

DSP

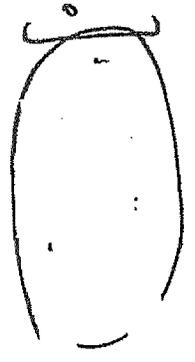
LCCDE:
Linear Constant Coefficient
Difference Equations

- SOLUTION
- Impulse Response
- Examples.

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DISCRETE TIME SYSTEMS DIFFERENCE EQUATIONS

SOLUTION TO THE DIFFERENCE EQN.



solve for $y(n)$ recursively, or algorithmically using ①

Solve in a manner similar to ODE:
 generate the homogeneous solution (natural response),
 and the particular solution (forced response),
 and combine the two.

SOLUTION STEPS:

1. Form of the homogeneous solution, $y_h(n)$ from the entire diff. equation
2. The particular solution $y_p(n) = y_h(n) + y_p(n)$
3. Total response: $y(n) = y_h(n) + y_p(n)$ using the constants in $y(n)$
4. Evaluate the constants in $y(n)$ using the initial conditions.

DISCRETE TIME SYSTEMS
DIFFERENCE EQUATIONS

DTSS-52
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Homogeneous Solution

to the homogeneous equation,

We develop the "solution" to the homogeneous equation, — (2)

$$\sum_{k=0}^N a_k y(n-k) = 0.$$

Let $y_h(n) = A \cdot \alpha^n$.

This must satisfy (2): — (3)

$$\sum_{k=0}^N a_k A \alpha^{(n-k)} = 0$$

or $a_0 + a_1 \alpha + a_2 \alpha^2 + \dots + a_N \alpha^N = 0$ — (4)

(4) is known as the characteristic equation, which has N-roots.

DISCRETE TIME SYSTEMS

DIFFERENCE EQUATIONS

Homogeneous Solution (continued).
 $d_1, d_2, d_3, \dots, d_N.$

Let us denote the n roots by $d_1, d_2, d_3, \dots, d_N.$
(Not all of them may be distinct. Or real.)

Real & distinct Roots:
$$y_h(n) = A_1 d_1^n + A_2 d_2^n + \dots + A_N d_N^n$$

Real, but P_i roots repeated:

$$y_h(n) = A_1 d_1^n + A_2 n d_1^{n-1} + \dots + A_{P_1} n^{P_1-1} d_1^{n-P_1+1} + \dots + A_{P_i+1} d_i^n + \dots + A_N d_N^n$$

DISCRETE TIME SYSTEMS

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Homogeneous Solutions (continued)

If the coefficients $\{a_i\}$ are real, we may have

Complex conjugate pair:

$$\begin{aligned} d_1 &= r_1 e^{j\theta} \\ d_2 &= d_1^* = r_1 e^{-j\theta} \end{aligned}$$

Then, corresponding to d_1 & d_2 , we write

$$y_h(n) = [A_1 \cos n\theta + A_2 \sin n\theta] \cdot r_1^n + A_3 d_3^n + A_4 d_4^n + \dots + A_n d_n^n$$

Repeated complex roots shall be handled the same way as repeated real roots.

DISCRETE TIME SYSTEMS
DIFF. EQNS.

PARTICULAR SOLUTION:

$$\sum_{k=0}^N a_k y(n-k) = \sum_{k=0}^M b_k x(n-k)$$

The RHS is known and is a weighted sum of $x(n)$ and its delayed versions. The particular solution $y_p(n)$ has the same form as the "forcing function" (i.e. the RHS of the equation).

- Forcing function
- constant $\longleftrightarrow C_1$ (constant)
 - β^n (β is root of ch. eqn.) $\longleftrightarrow C_1 \cdot \beta^n$
 - $\sin(n\theta + \phi)$ $\longleftrightarrow C_1 \sin(n\theta) + C_2 \cos(n\theta)$
 - n^p $\longleftrightarrow C_0 + C_1 n + \dots + C_p n^p$
 - $\delta(n)$ $\longleftrightarrow 0$

DIFFERENCE EQUATIONS

ZERO-INPUT & ZERO-STATE SOLUTIONS

* If we set $x(n) = 0$ and obtain the solution, or natural

We have the zero-input response $y_{ZI}(n)$, solve for the response, $y_{ZI}(n)$. To obtain $y_{ZI}(n)$ using the natural response.

The undetermined coefficients in $y_h(n)$ are set to zero, i.e. the initial conditions. It is also called the zero-state

* If the initial conditions are set to zero, i.e. the system is relaxed and we obtain zero-state proceed to

The difference equation, $y(n)$.

response or forced response, $y_{ZS}(n)$. Proceed for the

obtain $y_h(n)$ and $y_p(n)$, and solve for the $y(n) = y_h(n) + y_p(n)$

undetermined coefficients in $y_{ZS}(n) = y_h(n) + y_p(n)$. You may have to

with zero initial conditions. You may have to

generate the first few values of $y_{ZS}(n)$ from the dif. equation.

* $\rightarrow y(n) = y_{ZS}(n) + y_{ZI}(n) = y_h(n) + y_p(n)$.

DIFFERENCE EQUATIONS

IMPULSE RESPONSE

* Impulse response is obtained by setting $x(n) = \delta(n)$, and zeroing out all initial conditions. (i.e. relaxed

system).

* with $x(n) = \delta(n)$, $y_p(n) = 0$.

* Thus, $h(n) = y_h(n)$ which contains undetermined coefficients, we generate the first few values (initial conditions) of $h(n)$ recursively from the difference equation that is to be satisfied by $h(n)$. From the resulting linear equations, we solve for the undetermined coefficients.

* We illustrate this with an example.

