

DISCRETE TIME SYSTEMS

SIGNALS

What are the various ways in which $x(n)$ can be represented?

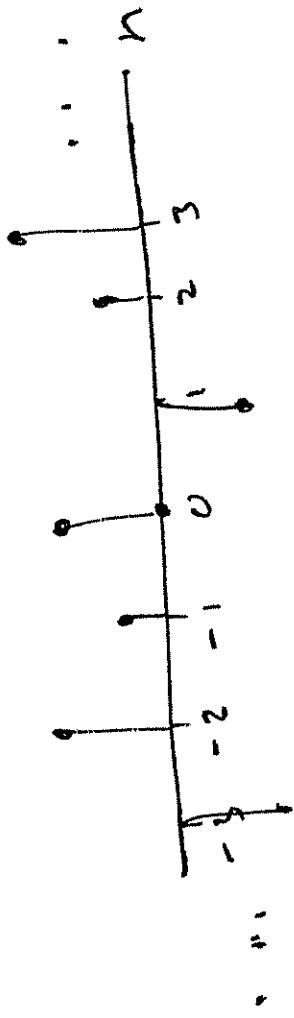
→ Analytical

$$x(n) = \frac{1}{2} \cos 3\pi n$$

$$x(n) = \left\{ \dots, 0, \frac{1}{4}, \frac{1}{8}, 1, -2, 1, -\frac{1}{2}, 0, -1, \dots \right\}$$

↑
($n=0$)

→ Graphical



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PERIODIC SIGNALS:

The signal $x(m)$ is periodic if

$$x(m+N) = x(m) \quad \text{for some integer } N > 0.$$

The smallest N that satisfies the above condition is the fundamental period, and

$$\frac{2\pi}{N} = \omega_0 \text{ (radians)}$$

is the "fundamental frequency". [Note: ω_0 is in radians, not radians/sec, N is # of samples]

not periodic $x(m)$, which

BE CAREFUL of seemingly periodic.
may or may not be periodic.

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There are only N distinct harmonics of the discrete time periodic signal.

Discrete Time Impulse Signal

$$\delta(n) = \begin{cases} 1 & n=0 \\ 0 & n \neq 0 \end{cases}$$

$$\delta(n-k) = \begin{cases} 1 & n=k \\ 0 & n \neq k \end{cases}$$

Unit step function

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

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$$\boxed{u(n) - u(n-1) = \delta(n)}$$

$$\sum_{k=-\infty}^n \delta(k) = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases}$$

$$= u(n).$$

Sampling
property

$$\boxed{x(n) \cdot \delta(n-k) = x(k) \cdot \delta(n-k)}$$

$$x(n) = \dots x(-2) \delta(n+2) + x(-1) \delta(n+1) + \dots + x(1) \delta(n-1) + x(0) \delta(n)$$

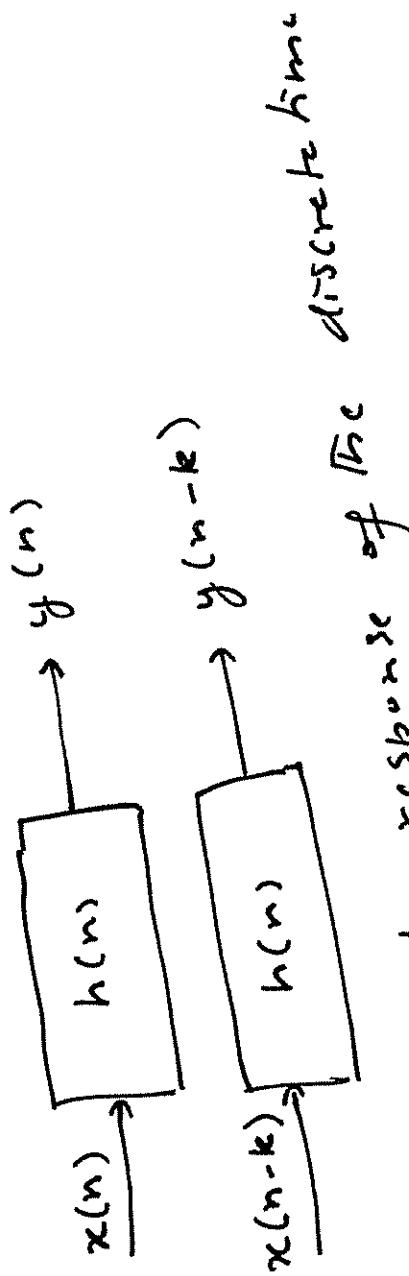
$$x(n) = \sum_{k=-\infty}^{\infty} x(k) \cdot \delta(n-k)$$

