# DSP

## Additional Material on Bandpass Sampling

Highly Readable Material

BANDPASS SAMPLING

(A highly readable & practical version - from Mitra's DSP book)

## 5.3 Sampling of Bandpass Signals

The conditions developed in Section 5.2.1 for the unique representation of a continuous-time signal by the discrete-time signal obtained by uniform sampling assumed that the spectrum of the continuous-time signal is bandlimited in the frequency range from dc to some frequency $\Omega_m$. Such continuous-time signals are commonly referred to as *lowpass* signals. There are applications where the continuous-time signal is bandlimited to a higher range $\Omega_L \le |\Omega| \le \Omega_H$, where $\Omega_L > 0$. Such a signal is usually referred to as a *bandpass* signal and is often obtained by modulating a lowpass signal. We can of course sample such a bandpass continuous-time signal with a sampling rate greater than twice the highest frequency, i.e., by ensuring

$$\Omega_T \ge 2\Omega_H,$$

to prevent aliasing. However, in this case, due to the bandpass spectrum of the continuous-time signal, the spectrum of the discrete-time signal obtained by sampling will have spectral gaps with no signal components present in these gaps. Moreover, if $\Omega_H$ is very large, the sampling rate also has to be very large which may not be practical in some situations.

We outline next a more practical and efficient approach [Por97]. Let $\Delta\Omega = \Omega_H - \Omega_L$ define the *bandwidth* of the bandpass signal. Assume first that the highest frequency $\Omega_H$ contained in the signal is an integer multiple of the bandwidth, i.e.,

$$\Omega_H = M(\Delta\Omega).$$

We choose the sampling frequency $\Omega_T$ to satisfy the condition

$$\Omega_T = 2(\Delta\Omega) = \frac{2\Omega_H}{M}, \tag{5.23}$$

which is smaller than $2\Omega_H$, the Nyquist rate. Substituting Eq. (5.23) in Eq. (5.9) we arrive at the expression for the Fourier transform $G_p(j\Omega)$ of the impulse-sampled signal $g_p(t)$:

$$G_p(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a\left(j(\Omega - 2k(\Delta\Omega))\right). \tag{5.24}$$

As before, $G_p(j\Omega)$ consists of a sum of the original Fourier transform $G_a(j\Omega)$ and replicas of $G_a(j\Omega)$ shifted by integer multiples of twice the bandwidth $\Delta\Omega$, and then scaled by $1/T$. The amount of the shift for each value of $k$ ensures that there will be no overlap between all shifted replicas, and hence no aliasing. Figure 5.11 shows the spectrum of the original continuous-time signal $g_a(t)$ and that of the sampled version $g_p(t)$, sampled at the rate given by Eq. (5.23) for $M = 4$. As can be seen from this figure, $g_a(t)$ can be recovered from $g_p(t)$ by passing the latter through an ideal bandpass filter with a passband given by $\Omega_L \le |\Omega| \le \Omega_H$ and a gain of $T$.

BANDPASS SAMPLING (Continued)
— from Mitra's DSP book

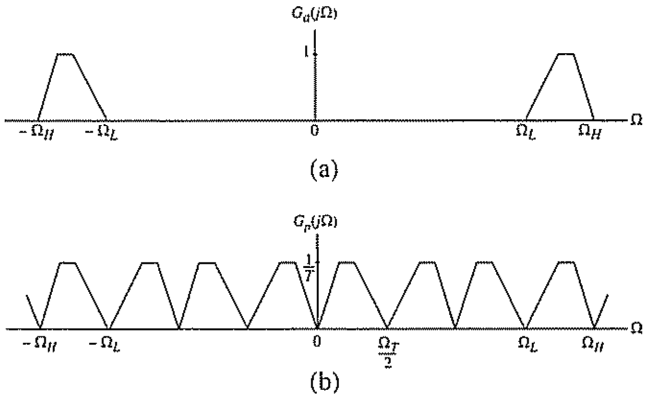


**Figure 5.11**: Illustration of the effect in the frequency-domain of sampling below the Nyquist rate a bandpass signal with highest frequency that is an integer multiple of its bandwidth: (a) spectrum of original bandpass signal, and (b) spectrum of sampled bandpass signal