Impulse Response from LCCDE: "DAMPED SINUSOID"

Second order LCCDE with complex characteristic roots have the form as below:

$$y(n) - 2r \cos \theta y(n-1) + r^2 y(n-2) = \text{function of } x(n). \quad \text{--- (1)}$$

The characteristic equation is given by

$$\lambda^2 - 2r \cos \theta \lambda + r^2 = 0$$

and the roots are given by

$$\lambda_1 = r e^{j\theta} \quad \text{and} \quad \lambda_2 = \lambda_1^* = r e^{-j\theta}$$

The homogeneous solution is:

$$y_h(n) = A (r e^{j\theta})^n + B (r e^{-j\theta})^n, \quad n \geq 0 \quad \text{--- (2)}$$

With $x(n) = \delta(n)$, the particular solution $y_p(n) = 0$. The total solution, with zero valued initial conditions, gives the impulse response with the undetermined coeffs, A & B in (2).

$$h(n) = y(n) = y_p(n) + y_h(n) = A (r e^{j\theta})^n + B (r e^{-j\theta})^n, \quad n \geq 0 \quad \text{--- (3)}$$

From the LCCDE given by (1), we generate the values of $y(0)$ and $y(1)$; we generate the values of $y(0)$ & $y(1)$ from (3), but they will contain the coeffs A & B. (For our case, we will find that $B = A^*$). From this information we can solve for A & B, and thus obtain

$$h(n) = A (r e^{j\theta})^n + B (r e^{-j\theta})^n, \quad n \geq 0$$

which can be simplified further as shown in the following example.

Impulse Response from LCCDE:

"Damped Sinusoid"

(2/3)

Example: Determine the impulse response of the system

$$y(n) - y(n-1) + \frac{1}{2}y(n-2) = x(n) - \frac{1}{2}x(n-1) \quad (4)$$

$n \geq 0$

$$y(-1) = y(-2) = 0.$$

Solution:

* Even if the initial conditions are non-zero, we need to set them to be zero. (why?)

* $x(n) = \delta(n)$ (why?)

From (4), we identify that

$$2r \cos \theta = 1 \quad \text{and} \quad r^2 = \frac{1}{2} \Rightarrow \boxed{r = \frac{1}{\sqrt{2}}, \theta = e^{j\pi/4}}$$

Thus

$$y_h(n) = A \left(\frac{1}{\sqrt{2}} e^{j\pi/4} \right)^n + B \left(\frac{1}{\sqrt{2}} e^{-j\pi/4} \right)^n, \quad n \geq 0.$$

$y_p(n) = 0$ because $x(n) = \delta(n)$.

$$\Rightarrow h(n) = y(n) = y_p(n) + y_h(n) = A \left(\frac{1}{\sqrt{2}} e^{j\pi/4} \right)^n + B \left(\frac{1}{\sqrt{2}} e^{-j\pi/4} \right)^n \quad (5) \quad n \geq 0.$$

	From (4)	From (5)
$y(0)$	1	$A + B \Rightarrow \boxed{A + B = 1} \quad (6)$
$y(1)$	$\frac{1}{2}$	$A \cdot \frac{1}{\sqrt{2}} e^{j\pi/4} + B \cdot \frac{1}{\sqrt{2}} e^{-j\pi/4}$ $\Rightarrow \boxed{A e^{j\pi/4} + B e^{-j\pi/4} = \frac{1}{\sqrt{2}}} \quad (7)$

(6) & (7) can be solved for A & B

$$\Rightarrow A = \frac{1}{2} \quad B = \frac{1}{2}$$

using the above values of A & B in (5), we get

$$\boxed{h(n) = \left(\frac{1}{\sqrt{2}} \right)^n \cos\left(\frac{\pi}{4}n\right)} \leftarrow \text{Damped Sinusoid}$$

Impulse Response: Example 3/3

$$y(n) - \frac{1}{4}y(n-2) = x(n) \quad \text{--- (1)}$$

$$x(n) = \delta(n), \quad y(-1) = y(-2) = 0$$

With $x(n) = \delta(n)$, $y_p(n) = 0$.

From (1), the characteristic eqn. is

$$\lambda^2 - \frac{1}{4} = 0 \Rightarrow \lambda_1 = \frac{1}{2}, \quad \lambda_2 = -\frac{1}{2}$$

$$y_h(n) = A\left(\frac{1}{2}\right)^n + B\left(-\frac{1}{2}\right)^n, \quad n \geq 0$$

$$h(n) = y(n) = y_p(n) + y_h(n) = A\left(\frac{1}{2}\right)^n + B\left(-\frac{1}{2}\right)^n \quad n \geq 0 \quad \text{--- (2)}$$

	From (1)	From (2)
$y(0)$	1	$(A+B) \Rightarrow \boxed{A+B=1}$ --- (3)
$y(1)$	0	$\frac{1}{2}(A-B) \Rightarrow \boxed{A-B=0}$ --- (4)

From (3) & (4), we have

$$\boxed{A=B=\frac{1}{2}}$$

Thus,

$$\boxed{h(n) = \left[\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)^n \right] u(n)}$$

LCCDE : EXAMPLE 1

$$y(n) - \frac{1}{6}y(n-1) - \frac{1}{6}y(n-2) = x(n), \quad n \geq 0$$
$$y(-1) = 0, \quad y(-2) = 12, \quad x(n) = 4u(n)$$

SOLUTIONS:

- Classical Method: $y_h(n)$, $y_p(n)$
- Zero Input, zero state solutions
- z-transform

DISCRETE TIME SYSTEMS DIFFERENCE EQUATIONS

Homogeneous Solution (cont'd)

Example:

$$y(n) - \frac{1}{6}y(n-1) - \frac{1}{6}y(n-2) = 0$$

The characteristic equation is:

$$1 - \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2} = 0$$

or $z^2 - \frac{1}{6}z - 1 = 0 \implies$

$\begin{aligned} \alpha_1 &= \frac{1}{2} \\ \alpha_2 &= -\frac{1}{3} \end{aligned}$

The homogeneous solution is:

$$y_h(n) = A_1\left(\frac{1}{2}\right)^n + A_2\left(-\frac{1}{3}\right)^n$$

We solve for A_1 & A_2 from the total solution $y(n)$ using the initial conditions.

DISCRETE TIME SYSTEMS DIFFERENCE EQUATIONS

Particular Solution (Continued)

Example: (continuation of previous example)

$$y(n) - \frac{1}{6}y(n-1) - \frac{1}{6}y(n-2) = x(n)$$

$$x(n) = 4u(n)$$

(because $x(n)$ is constant)

we assume that $y_p(n) = c_1$

Substituting $y_p(n)$ into the difference equation, we

$$n > 0$$

$$\text{get: } c_1 - \frac{c_1}{6} - \frac{c_1}{6} = 4$$

$$\text{or } y_p(n) = 6.$$

$$\Rightarrow \boxed{c_1 = 6}$$

(Read Examples 6.5.4 & 6.5.5)

DISCRETE TIME SYSTEMS DIFF. EQNS.

