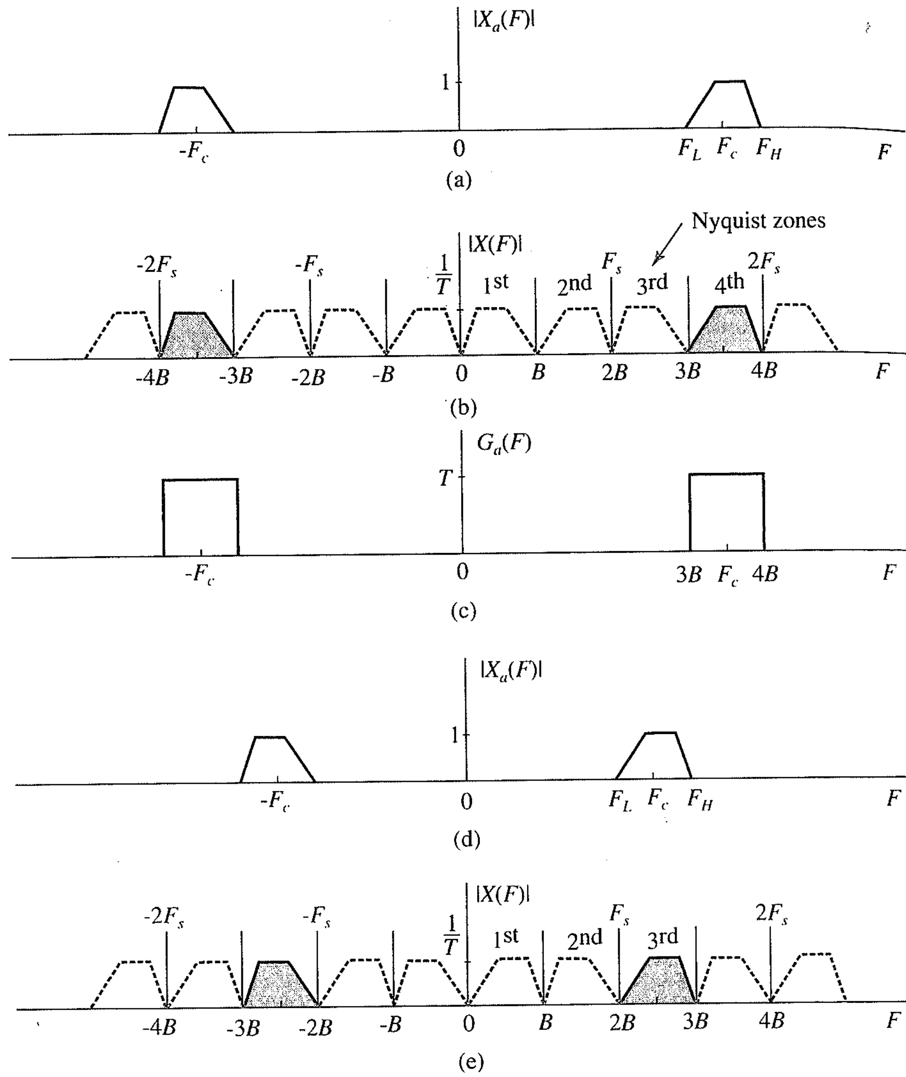

Figure 6.4.1 Illustration of bandpass signal sampling for integer band positioning.

Arbitrary Band Positioning. Consider now a bandpass signal with arbitrarily p sitioned spectral bands, as shown in Figure 6.4.2. To avoid aliasing, the sampli frequency should be such that the $(k-1)$th and kth shifted replicas of the "negativ spectral band do not overlap with the "positive" spectral band. From Figure 6.4.2(we see that this is possible if there is an integer k and a sampling frequency F_s th satisfy the following conditions:

$$2F_H \leq kF_s \qquad (6.4.$$

$$(k-1)F_s \leq 2F_L \qquad (6.4.$$

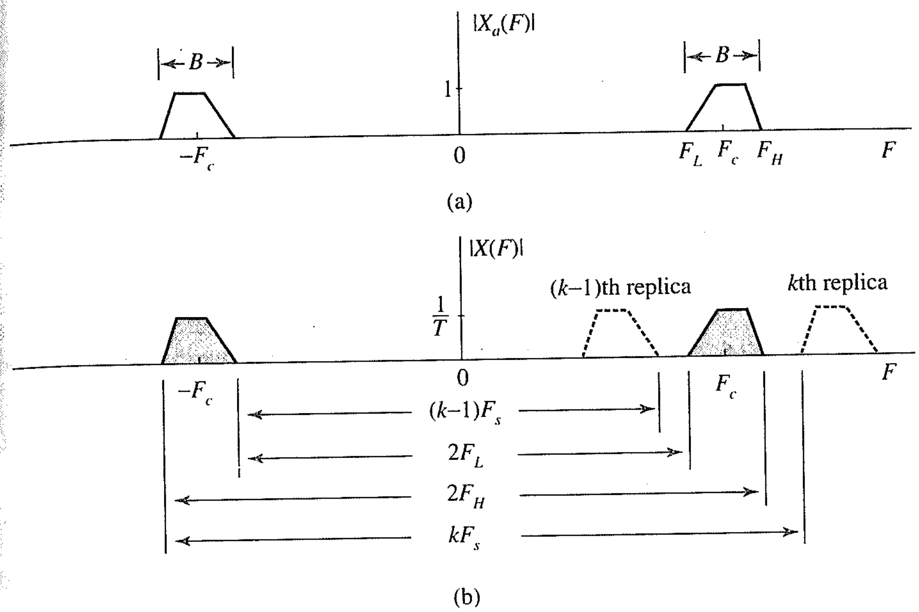

Figure 6.4.2 Illustration of bandpass signal sampling for arbitrary band positioning.

which is a system of two inequalities with two unknowns, k and F_s. From (6.4.4) and (6.4.5) we can easily see that F_s should be in the range

$$\frac{2F_H}{k} \leq F_s \leq \frac{2F_L}{k-1} \tag{6.4.6}$$

To determine the integer k we rewrite (6.4.4) and (6.4.5) as follows:

$$\frac{1}{F_s} \leq \frac{k}{2F_H} \tag{6.4.7}$$

$$(k-1)F_s \leq 2F_H - 2B \tag{6.4.8}$$

By multiplying (6.4.7) and (6.4.8) by sides and solving the resulting inequality for k we obtain

$$k_{\max} \leq \frac{F_H}{B} \tag{6.4.9}$$

The maximum value of integer k is the number of bands that we can fit in the range from 0 to F_H, that is

$$k_{\max} = \left\lfloor \frac{F_H}{B} \right\rfloor \tag{6.4.10}$$

where $\lfloor b \rfloor$ denotes the integer part of b. The minimum sampling rate required to avoid aliasing is $F_{s\max} = 2F_H/k_{\max}$. Therefore, the range of acceptable uniform sampling rates is determined by

$$\frac{2F_H}{k} \leq F_s \leq \frac{2F_L}{k-1} \tag{6.4.11}$$

where k is an integer number given by

$$1 \le k \le \left\lfloor \frac{F_H}{B} \right\rfloor \qquad (6.4.12)$$

As long as there is no aliasing, reconstruction is done using (6.4.2) and (6.4.3), which are valid for both integer and arbitrary band positioning.

Choosing a Sampling Frequency. To appreciate the implications of conditions (6.4.11) and (6.4.12), we depict them graphically in Figure 6.4.3, as suggested by Vaughan et al. (1991). The plot shows the sampling frequency, normalized by B, as a function of the band position, F_H/B. This is facilitated by rewriting (6.4.11) as follows:

$$\frac{2}{k}\frac{F_H}{B} \le \frac{F_s}{B} \le \frac{2}{k-1}\left(\frac{F_H}{B} - 1\right) \qquad (6.4.13)$$

The shaded areas represent sampling rates that result in aliasing. The allowed rang[e] of sampling frequencies is inside the white wedges. For $k = 1$, we obtain $2F_H$ [≤] $F_s \le \infty$, which is the sampling theorem for lowpass signals. Each wedge in the plo[t] corresponds to a different value of k.