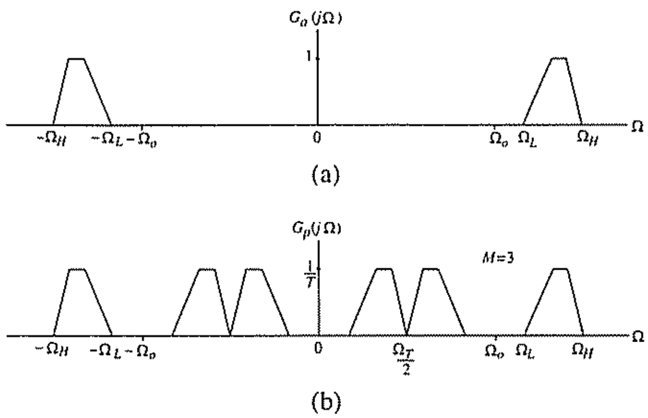


**Figure 5.12**: Illustration of the effect in the frequency-domain of sampling below the Nyquist rate a bandpass signal with highest frequency that is not an integer multiple of its bandwidth: (a) spectrum of original bandpass signal, and (b) spectrum of sampled bandpass signal

Note that any of the replicas in the lower frequency bands can be retained by passing $g_p(t)$ through bandpass filters with passbands $\Omega_L - k(\Delta\Omega) \leq |\Omega| \leq \Omega_H - k(\Delta\Omega)$, $1 \leq k \leq M-1$ providing a translation of the original bandpass signal to lower frequency ranges. If the bandpass signal has been obtained by modulating a lowpass signal, then the latter can be recovered by passing $g_p(t)$ through a lowpass filter with passband $0 \leq |\Omega| \leq \Omega_H$ which retains the replica in the baseband. This approach is often employed in digital radio receivers.

If $\Omega_H$ is not an integer multiple of the bandwidth $\Omega_H - \Omega_L$, we can artificially extend the bandwidth either to the right or to the left so that the highest frequency contained in the bandpass signal is an integer multiple of the extended bandwidth. For example, if we extend the bandwidth to the left by assuming the lowest frequency contained in the bandpass signal to be $\Omega_o$, then $\Omega_o$ is chosen such that the extended bandwidth $\Omega_H - \Omega_o$ is an integer multiple of $\Omega_H$. In both cases the spectrum of the sampled signal obtained by sampling $g_a(t)$ will have small spectral gaps between the replicas. This is illustrated in Figure 5.12 when the bandwidth is extended to the left and $M$ is chosen as 3.

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Bandpass Sampling (from Taub & Schilling's book, Principles of Communication Systems)

## Bandpass Signals

For a signal $m(t)$ whose highest-frequency spectral component is $f_M$, the sampling frequency $f_s$ must be no less than $f_s = 2f_M$ only if the lowest-frequency spectral component of $m(t)$ is $f_L = 0$. In the more general case, where $f_L \neq 0$, it may be that the sampling frequency need be no larger than $f_s = 2(f_M - f_L)$. For example, if the spectral range of a signal extends from 10.0 to 10.1 MHz, the signal may be recovered from samples taken at a frequency $f_s = 2(10.1 - 10.0) = 0.2$ MHz.