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Total Solution:

Total solution, $y(n)$, is the sum of homogeneous

and particular solutions.

$$y(n) = y_h(n) + y_p(n)$$

coefficients.
initial conditions.

But $y_h(n)$ contains undetermined

These y_h are determined by using the

$$x(n) = 4u(n)$$

Example: (continued)

$$y(n) - \frac{1}{6}y(n-1) - \frac{1}{6}y(n) = x(n),$$

$$y(n) - \frac{1}{6}y(n-1) - \frac{1}{6}y(n) = 12.$$

$$y(-1) = 0,$$

$$y_p(n)$$

$$y(n) = y_h(n) + y_p(n)$$

$$= A_1 \left(\frac{1}{2}\right)^n + A_2 \left(-\frac{1}{3}\right)^n + 6.$$

Total Solution

$$y(n) = A_1 \left(\frac{1}{2}\right)^n + A_2 (-\frac{1}{3})^n + 6 \quad n \geq 0 \quad \text{--- (1) (Total Soln.)}$$

$$y(n) = \frac{1}{6} y(n-1) + \frac{1}{6} y(n-2) + x(n), \quad n \geq 0 \quad \text{--- (2) LCCDE}$$

$$x(n) = 4u(n), \quad y(-1) = 0, \quad y(-2) = 12$$

n	LCCDE (2)	Total Soln. (1)
0	$y(0) = \frac{1}{6} y(-1) + \frac{1}{6} y(-2) + x(0)$ $= 6 \quad \frac{1}{6} y(-1) + \frac{1}{6} y(-2) + x(0) = 5$	$y(0) = A_1 + A_2 + 6 \quad \text{--- (3)}$
1	$y(1) = \frac{1}{6} y(0) + \frac{1}{6} y(-1) + x(1) = 5$	$y(1) = \frac{A_1}{2} - \frac{A_2}{3} + 6 \quad \text{--- (4)}$

(3) & (4) can be rewritten as

$$A_1 + A_2 = 0 \quad \text{--- (3)}$$

$$\frac{A_1}{2} - \frac{A_2}{3} = -1 \quad \text{--- (4)}$$

Solving for A_1 & A_2 we get

$$A_1 = -6/5 \quad \& \quad A_2 = 6/5.$$

Thus, the total solution is:

$$y(n) = \left[\left(-\frac{6}{5}\right) \left(\frac{1}{2}\right)^n + \left(\frac{6}{5}\right) \left(-\frac{1}{3}\right)^n + 6 \right] u(n)$$

Zero state, Zero Input Response (Example)

(1/3)

$$y(n) - \frac{1}{6} y(n-1) - \frac{1}{6} y(n-2) = x(n) \quad \text{--- (1)} \quad n \geq 0$$

$$y(-1) = 0 \quad y(-2) = 12$$

$$x(n) = 4u(n)$$

Zero-Input Response, $y_{zi}(n)$ [Natural Response]

Characteristic eqn: $d^2 - \frac{1}{6}d - \frac{1}{6} = 0 \Rightarrow$

$$d_1 = \frac{1}{2}, \quad d_2 = \left(-\frac{1}{3}\right)$$

$$y_{zi}(n) = A_1 \left(\frac{1}{2}\right)^n + A_2 \left(-\frac{1}{3}\right)^n, \quad n \geq 0 \quad \text{--- (2)}$$

A_1 & A_2 are solved using the given initial conditions.

The homogeneous eqn. is given by

$$y_{zi}(n) - \frac{1}{6} y_{zi}(n-1) - \frac{1}{6} y_{zi}(n-2) = 0 \quad n \geq 0 \quad \text{--- (3)}$$

or $y_{zi}(n) = \frac{1}{6} y_{zi}(n-1) + \frac{1}{6} y_{zi}(n-2)$

using $y_{zi}(0) = \frac{1}{6} y_{zi}(-1) + \frac{1}{6} y_{zi}(-2) = 2$ } --- (4)
 $y_{zi}(1) = \frac{1}{6} y_{zi}(0) + \frac{1}{6} y_{zi}(-1) = \frac{1}{3}$ }

But from (2)

$$y_{zi}(0) = A_1 + A_2 \quad \text{--- (5)}$$

$$y_{zi}(1) = \frac{A_1}{2} - \frac{A_2}{3}$$

Equating (4) & (5), we have

$$A_1 + A_2 = 2$$

$$\frac{A_1}{2} - \frac{A_2}{3} = \frac{1}{3}$$

$$\Rightarrow \boxed{A_1 = 1.2, \quad A_2 = 0.8}$$

Zero-state, zero-input Response Example (continued)

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Thus, the zero input response is given by

$$y_{zi}(n) = \left[1.2 \left(\frac{1}{2}\right)^n + 0.8 \left(-\frac{1}{3}\right)^n \right] u(n) \quad \text{--- (6)}$$

Zero-state (forced) Response, $y_{zs}(n)$

$$y_{zs}(n) = y_p(n) + y_h(n) \quad \text{--- (7)}$$

\uparrow particular soln \uparrow homogeneous solution with undetermined cpts

Because the input $x(n) = 4u(n)$, we choose $y_p(n) = c$, $n \geq 0$, and use (1) to solve for c .

$$c - \frac{1}{6}c - \frac{1}{6}c = 4 \Rightarrow \boxed{c = 6}$$

$$y_h(n) = k_1 \left(\frac{1}{2}\right)^n + k_2 \left(-\frac{1}{3}\right)^n$$

where k_1 & k_2 are undetermined cpts.

$$y_{zs}(n) = 6 + k_1 \left(\frac{1}{2}\right)^n + k_2 \left(-\frac{1}{3}\right)^n \quad n \geq 0 \quad \text{--- (8)}$$

We generate new initial conditions,

$y_{zs}(0)$ and $y_{zs}(1)$ from (1), using zero

initial condition ("zero state"), and then use these values along with (8) to solve for k_1 & k_2 .