To determine the allowed sampling frequencies, for a given F_H and B, we dra[w] a vertical line at the point determined by F_H/B. The segments of the line within th[e] allowed areas represent permissible sampling rates. We note that the theoreticall[y] minimum sampling frequency $F_s = 2B$, corresponding to integer band positionin[g]

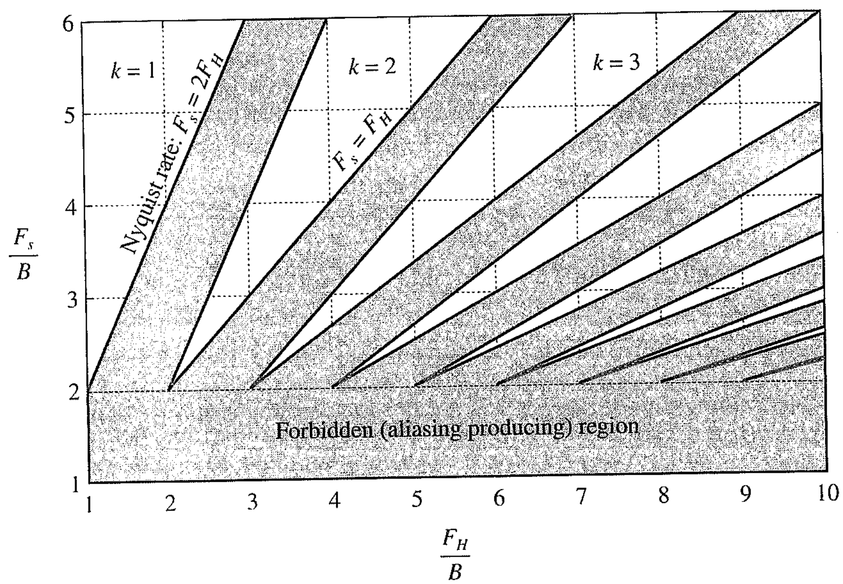

Figure 6.4.3 Allowed (white) and forbidden (shaded) sampling frequency regions for bandpass signals. The minimum sampling frequency $F_s = 2B$, which corresponds to the corners of the alias-free wedges, is possible for integer-positioned bands only.

(6.4.12)

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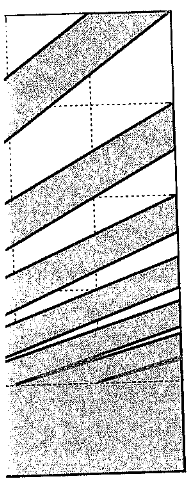

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occurs at the tips of the wedges. Therefore, any small variation of the sampling rate or the carrier frequency of the signal will move the sampling frequency into the forbidden area. A practical solution is to sample at a higher sampling rate, which is equivalent to augmenting the signal band with a guard band $\Delta B = \Delta B_L + \Delta B_H$. The augmented band locations and bandwidth are given by

$$F_L' = F_L - \Delta B_L \tag{6.4.14}$$

$$F_H' = F_H + \Delta B_H \tag{6.4.15}$$

$$B' = B + \Delta B \tag{6.4.16}$$

The lower-order wedge and the corresponding range of allowed sampling are given by

$$\frac{2F_H'}{k'} \leq F_s \leq \frac{2F_L'}{k'-1} \quad \text{where} \quad k' = \left\lfloor \frac{F_H'}{B'} \right\rfloor \tag{6.4.17}$$

The k'th wedge with the guard bands and the sampling frequency tolerances are illustrated in Figure 6.4.4. The allowable range of sampling rates is divided into values above and below the practical operating points as

$$\Delta F_s = \frac{2F_L'}{k'-1} - \frac{2F_H'}{k'} = \Delta F_{sL} + \Delta F_{sH} \tag{6.4.18}$$

From the shaded orthogonal triangles in Figure 6.4.4, we obtain

$$\Delta B_L = \frac{k'-1}{2} \Delta F_{sH} \tag{6.4.19}$$

$$\Delta B_H = \frac{k'}{2} \Delta F_{sL} \tag{6.4.20}$$

which shows that symmetric guard bands lead to asymmetric sampling rate tolerance.