"Shifting property"

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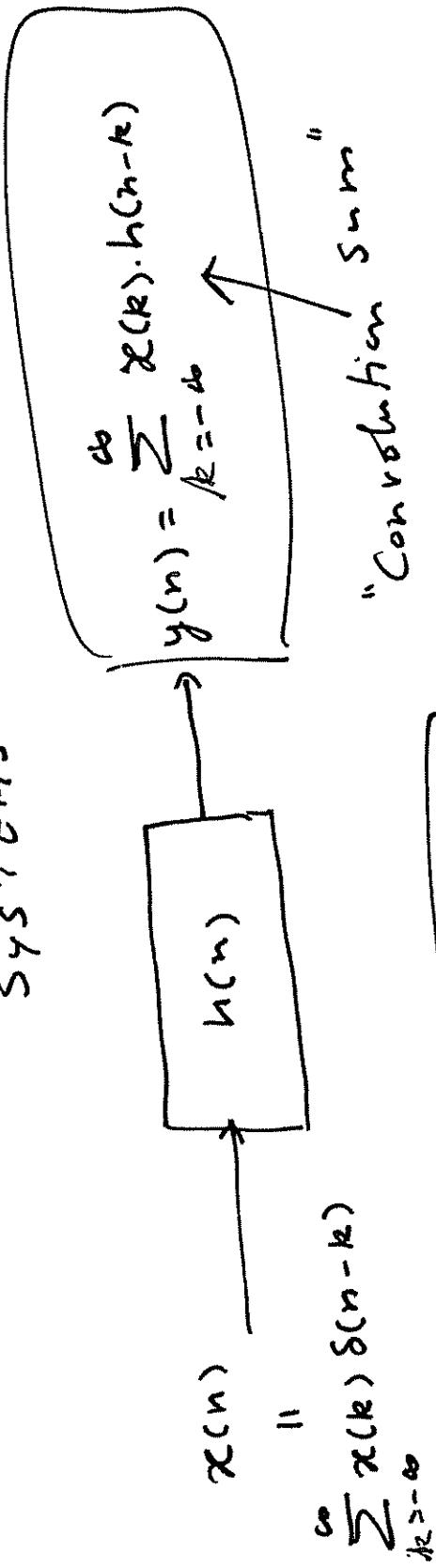
we will focus on "shift-invariant" linear systems in the discrete time domain.



$h(n)$ is the impulse response of the discrete time system.
Shift invariant linear systems, for continuous time signals, for
analogous to continuous signals, we can write
Note: $x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$

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$y(n) = \sum_{k=-\infty}^{\infty} x(k) \cdot h(n-k)$

"Convolution sum"

$$y(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k)$$

"time" variable
 n - is the running variable of summation.
 k - is the variable of summation.

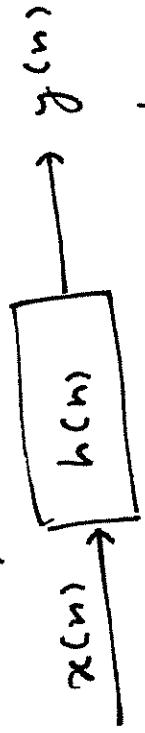
The convolution sum is:

- Commutative
- Distributive
- Associative

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Example: Step Response calculation



If $x(n) = u(n)$, a unit step original, $y(n)$ is the step response of the shift-invariant linear system.