From (1),

$$y_{zs}(n) = \frac{1}{6} y_{zs}(n-1) + \frac{1}{6} y_{zs}(n-2) + 4$$

$$y_{zs}(0) = \frac{1}{6} y_{zs}(-1) + \frac{1}{6} y_{zs}(-2) + 4 = 4$$

$$y_{zs}(1) = \frac{1}{6} y_{zs}(0) + \frac{1}{6} y_{zs}(-1) + 4 = \frac{14}{3}$$

(9)

Zero-state, Zero-input Responses Example (continued)

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from (9),

$$\left. \begin{aligned} y_{zs}(0) &= 6 + k_1 + k_2 \\ y_{zs}(1) &= 6 + \frac{k_1}{2} - \frac{k_2}{3} \end{aligned} \right\} \text{--- (10)}$$

using (9) & (10), we have

$$\left. \begin{aligned} k_1 + k_2 &= -2 \\ \frac{k_1}{2} - \frac{k_2}{3} &= -\frac{8}{6} \end{aligned} \right\} \Rightarrow \boxed{\begin{aligned} k_1 &= -2.4 \\ k_2 &= 0.4 \end{aligned}} \text{--- (11)}$$

using (11) in (8), we get the zero-state (forced)

response:

$$\boxed{y_{zs}(n) = \left[6 + (-2.4)\left(\frac{1}{2}\right)^n + 0.4\left(-\frac{1}{3}\right)^n \right] u(n)} \text{--- (12)}$$

Total Response:

$$\begin{aligned} y(n) &= y_{zs}(n) + y_{zi}(n) \\ &= \left[6 + (-2.4)\left(\frac{1}{2}\right)^n + 0.4\left(-\frac{1}{3}\right)^n \right] u(n) \\ &\quad + \left[1.2\left(\frac{1}{2}\right)^n + 0.8\left(-\frac{1}{3}\right)^n \right] u(n) \\ &= \left[6 - (1.2)\left(\frac{1}{2}\right)^n + (1.2)\left(-\frac{1}{3}\right)^n \right] u(n) \end{aligned}$$

Note that this result is the same we had using classical method.

LCCDE EXAMPLE #2

LCCDE: $y(n) - \frac{1}{4}y(n-2) = x(n), n \geq 0$ — (1)
 $y(-1) = 1, y(-2) = 0$
 $x(n) = u(n).$

Classical Solution:

Homogeneous soln:

Characteristic Eqn:

$$\lambda^n - \frac{1}{4}\lambda^{n-2} = 0 \Rightarrow \lambda^2 - \frac{1}{4} = 0 \Rightarrow \boxed{\lambda_1 = \frac{1}{2}, \lambda_2 = -\frac{1}{2}}$$

$$\Rightarrow y_h(n) = A\left(\frac{1}{2}\right)^n + B\left(-\frac{1}{2}\right)^n, n \geq 0$$
 — (2)

A & B are the undetermined coefficients. We will solve for A & B from by using the initial condition [given in (1)] in the total solution, $y(n) = y_p(n) + y_h(n)$ where $y_p(n)$ is the particular soln.

Particular Solution:

Because $x(n) = u(n)$, (a constant for $n \geq 0$), we

Choose $y_p(n) = C, n \geq 0$ — (3)

using (3) in (1), we get:

$$C - \frac{1}{4}C = 1, n \geq 0$$

$$\Rightarrow \frac{3}{4}C = 1 \Rightarrow \boxed{C = \frac{4}{3}} \Rightarrow \boxed{y_p(n) = \frac{4}{3}, n \geq 0}$$
 — (4)

Total Soln:

$$y(n) = y_p(n) + y_h(n)$$

$$\Rightarrow y(n) = \frac{4}{3} + A\left(\frac{1}{2}\right)^n + B\left(-\frac{1}{2}\right)^n, n \geq 0$$
 — (5)

n	$y(n)$ from Eqn (1)	$y(n)$ from Eqn (5)	Simultaneous Eqn. to be solved
0	$y(0) = \frac{1}{4}y(-2) + x(0) = 1$	$y(0) = \frac{4}{3} + A + B \Rightarrow$	$A + B = -\frac{1}{3}$
1	$y(1) = \frac{1}{4}y(-1) + x(1) = \frac{5}{4}$	$y(1) = \frac{4}{3} + \frac{A}{2} - \frac{B}{2} \Rightarrow$	$\frac{A}{2} - \frac{B}{2} = \frac{5}{4} - \frac{4}{3} = \frac{5}{4} - \frac{4}{3}$
			$\Rightarrow \boxed{A = -\frac{1}{4}, B = -\frac{1}{2}}$ — (6)

Thus

$$y(n) = \left\{ \frac{4}{3} + (-\frac{1}{4}) \cdot (\frac{1}{2})^n + (-\frac{1}{12}) (-\frac{1}{2})^n \right\} u(n) \quad (6)$$

Zero-State / Zero-Input Response:

* The starting point is still $y_h(n)$ and $y_p(n)$ as in the classical method.

* Zero State Response:

The initial conditions are assumed to be zero, and we solve for A & B in (5). We generate new initial condition by computing them recursively from (1).

$$\begin{aligned} y(-1) = y(-2) &= 0. \\ \Rightarrow y(0) = x(0) &= 1. \\ y(1) &= \frac{1}{4}y(0) + x(1) = 1. \end{aligned}$$

$$\Rightarrow \left. \begin{aligned} A+B &= -\frac{1}{3} \\ \frac{A}{2} - \frac{B}{2} &= -\frac{1}{3} \end{aligned} \right\} \Rightarrow \frac{\begin{aligned} A+B &= -\frac{1}{3} \\ A-B &= -\frac{2}{3} \end{aligned}}{2A = -1} \Rightarrow \boxed{\begin{aligned} A &= -\frac{1}{2} \\ B &= \frac{1}{6} \end{aligned}}$$

$$\Rightarrow y_{zs}(n) = \frac{4}{3} + (-\frac{1}{2}) \left(\frac{1}{2}\right)^n + \left(\frac{1}{6}\right) \left(-\frac{1}{2}\right)^n, n \geq 0 \quad (7)$$

* Zero Input Response:

$y_{zi}(n) = y_h(n)$ with undetermined coeff evaluated from the given initial condition.