To establish the sampling theorem for such bandpass signals, let us select a sampling frequency $f_s = 2(f_M - f_L)$ and let us initially assume that it happens that the frequency $f_L$ turns out to be an integral multiple of $f_s$, that is, $f_L = nf_s$ with $n$ an integer. Such a situation is represented in Fig. 5.1-4. In part $a$ is shown the two-sided spectral pattern of a signal $m(t)$ with Fourier transform $M(j\omega)$. Here it has been arranged that $n = 2$; that is, $f_L$ coincides with the second harmonic of

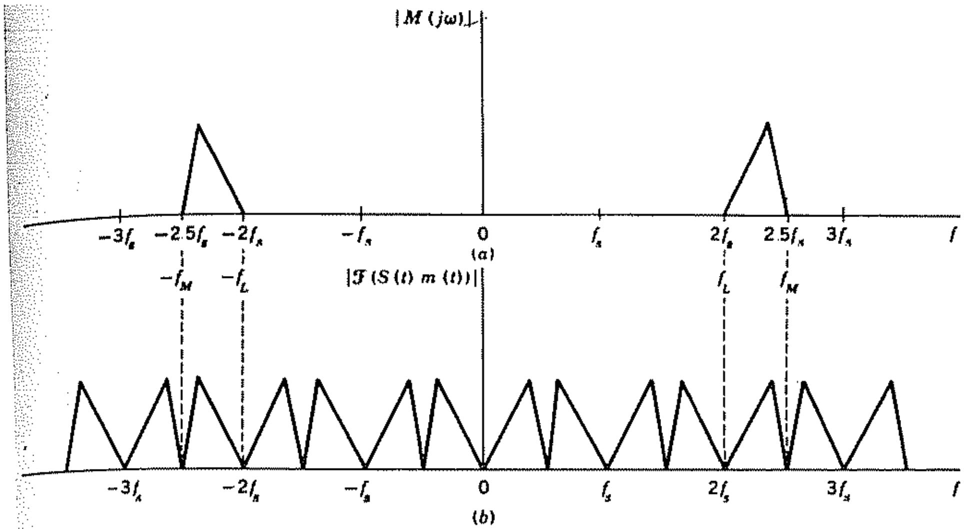


Figure 5.1-4 (a) The spectrum of a bandpass signal. (b) The spectrum of the sampled bandpass signal.

Bandpass Sampling - Continued
(from Taub & Schilling)

the sampling frequency, while the sampling frequency is exactly $f_s = 2(f_M - f_L)$. In part *b* is shown the spectral pattern of the sampled signal $S(t)m(t)$. The product of $m(t)$ and the dc term of $S(t)$ [Eq. (5.1-1)] duplicates in part *b* the form of the spectral pattern in part *a* and leaves it in the same frequency range from $f_L$ to $f_M$. The product of $m(t)$ and the spectral component in $S(t)$ of frequency $f_s(= 1/T_s)$ gives rise in part *b* to a spectral pattern derived from part *a* by shifting the pattern in part *a* to the right and also to the left by amount $f_s$. Similarly, the higher harmonics of $f_s$ in $S(t)$ give rise to corresponding shifts, right and left, of the spectral pattern in part *a*. We now note that if the sampled signal $S(t)m(t)$ is passed through a bandpass filter with arbitrarily sharp cutoffs and with passband from $f_L$ to $f_M$, the signal $m(t)$ will be recovered exactly.

In Fig. 5.1-4 the spectrum of $m(t)$ extends over the first half of the frequency interval between harmonics of the sampling frequency, that is, from $2.0f_s$ to $2.5f_s$. As a result, there is no spectrum overlap, and signal recovery is possible. It may also be seen from the figure that if the spectral range of $m(t)$ extended over the second half of the interval from $2.5f_s$ to $3.0f_s$, there would similarly be no overlap. Suppose, however, that the spectrum of $m(t)$ were confined neither to the first half nor to the second half of the interval between sampling-frequency harmonics. In such a case, there would be overlap between the spectrum patterns, and signal recovery would not be possible. Hence the minimum sampling frequency allowable is $f_s = 2(f_M - f_L)$ provided that either $f_M$ or $f_L$ is a harmonic of $f_s$.