$$\begin{aligned} y(n) &= x(n) * h(n) \\ &= \sum_{k=-\infty}^{\infty} h(k) x(n-k) \end{aligned}$$

$$= \sum_{k=-\infty}^{\infty} h(k) \cdot u(n-k) \quad \left| \begin{array}{l} \text{Note: } \\ u(n-k) = \begin{cases} 1 & n \geq k \\ 0 & \text{otherwise} \end{cases} \end{array} \right.$$

$$y(n) = \sum_{k=-\infty}^n h(k)$$

Step Response is the "integration" of "time"

$h(k)$ up to current

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Example (continued)

$$\delta(n) = u(n) - u(n-1).$$

Recall :

$$x(n) = u(n) - u(n-1).$$

$$\text{Let } x(n) = \sum_{k=-\infty}^{\infty} h(k) [u(n-k) - u(n-k-1)]$$

$$\begin{aligned} y(n) &= \sum_{k=-\infty}^{n-1} h(k) - \sum_{k=-\infty}^{n-1} h(k) = h(n). \\ &= \sum_{k=-\infty}^{n-1} h(k) \end{aligned}$$

What is the significance?

The impulse response of a system can be

The impulse response of the system is

Computed from the step response

where $y_s(n)$ is the response of the linear, shift-invariant system to unit step.

$y_s(n) = y_s(n-1) + h(n)$

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Useful Equations

$$\boxed{1} \quad \sum_{n=0}^{N-1} \alpha^n = \begin{cases} \frac{1-\alpha^N}{1-\alpha} & \alpha \neq 1 \\ N & \alpha = 1 \end{cases}$$

2

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha} \quad |\alpha| < 1$$

3

$$\sum_{n=k}^{\infty} \alpha^n = \frac{\alpha^k}{1-\alpha} \quad |\alpha| < 1$$

4

$$\sum_{n=0}^{\infty} n\alpha^n = \frac{\alpha}{(1-\alpha)^2} \quad |\alpha| < 1$$

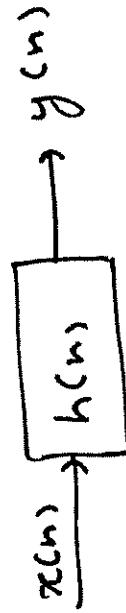
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Example:

$$x(n) = u(n)$$

$$\boxed{0 < d < 1}$$

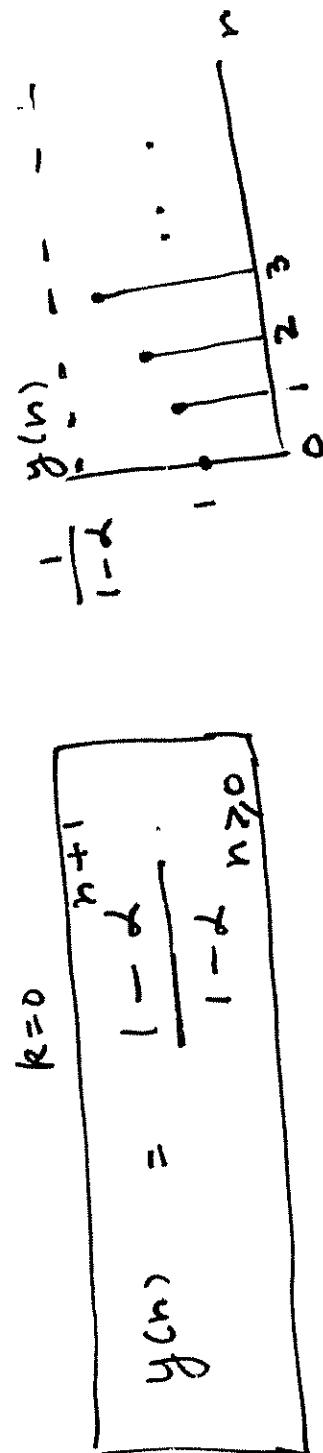


example, we can write:

$$\text{From previous example, } h(n) = \sum_{k=-\infty}^n d^k u(k)$$

$$y(n) = \sum_{k=0}^{n+1} d^k$$

$\left[\text{because } u(k) = 1, k \geq 0, \right]$



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EXAMPLE:

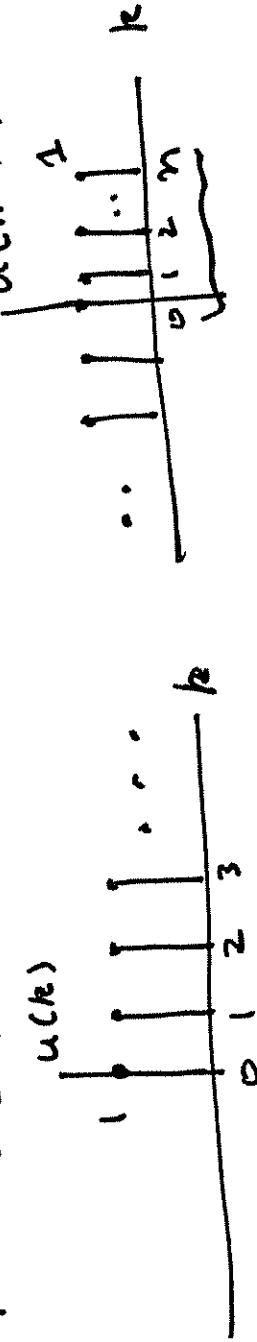
$$x(n) = \delta^n u(n) \quad h(n) = \beta^n u(n)$$



$$y(n) = \sum_{k=-\infty}^{\infty} x(k) \cdot h(n-k)$$

$$= \sum_{k=-\infty}^{\infty} \delta(k) u(k) \beta^{(n-k)} \\ = \sum_{k=-\infty}^{\infty} u(k) u(-(k-n))$$

$$\text{Note: } u(k) \cdot u(n-k) = u(k) u(-(k-n))$$



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Example (continued)

Therefore $u(k) \cdot u(n-k) = \begin{cases} 1 & k=0, 1, 2, \dots, n \\ 0 & \text{otherwise.} \end{cases}$

$$\begin{aligned} y(n) &= \sum_{k=-\infty}^{\infty} d u(k) \beta^{n-k} u(n-k) \\ &= \sum_{k=0}^n d \beta^{n-k} = \beta^n \cdot \sum_{k=0}^n (\beta \beta^{-1})^k \\ &= \end{aligned}$$

case (i) $\boxed{d = \beta \cdot \beta^n \cdot (1+n)} \rightarrow n \geq 0$
 Then $\boxed{y(n) = \beta^n \cdot (1+n) \cdot u(n)}$ or $\boxed{y(n) = \beta^n (1+n) \cdot u(n)}$

case (ii) $d \neq \beta$.

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Example: (continued)

$$\alpha \neq \beta: \quad \beta^n \sum_{k=0}^n (\alpha \beta^{-1})^k \quad n \geq 0$$

$$y(n) = \boxed{\frac{1}{\beta - \alpha} \cdot (\beta^{n+1} - \alpha^{n+1}) u(n)}$$

$y(n)$ is the step response.

As a special case, let $\alpha = 1 \Rightarrow y(n)$ is

$$y(n) = \frac{1}{1 - \beta^{n+1}} \left[\beta^{n+1} - 1 \right] u(n) \\ = \frac{1 - \beta^{n+1}}{1 - \beta} u(n)$$

a result that we obtained earlier.

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Example:

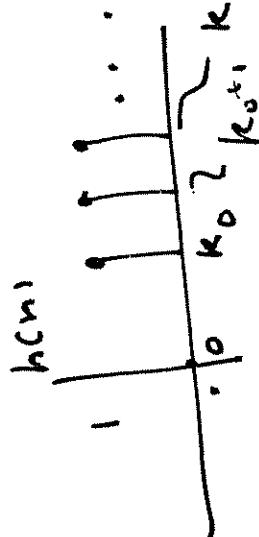
$$x(n) = u(n), \quad h(n) = u(n)$$

From Fig. previous example, we have

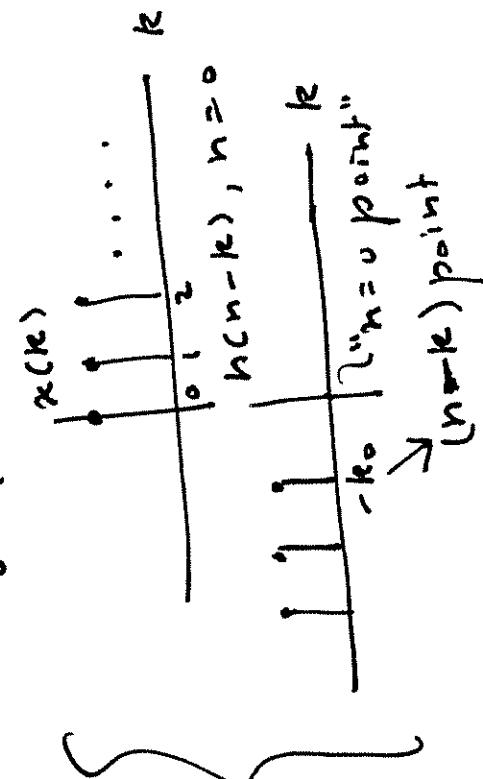
$$\boxed{y(n) = (1+n) \cdot u(n)}$$

Example:

$$h(n) = u(n-k)$$

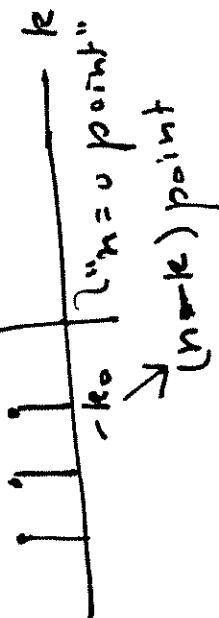


$$x(n) = u(n)$$



T

$y(n) = 0$
if "time" overlap
no "time" overlap



$y(n) = u(n-k)$
(n=k) point

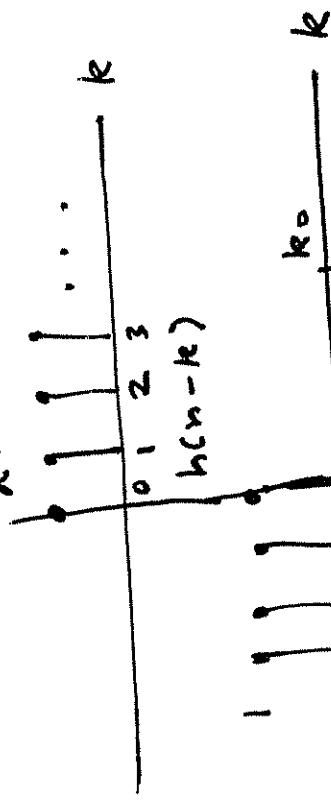
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Example (continued)

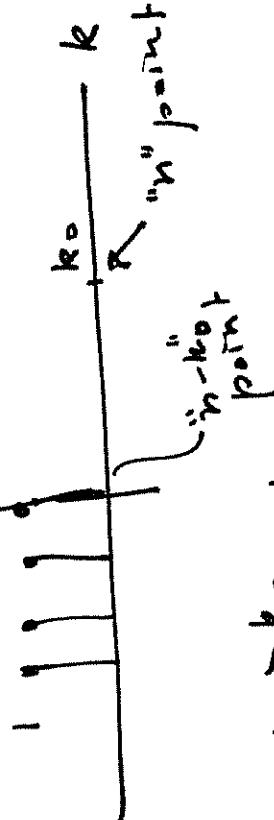
II.