$$\Rightarrow \left. \begin{aligned} y_{zi}(n) &= k_1 \left(\frac{1}{2}\right)^n + k_2 \left(-\frac{1}{2}\right)^n, n \geq 0. \\ y_{zi}(0) &= k_1 + k_2 = 0 \\ y_{zi}(1) &= \frac{k_1}{2} - \frac{k_2}{2} = \frac{1}{4} \end{aligned} \right\} \Rightarrow \boxed{\begin{aligned} k_1 &= \frac{1}{4} \\ k_2 &= -\frac{1}{4} \end{aligned}}$$

$$\Rightarrow \boxed{y_{zi}(n) = \left(\frac{1}{4}\right)\left(\frac{1}{2}\right)^n + \left(-\frac{1}{4}\right)\left(-\frac{1}{2}\right)^n, n \geq 0.} \quad (8)$$

Total Soln.:

$$y(n) = y_{zo}(n) + y_{zi}(n)$$

$$= \frac{4}{3} + \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)^n + \left(\frac{1}{6}\right)\left(-\frac{1}{2}\right)^n$$

$$+ \left(\frac{1}{4}\right)\left(\frac{1}{2}\right)^n + \left(-\frac{1}{4}\right)\left(-\frac{1}{2}\right)^n, n \geq 0$$

$$= \frac{4}{3} + \left(-\frac{1}{4}\right)\left(\frac{1}{2}\right)^n + \left(-\frac{1}{12}\right)\left(-\frac{1}{2}\right)^n, n \geq 0 \quad (9)$$

Note that (9) is the same as (6)

$$\Rightarrow y(n) = y_p(n) + y_h(n) = y_{zo}(n) + y_{zi}(n)$$

But $y_p(n) \neq y_{zo}(n)$ & $y_h(n) \neq y_{zi}(n)$.

LCCDE Example #3

3. (20 points)

Determine the output $y(n)$, $n \geq 0$, of the DT system described the following LCCDE:

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = x(n), \quad n \geq 0. \quad \text{--- (1)}$$

The initial conditions are zero, i.e. $y(-1) = y(-2) = 0$, and $x(n) = (\frac{1}{3})^n u(n)$.

Particular solution: $y_p(n)$ with $x(n) = (\frac{1}{3})^n u(n)$.

$$y_p(n) = k \cdot (\frac{1}{3})^n, \quad n \geq 0 \quad \text{--- (2)}$$

Substitute (2) in (1) \Rightarrow

$$k \cdot (\frac{1}{3})^n - \frac{3}{4}k \cdot (\frac{1}{3})^{n-1} + \frac{1}{8}k \cdot (\frac{1}{3})^{n-2} = (\frac{1}{3})^n, \quad n \geq 0. \quad \text{--- (3)}$$

Setting $n=2$ in (3), we get

$$k \cdot (\frac{1}{3})^2 - \frac{3}{4}k \cdot (\frac{1}{3}) + \frac{1}{8}k \cdot (\frac{1}{3})^0 = (\frac{1}{3})^2$$

$$\Rightarrow \frac{k}{9} - \frac{k}{4} + \frac{k}{8} = \frac{1}{9} \Rightarrow \frac{k}{9} - \frac{k}{8} = \frac{1}{9}$$

$$\Rightarrow \frac{-1}{72}k = \frac{1}{9} \Rightarrow \boxed{k = -8}$$

$$\text{Thus, } \boxed{y_p(n) = -8 \left(\frac{1}{3}\right)^n, \quad n \geq 0} \quad \text{--- (4)}$$

Homogeneous solution, $y_h(n)$

From (1), the characteristic equation is:

$$\lambda^2 - \frac{3}{4}\lambda + \frac{1}{8} = 0 \Rightarrow \lambda_1 = \frac{1}{2}, \quad \lambda_2 = \frac{1}{4}$$

$$\Rightarrow \boxed{y_h(n) = A \cdot \left(\frac{1}{2}\right)^n + B \cdot \left(\frac{1}{4}\right)^n, \quad n \geq 0} \quad \text{--- (5)}$$

Total solution

$$\Rightarrow y(n) = y_p(n) + y_h(n)$$

$$\Rightarrow \boxed{y(n) = -8 \cdot \left(\frac{1}{3}\right)^n + A \cdot \left(\frac{1}{2}\right)^n + B \cdot \left(\frac{1}{4}\right)^n, \quad n \geq 0.} \quad (6)$$

n	LCCDE (1)	Total solution (6)
0	$y(0) = \frac{3}{4}y(-1) - \frac{1}{8}y(-2) + x(0)$ $= x(0) = 1$	$y(0) = -8 + A + B. \quad (7)$
1	$y(1) = \frac{3}{4}y(0) - \frac{1}{8}y(-1) + x(1)$ $= \frac{3}{4} + \frac{1}{3} = \frac{13}{12}$	$y(1) = -\frac{8}{3} + \frac{A}{2} + \frac{B}{4} \quad (8)$

From (7) & (8), we have:

$$-8 + A + B = 1 \quad \Rightarrow$$

$$-\frac{8}{3} + \frac{A}{2} + \frac{B}{4} = \frac{13}{12} \quad \Rightarrow$$

$$A + B = 9 \quad (9)$$

$$6A + 3B = 45 \quad (10)$$

which can be solved to yield $\boxed{A=6, B=3}$

Total solution:

$$\boxed{y(n) = \left\{ -8 \left(\frac{1}{3}\right)^n + 6 \left(\frac{1}{2}\right)^n + 3 \left(\frac{1}{4}\right)^n \right\} u(n)}$$

Q7 (10 points)

Find the output $y(n)$ of the causal LSI system given by the following difference equation:

$$y(n] - (3/4)y(n-1) + (1/8)y(n-2) = x(n]$$

for input $x(n) = (1/3)^n u(n)$. Assume zero initial conditions.