If neither $f_M$ nor $f_L$ is a harmonic of $f_s$, a more general analysis is required. In Fig. 5.1-5*a* we have reproduced the spectral pattern of Fig. 5.1-4. The positive-frequency part and the negative-frequency part of the spectrum are called PS and NS respectively. Let us, for simplicity, consider separately PS and NS and the

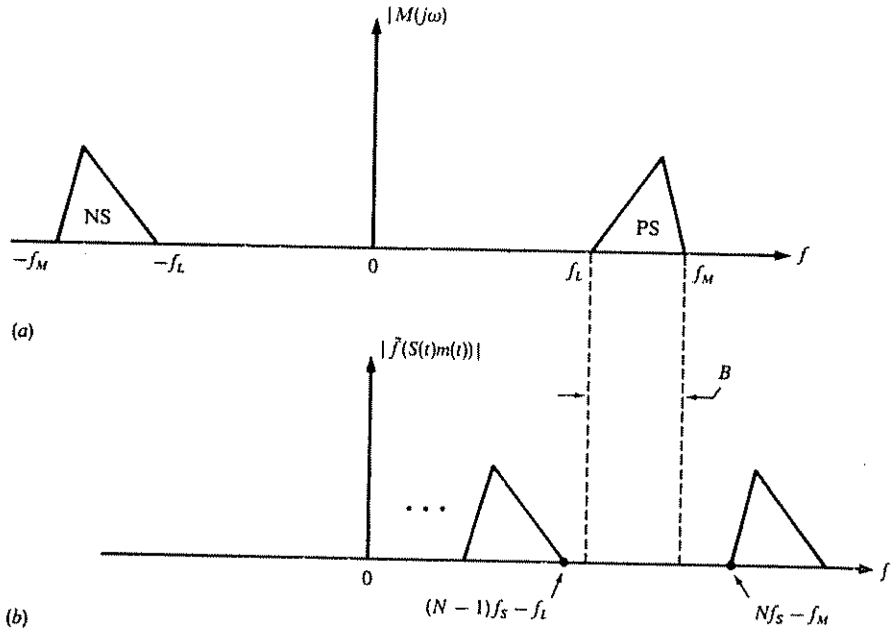


**Figure 5.1-5** (*a*) Spectrum of the bandpass signal (*b*) Spectrum of NS shifted by the $(N-1)$st and the $N$th harmonic of the sampling waveform

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Bandpass Sampling — Continued
(Taub & Schilling)

manner in which they are shifted due to the sampling and let us consider initially what constraints must be imposed so that we cause no overlap over, say, PS.

The product of $m(t)$ and the dc component of the sampling waveform leaves PS unmoved and it is this part of the spectrum which we propose to selectively draw out to reproduce the original signal. If we select the minimum value of $f_s$ to be $f_s = 2(f_H - f_L) = 2B$ then the shifted PS patterns will not overlap PS. The NS will also generate a series of shifted patterns to the left and to the right. The left shiftings cannot cause an overlap of PS. However, the right shiftings of NS might cause an overlap and these right shiftings of NS are the only possible source of such overlap over PS.

Shown in Fig. 5.1-5*b* are the right shifted patterns of NS due to the ($N$-1)st and $N$th harmonics of the sampling waveform. It is clear that to avoid overlap it is necessary that

$$(N-1)f_s - f_L \le f_L \tag{5.1-3}$$

and

$$Nf_s - f_M \ge f_M \tag{5.1-4}$$

so that, with $B \equiv f_M - f_L$ we have

$$(N-1)f_s \le 2(f_M - B) \tag{5.1-5}$$

and

$$Nf_s \ge 2f_M \tag{5.1-6}$$

If we let $k \equiv f_M/B$, Eqs. (5.1-5) and (5.1-6) become

$$f_s \le 2B\left(\frac{k-1}{N-1}\right) \tag{5.1-7}$$

and

$$f_s \ge 2B\left(\frac{k}{N}\right) \tag{5.1-8}$$

in which $k \ge N$ since $f_s \ge 2B$. Equations (5.1-7) and (5.1-8) establish the constraint which must be observed to avoid an overlap on PS. It is clear from the symmetry of the initial spectrum and the symmetry of the shiftings required that this same constraint assures that there will be no overlap on NS.