$$n = k_0 \cdot \chi(k)$$

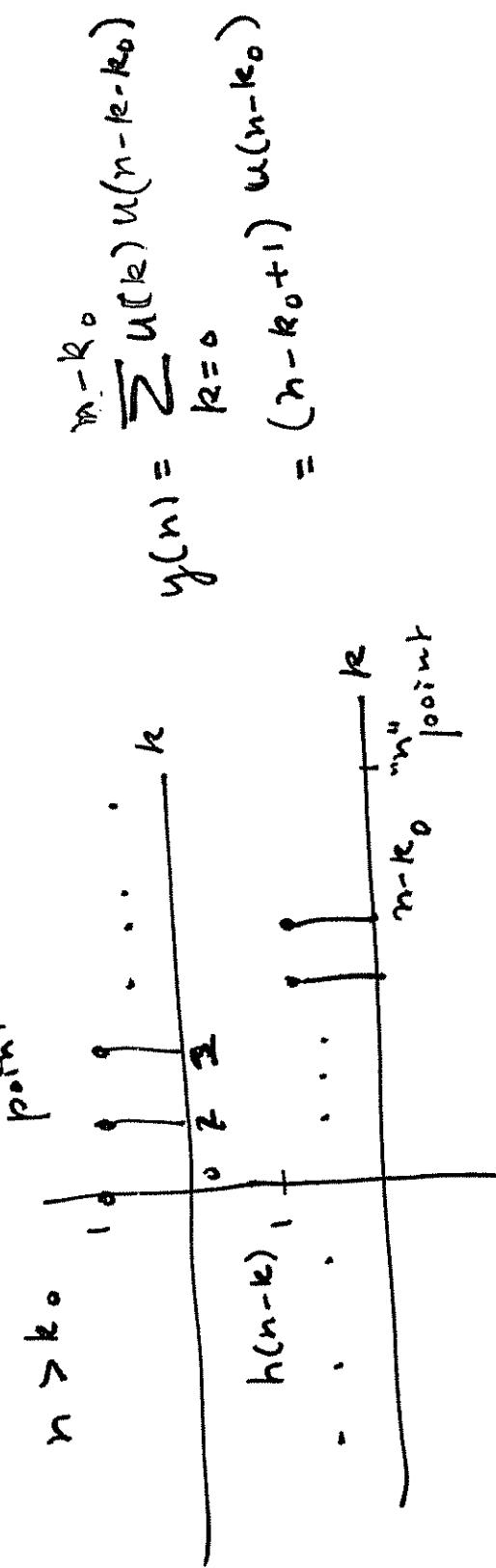


$$y(n) = x(0) \cdot h(k_0 - k)$$

$$\begin{aligned} &= 1 \cdot 1 = 1 \\ &= u(k) \cdot u(n - k - k_0) \\ &= u(n - k_0). \end{aligned}$$



III



6.2.1

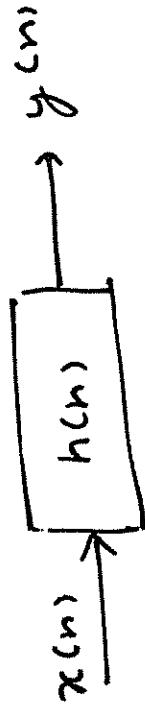
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DISCRETE CONVOLUTION

Simpler, tabular or graphical approach is often used to obtain the convolution sum in the case of finite sequences made up $x(n)$ and $h(n)$.

Cases.



$$\begin{aligned}
 y(n) &= \sum_{k=0}^{\infty} x(k)h(n-k) \\
 h(n) &= \{h(0), h(1), \dots, h(n)\} \\
 x(n) &= \{x(0), x(1), x(2), \dots, x(n)\}
 \end{aligned}$$

$$\begin{aligned}
 y(n) &= x(0)h(n) + x(1)h(n-1) + x(2)h(n-2) + \dots + x(n)h(0) \\
 &= x(0) \cdot h(n) + x(1)h(n-1) + x(2)h(n-2)
 \end{aligned}$$

DISCRETE TIME SYSTEMS

DISCRETE CONVOLUTION

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Note:

Every term on the RHS is of the form $x(k) \cdot h(n-k)$, i.e. its arguments sum up to n , the "time instance" at which the output is calculated. There are at most " p " (assuming $y(n) > p$) terms. But $y(n)$ needs to be calculated for each n ,