Taking z-transforms on both sides,

we get

$$Y(z) - 3/4 z^{-1} Y(z) + 1/8 z^{-2} Y(z) = X(z)$$

$$\Rightarrow Y(z) = \frac{1}{[1 - 3/4 z^{-1} + 1/8 z^{-2}]} \cdot X(z) \quad \text{--- (2)}$$

$$X(z) = z [(1/3)^n u(n)] = \frac{z}{z - 1/3}$$

Therefore,

$$Y(z) = \frac{z^3}{(z^2 - 3/4 z + 1/8)(z - 1/3)}$$

$$= \frac{z^3}{(z - 1/2)(z - 1/4)(z - 1/3)} \quad \text{--- (3)}$$

$$\frac{Y(z)}{z} = \frac{A}{z - 1/2} + \frac{B}{z - 1/4} + \frac{C}{z - 1/3} \quad \text{--- (4)}$$

$$A \cdot z = 1/2$$

$$B \cdot z = 1/4 = 8$$

$$A = \frac{(1/2)^2}{(1/2 - 1/4)(1/2 - 1/3)} = \frac{(1/4)}{(1/4)(1/6)} = 6 //$$

$$B = \frac{(1/4)^2}{(1/4 - 1/2)(1/4 - 1/3)} = \frac{1/16}{(-1/4)(-1/12)} = \frac{1}{16} \cdot 4 \cdot 12 = 3 //$$

(Space for continuation of Q7)

$$c/3 = \frac{1}{3} \quad c = \frac{(1/3)^2}{(1/3 - 1/2)(1/3 - 1/4)} = \frac{-1/9}{(-1/6)(1/12)}$$
$$= -\frac{1}{9} \times \frac{2}{6} \times \frac{4}{12} = -8$$

(4) can be rewritten as:

$$Y(z) = \frac{6z}{(z-1/2)} + \frac{3z}{(z-1/4)} - \frac{8z}{(z-1/3)} \quad \text{--- (5)}$$

Because $y(n)$ is given as causal,
we write the inverse transform of (5) as:

$$y(n) = \left\{ 6\left(\frac{1}{2}\right)^n + 3\left(\frac{1}{4}\right)^n - 8\left(\frac{1}{3}\right)^n \right\} u(n)$$

LCCDE Example # 4

5. (20 points)

A causal, LTI system is described by the following LCCDE:

$$y(n] - 0.5 y[n-1] = x[n], n \geq 0 \text{ with } y[-1] = 1, \text{ and } x[n] = (1/3)^n u[n]. \quad \textcircled{1}$$

Determine the zero state and zero input responses. (10 points) Identify the homogenous and particular solution. (10 points)

Taking the one-sided z-transform of the LCCDE we get

$$Y(z) - \frac{1}{2} [z^{-1} Y(z) + y[-1]] = X(z)$$

$$Y(z) = \underbrace{\frac{X(z)}{(1 - \frac{1}{2} z^{-1})}}_{Y_{zs}(z)} + \underbrace{\frac{\frac{1}{2} y[-1]}{(1 - \frac{1}{2} z^{-1})}}_{Y_{zi}(z)}$$

Zero state:

$$X(z) = \frac{1}{(1 - \frac{1}{3} z^{-1})}$$

$$Y(z) = \frac{1}{(1 - \frac{1}{2} z^{-1})(1 - \frac{1}{3} z^{-1})} = \frac{A}{(1 - \frac{1}{2} z^{-1})} + \frac{B}{(1 - \frac{1}{3} z^{-1})}$$

$$A = Y(z) \cdot (1 - \frac{1}{2} z^{-1}) \Big|_{z^{-1}=2} = \frac{1}{1 - \frac{2}{3}} = \boxed{3}$$

$$B = Y(z) \cdot (1 - \frac{1}{3} z^{-1}) \Big|_{z^{-1}=3} = \frac{1}{1 - \frac{3}{2}} = \boxed{-2}$$

$$Y_{zs}(z) = 3 \left(\frac{1}{2}\right)^n u[n] + (-2) \left(\frac{1}{3}\right)^n u[n]$$

Zero input:

$$Y_{zi}(z) = \frac{\frac{1}{2} y[-1]}{(1 - \frac{1}{2} z^{-1})} \Rightarrow$$

$$Y_{zi}(z) = \frac{1}{2} \left(\frac{1}{2}\right)^n u[n]$$

Total solution:

$$y(n) = y_{zs}(n) + y_{zi}(n) \\ = \left[(-2) \left(\frac{1}{3} \right)^n + \left(\frac{7}{2} \right) \left(\frac{1}{2} \right)^n \right] u(n)$$

$y_p(n)$ depends on the input

$$\begin{aligned} y_p(n) &= -2 \cdot \left(\frac{1}{3} \right)^n u(n) \\ y_h(n) &= \left(\frac{7}{2} \right) \left(\frac{1}{2} \right)^n u(n) \end{aligned}$$

6. (20 points)

A causal, LTI system is described by the following LCCDE:

$$y(n) - 0.5 y(n-1) = x(n], n \geq 0 \text{ with } y(-1) = 1, \text{ and } x(n) = (1/3)^n u(n).$$

Determine $y(n)$, the total response of the system. (15 points). What is the response of the system if the initial condition is changed to $y(-1) = 2$? (5 points)

Taking 1-sided z -transform of the LCCDE, we have:

$$Y(z) - \frac{1}{2} [z^{-1} Y(z) + y(-1)] = X(z)$$

$$Y(z) = \frac{X(z)}{(1 - \frac{1}{2} z^{-1})} + \frac{\frac{1}{2} y(-1)}{(1 - \frac{1}{2} z^{-1})}$$

\uparrow Zero state \uparrow Zero input

Zero state response:

$$Y_{zs}(z) = \frac{1}{(1 - \frac{1}{2} z^{-1})} \cdot \frac{1}{(1 - \frac{1}{3} z^{-1})} = \frac{3}{(1 - \frac{1}{2} z^{-1})} + \frac{-2}{(1 - \frac{1}{3} z^{-1})}$$

$$\Rightarrow \boxed{y_{zs}(n) = [3 \left(\frac{1}{2}\right)^n - 2 \left(\frac{1}{3}\right)^n] u(n)}$$

Zero Input Response

$$Y_{zi}(z) = \frac{\frac{1}{2} y(-1)}{(1 - \frac{1}{2} z^{-1})} = \frac{\frac{1}{2}}{(1 - \frac{1}{2} z^{-1})}$$

$$\Rightarrow \boxed{y_{zi}(n) = \frac{1}{2} \left(\frac{1}{2}\right)^n u(n)}$$

Total Response:

$$\boxed{y_t(n) = y_{zs}(n) + y_{zi}(n) = \left[\frac{7}{2} \left(\frac{1}{2}\right)^n - 2 \left(\frac{1}{3}\right)^n \right] u(n)}$$

Change in the initial condition changes $y_{zi}(n)$. The

new $y_{zi_new}(n) = 2 \cdot y_{zi}(n) = 2 \cdot \frac{1}{2} \left(\frac{1}{2}\right)^n u(n) = \left(\frac{1}{2}\right)^n u(n)$

New Total output

$$\boxed{y_t(n) = y_{zs}(n) + y_{zi_new}(n) = \left[4 \left(\frac{1}{2}\right)^n - 2 \left(\frac{1}{3}\right)^n \right] u(n)}$$

LCCDE Example #5

2. A causal, LTI system is described by the following LCCDE:

$$y(n) - 0.25 y(n-1) = x(n), \quad n \geq 0 \text{ with } y(-1) = 1, \text{ and } x(n) = u(n) \quad (1)$$

Determine the transient and steady state responses. Identify the zero input and zero state responses. Identify the homogeneous and particular solutions.