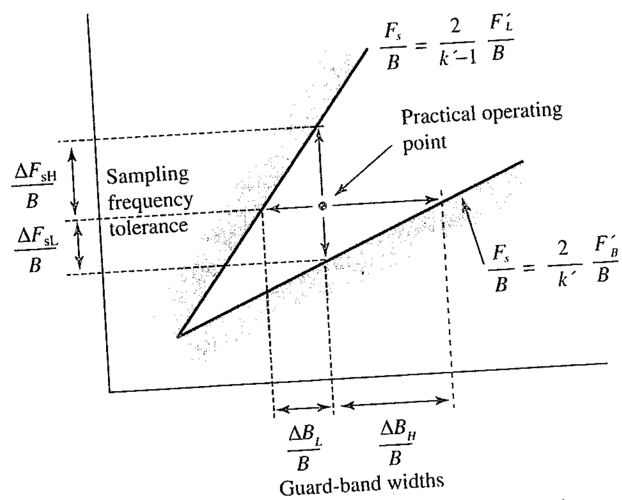

Figure 6.4.4
Illustration of the relationship between the size of guard bands and allowed sampling frequency deviations from its nominal value for the kth wedge.

If we choose the practical operating point at the vertical midpoint of tl
the sampling rate is

$$F_s = \frac{1}{2}\left(\frac{2F'_H}{k'} + \frac{2F'_L}{k'-1}\right)$$

Since, by construction, $\Delta F_{sL} = \Delta F_{sH} = \Delta F_s/2$, the guard bands become

$$\Delta B_L = \frac{k'-1}{4}\Delta F_s$$

$$\Delta B_H = \frac{k'}{4}\Delta F_s$$

We next provide an example that illustrates the use of this approach.

EXAMPLE 6.4.1

Suppose we are given a bandpass signal with $B = 25$ kHz and $F_L = 10{,}702.5$ kH:
(6.4.10) the maximum wedge index is

$$k_{\max} = \lfloor F_H/B \rfloor = 429$$

This yields the theoretically minimum sampling frequency

$$F_s = \frac{2F_H}{k_{\max}} = 50.0117 \text{ kHz}$$

To avoid potential aliasing due to hardware imperfections, we wish to use two guard ba:
$\Delta B_L = 2.5$ kHz and $\Delta B_H = 2.5$ kHz on each side of the signal band. The effective band
of the signal becomes $B' = B + \Delta B_L + \Delta B_H = 30$ kHz. In addition, $F'_L = F_L - \Delta B_L = 1$
kHz and $F'_H = F_H + \Delta B_H = 10{,}730$ kHz. From (6.4.17), the maximum wedge index is

$$k'_{\max} = \lfloor F'_H/B' \rfloor = 357$$

Substitution of $k_{\max}$ into the inequality in (6.4.17) provides the range of acceptable sam|
frequencies

$$60.1120 \text{ kHz} \le F_s \le 60.1124 \text{ kHz}$$

A detailed analysis on how to choose in practice the sampling rate for bandp
signals is provided by Vaughan et al. (1991) and Qi et al. (1996).

6.4.2 Interleaved or Nonuniform Second-Order Sampling

Suppose that we sample a continuous-time signal $x_a(t)$ with sampling rate $F_i = 1$,
at time instants $t = nT_i + \Delta_i$, where Δ_i is a fixed time offset. Using the sequence

$$x_i(nT_i) = x_a(nT_i + \Delta_i), \quad -\infty < n < \infty \qquad (6.4.2$$