Equations (5.1-7) and (5.1-8) have been plotted in Fig. 5.1-6 for several values of $N$. The shaded regions are the regions where the constraints are satisfied, while in the unshaded regions the constraints are not satisfied and overlap will occur. As an example of the use of these plots consider a case in which a baseband signal has a spectrum which extends from $f_L = 2.5$ kHz to $f_M = 3.5$ kHz. Here $B = 1$ kHz and $k = f_M/B = 3.5$. On the plot of Fig. 5.1-6 we have accordingly erected a dashed vertical line at $k = 3.5$. We observe that for this value of $k$, the

selection of a sampling frequency $f_s = 2B = 2$ kHz brings us to a point in an overlap region. As $f_s$ is increased there is a small range of $f_s$, corresponding to $N = 3$, where there is no overlap. Further increase in $f_s$ again takes us to an overlap region, while still further increase in $f_s$ provides a nonoverlap range, corresponding to $N = 2$ (from $f_s = 3.5B$ to $f_s = 5B$). Increasing $f_s$ further we again enter an overlap region while at $f_s = 7B$ we enter the nonoverlap region for $N = 1$. When $f_s \ge 7B$ we do not again enter an overlap region. (This is the region where $f_s \ge 2f_M$; that is, we assume we have a lowpass rather than a bandpass signal.)

Bandpass Sampling - Continued
(Taub & Schilling)

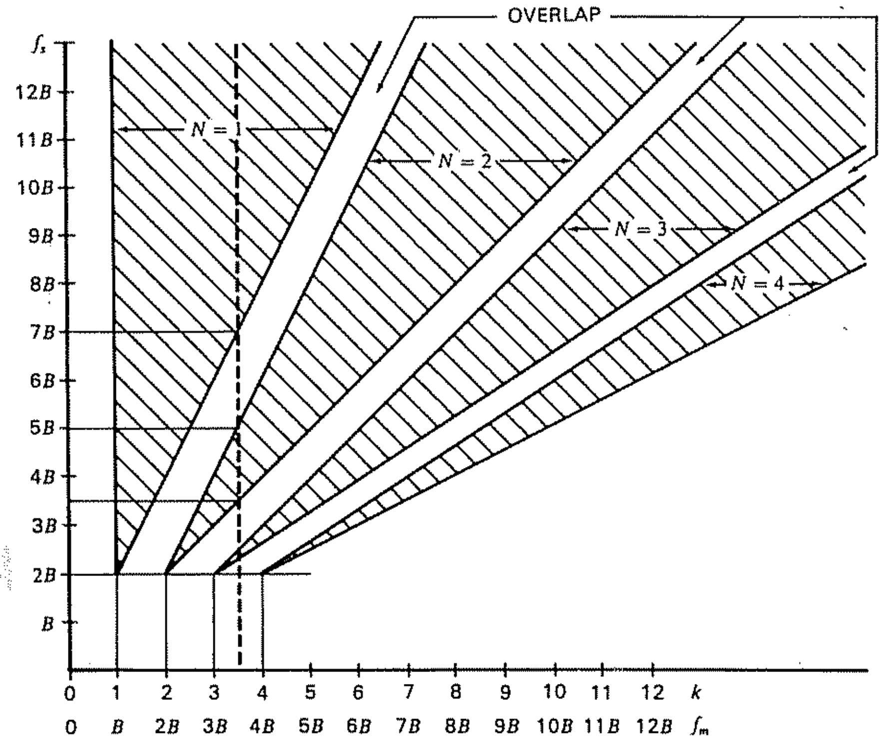


**Figure 5.1-6** Showing the regions (shaded) in which both Eqs (5.1-7) and (5.1-8) are satisfied