Taking 1-sided z-transform of the LCCDE in (1), we get:

$$Y(z) - \frac{1}{4} [z^{-1} Y(z) + y(-1)] = X(z)$$

$$\Rightarrow Y(z) = \underbrace{\frac{X(z)}{(1 - \frac{1}{4} z^{-1})}}_{Y_{zs}(z)} + \underbrace{\frac{\frac{1}{4} y(-1)}{(1 - \frac{1}{4} z^{-1})}}_{Y_{zi}(z)} \quad (2)$$

$$= \frac{1}{(1 - z^{-1})(1 - \frac{1}{4} z^{-1})} + \frac{\frac{1}{4} y(-1)}{(1 - \frac{1}{4} z^{-1})}$$

$$= \underbrace{\frac{A}{(1 - z^{-1})}}_{Y_{ss}(z)} + \underbrace{\frac{B}{(1 - \frac{1}{4} z^{-1})}}_{Y_{tr}(z)} + \frac{\frac{1}{4} y(-1)}{(1 - \frac{1}{4} z^{-1})} \quad (3)$$

$Y_p(z)$ $Y_h(z)$

Zero Input Response:

$$y_{zi}(n) = \frac{1}{4} y(-1) \left(\frac{1}{4}\right)^n u(n) = \boxed{\left(\frac{1}{4}\right) \cdot \left(\frac{1}{4}\right)^n u(n)} \quad (4)$$

ZERO INPUT RESP

Zero State Response

$$Y_{zo}(z) = \frac{A}{(1 - z^{-1})} + \frac{B}{(1 - \frac{1}{4} z^{-1})} = \frac{1}{(1 - z^{-1})(1 - \frac{1}{4} z^{-1})}$$

$$A = Y_{zs}(z) \cdot (1 - z^{-1}) \Big|_{z^{-1}=1} = \frac{1}{(1 - 1/4)} = \boxed{\frac{4}{3}}$$

$$B = Y_{zs}(z) \cdot (1 - \frac{1}{4}z^{-1}) \Big|_{z^{-1}=4} = \frac{1}{(1 - 4)} = \boxed{-\frac{1}{3}}$$

$$\Rightarrow \boxed{y_{zs}(n) = \frac{4}{3}u(n) + (-\frac{1}{3})\left(\frac{1}{4}\right)^n u(n)} \quad \text{ZERO STATE RESPONSE} \quad (5)$$

Transient & Steady State Responses

From (3), (4) & (5), we identify these responses:

Transient Response:

$$\boxed{y_{tr}(n) = \left(\frac{1}{4} - \frac{1}{3}\right)\left(\frac{1}{4}\right)^n u(n) = \left(-\frac{1}{12}\right)\left(\frac{1}{4}\right)^n u(n)}$$

Steady State Response:

$$\boxed{y_{ss}(n) = \frac{4}{3}u(n)}$$

Homogeneous & Particular Solutions.

$$\boxed{y_h(n) = y_{tr}(n) \quad \text{and} \quad y_p(n) = y_{ss}(n)}$$

LCCDE Example # 6

3. An LTI system is described the following LCCDE:

$$y(n) - 5/6 y(n-1) + 1/6 y(n-2) = x(n) \quad n \geq 0 \quad \text{--- (1)}$$

(a) Find the zero-input, zero-state and the total solutions $y_{zi}(n)$, $y_{zs}(n)$ and $y(n)$, when $x(n) = (1/5)^n u(n)$, $y(-1) = 6$ and $y(-2) = 25$

(b) Find the total solution if the initial conditions are changed to $y(-1) = 3$ and $y(-2) = 12.5$

Taking the 1-sided z-transform of (1), we get:

$$Y(z) - \frac{5}{6} [z^{-1} Y(z) + y(-1)] + \frac{1}{6} [z^{-2} Y(z) + z^{-1} y(-1) + y(-2)] = X(z) \quad \text{--- (2)}$$

$$Y(z) \left[1 - \frac{5}{6} z^{-1} + \frac{1}{6} z^{-2} \right] + \left[-\frac{5}{6} y(-1) + \frac{1}{6} y(-2) + \frac{1}{6} z^{-1} y(-1) \right] = X(z) \quad \text{--- (3)}$$

$$Y(z) = \frac{X(z)}{\underbrace{\left[1 - \frac{5}{6} z^{-1} + \frac{1}{6} z^{-2} \right]}_{Y_{zs}(z)}} + \frac{\left[\frac{5}{6} y(-1) - \frac{1}{6} y(-2) - \frac{1}{6} z^{-1} y(-1) \right]}{\underbrace{\left[1 - \frac{5}{6} z^{-1} + \frac{1}{6} z^{-2} \right]}_{Y_{zs}(z)}} \quad \text{--- (4)}$$

Zero state Response, $y_{zs}(n)$:

$$X(z) = \frac{1}{\left(1 - \frac{1}{5} z^{-1} \right)}$$

$$Y_{zs}(z) = \frac{1}{\left(1 - \frac{1}{5} z^{-1} \right) \left(1 - \frac{1}{3} z^{-1} \right) \left(1 - \frac{1}{2} z^{-1} \right)}$$

$$= \frac{1}{\left(1 - \frac{1}{5} z^{-1} \right)} + \frac{-5}{\left(1 - \frac{1}{3} z^{-1} \right)} + \frac{5}{\left(1 - \frac{1}{2} z^{-1} \right)}$$

$$\Rightarrow \boxed{y_{zs}(n) = \left[\left(\frac{1}{5} \right)^n - 5 \left(\frac{1}{3} \right)^n + 5 \left(\frac{1}{2} \right)^n \right] u(n)} \quad \text{--- (5)}$$

#3 Continued

Zero input Response, $y_{zi}(n)$:

With $y(-1) = 6$ and $y(-2) = 25$, we have

$$Y_{zi}(z) = \frac{(5/6 - z^{-1})}{(1 - \frac{1}{3}z^{-1})(1 - \frac{1}{2}z^{-1})}$$

$$= \frac{13/3}{(1 - \frac{1}{3}z^{-1})} + \frac{-7/2}{(1 - \frac{1}{2}z^{-1})}$$

$$y_{zi}(n) = \left[\left(\frac{13}{3}\right)\left(\frac{1}{3}\right)^n + \left(-\frac{7}{2}\right)\left(\frac{1}{2}\right)^n \right] u(n) \quad \text{--- (6)}$$

Total Response, $y(n)$:

$$y(n) = y_{zs}(n) + y_{zi}(n) \rightarrow \text{(7)}$$

Combining (5) & (6) as per (7), we have:

$$y(n) = \left[\left(\frac{1}{5}\right)^n - \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^n + \left(\frac{3}{2}\right)\left(\frac{1}{2}\right)^n \right] u(n) \quad \text{--- (8)}$$

Part b: Note that the initial conditions are $\frac{1}{2}$ of that given in part (a). This implies that $y_{zi}(n)$ is halved.

$$y_{zi \text{ new}}(n) = \frac{1}{2} \left[\left(\frac{13}{3}\right)\left(\frac{1}{3}\right)^n + \left(-\frac{7}{2}\right)\left(\frac{1}{2}\right)^n \right] u(n) \quad \text{(9)}$$

Combining (5) & (9) as per (7), the new total response is given by:

$$y_{\text{new}}(n) = \left[\left(\frac{1}{5}\right)^n - \left(\frac{17}{6}\right)\left(\frac{1}{3}\right)^n + \left(\frac{13}{4}\right)\left(\frac{1}{2}\right)^n \right] u(n) \quad \text{--- (10)}$$

LCCDE Example #7

$$y(n) - \frac{1}{2}y(n-1) = x(n), \quad n \geq 0 \quad \text{--- (1)}$$

$$y(-1) = 4, \quad x(n) = 5 \cos(2\pi \frac{1}{4}n) u(n) \quad \text{--- (2)}$$

Solution:

Homogeneous soln.:

$$y_h(n) = A \cdot (\frac{1}{2})^n, \quad n \geq 0 \quad \text{--- (3)}$$

Particular soln.

$$y_p(n) = K_1 \cos(2\pi \frac{1}{4}n) + K_2 \sin(2\pi \frac{1}{4}n), \quad n \geq 0 \quad \text{--- (4)}$$

(3) must satisfy (1) \Rightarrow

$$\begin{aligned} & [K_1 \cos(2\pi \frac{1}{4}n) + K_2 \sin(2\pi \frac{1}{4}n)] \\ & - \frac{1}{2} [K_1 \cos(2\pi \frac{1}{4}(n-1)) + K_2 \sin(2\pi \frac{1}{4}(n-1))] \\ & = 5 \cos(2\pi \frac{1}{4}n), \quad n \geq 0 \quad \text{--- (5)} \end{aligned}$$

Evaluating (5) at $n=0$, we get:

$$K_1 + \frac{1}{2}K_2 = 5 \quad \text{--- (6)}$$

Evaluating (5) at $n=1$, we get:

$$K_2 - \frac{1}{2}K_1 = 0 \quad \text{--- (7)}$$

Solving for K_1 & K_2 from (6) & (7), we get:

$$\boxed{K_1 = 4, \quad K_2 = 2}$$

Total soln:

$$y(n) = y_p(n) + y_h(n) = A \cdot (\frac{1}{2})^n + 4 \cos(2\pi \frac{1}{4}n) + 2 \sin(2\pi \frac{1}{4}n), \quad n \geq 0 \quad \text{--- (8)}$$

Evaluating (1) & (8) at $n=0$, we solve for A \Rightarrow $\boxed{A=3}$

Answer:

$$y(n) = [3(\frac{1}{2})^n + 4 \cos(2\pi \frac{1}{4}n) + 2 \sin(2\pi \frac{1}{4}n)] u(n)$$

LCCDE Example #8

$$y(n) - \frac{1}{2}y(n-1) = x(n), \quad n \geq 0 \quad \text{--- (1)}$$

$$y(-1) = 0, \quad x(n) = \left(\frac{1}{2}\right)^n u(n).$$

~~Note that the char. root is~~
Homogeneous Eqn:

$$y_h(n) = A \cdot \left(\frac{1}{2}\right)^n, \quad n \geq 0. \quad \text{--- (2)}$$

Particular Soln.

We may be tempted to choose
 $y_p(n) = k \cdot \left(\frac{1}{2}\right)^n, \quad n \geq 0$, but this is not
 distinct (linearly independent) of $y_h(n)$. In
 such a case, we must choose:

$$y_p(n) = k \cdot n \left(\frac{1}{2}\right)^n u(n) \quad \text{--- (3)}$$

But (3) must satisfy (1).

$$\Rightarrow k \cdot n \left(\frac{1}{2}\right)^n - \frac{1}{2} k (n-1) \left(\frac{1}{2}\right)^{n-1} = \left(\frac{1}{2}\right)^n, \quad n \geq 0$$

Evaluating the above at $n=1$, we get

$$k \cdot \frac{1}{2} = \frac{1}{2} \Rightarrow \boxed{k=1}$$

$$\text{Thus } y_p(n) = n \cdot \left(\frac{1}{2}\right)^n, \quad n \geq 0. \quad \text{--- (4)}$$

Total soln.

$$y(n) = y_p(n) + y_h(n) = n \left(\frac{1}{2}\right)^n + A \cdot \left(\frac{1}{2}\right)^n, \quad n \geq 0 \quad \text{--- (5)}$$

n	From (1)	From (5)
0	$y(0) = 1$	$y(0) = A \Rightarrow \boxed{A=1}$

Thus, the total soln. is:

$$\boxed{y(n) = \left[\left(\frac{1}{2}\right)^n + n \left(\frac{1}{2}\right)^n \right] u(n)}$$