Example: (Ex. 6.3.5)
 $x(n) = \{1, 3, -1, 2\}$
 $h(n) = \{1, 2, 0, -1, 1\}$

DISCRETE TIME SIGNALS

DISCRETE CONVOLUTION

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Method A: (~~Table given in the book~~)

<u>n</u>	0	1	2	3	4	5	6	7	<u>$y(n)$</u>
$x(n)$	1	3	-1	-2	1	3	-1	-2	$h(0) x(n)$
$x(n-1)$		1			1	3	-1	-2	$h(1) x(n-1)$
$x(n-2)$			6		-2	-4			$h(2) x(n-2)$
$x(n-3)$				1	3	-1	-2		$h(3) x(n-3)$
$x(n-4)$					-1	-2			$h(4) x(n-4)$

DISCRETE TIME SYSTEMS

DIFFERENCE EQUATIONS

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Continuous Time Systems are described by ordinary differential equations. It is natural to expect, then, that discrete time systems are described by discrete difference equations.

$$\begin{aligned} & a_0 y(n) + a_1 y(n-1) + \dots + a_N y(n-N) \\ & = b_0 x(n) + b_1 x(n-1) + \dots + b_M x(n-M) \end{aligned}$$

$x(n)$ is the input and $y(n)$ the output.

$x(n)$ is like input and $y(n)$ output of a differential equation to a difference approximation depending on discrete time variables $\{a_i\}$ and $\{b_i\}$, make the coefficients depend on "time" / i.e. " n ", making the coefficients dependent on shift - invariant or shift - varying system

DISCRETE-TIME SYSTEMS

DIFF. EQUATIONS

If the coefficients do not depend on $x(n)$ and $y(n)$, then the system is linear. That is, if

the principle of superposition obeys the principle of superposition if $y(n), \underline{does not depend}$ on current output, $y(n), \underline{k > 0}$, then if the input values, $x(n+k), k > 0$, on the far have causal. This also implies that the system is zero for $n < 0$. The impulse response, $h(n)$, is shift invariant

- Examples:
- $y(n) = x^2(n)$ is nonlinear, causal, shift varying.
 - $y(n) = 3^n + 2 x(n)$ is nonlinear, causal.
 - $y(n) = 2 \sin(n) \cdot x(n)$ is linear, causal.

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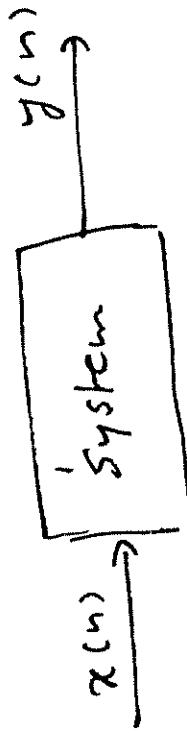
DIFFERENCE EQUATIONS

Examples (continued)

- $y(n+2) + y(n+1) = x(n+5)$ is linear, shift-invariant,
but noncausal (why?)
- $y(n+7) + y(n+5) = x(n+4)$ is linear, shift-invariant,
but causal!
- $y(n) = 2x(n) + x(n) \cdot y(n-1)$ is nonlinear, shift invariant,
causal.
- $y(n) = 2x(2n) + y(n-1)$ is linear, but shift varying
due to $x(2n)$ term.
- $y(n) = 2^n x(n) + y(n-1)$ is linear, shift varying,
causal.
- $y(n) = 2^{x(n)} x(n)$ is nonlinear, shift-invariant,
causal.
- $y(n) = 2 \cdot x(n)$

① DISCRETE-TIME SYSTEMS

DIFFERENCE EQUATIONS



we will consider only linear, shift-invariant systems. When necessary, we will limit our discussion to causal systems.

$$\boxed{\sum_{k=0}^N a_k y(n-k) = \sum_{k=0}^M b_k x(n-k)} \quad -\textcircled{1}$$

which the current system in which the weighted combinations described in (i) weight the current output $y(n)$ depend on (i) weight of $x(n)$, and (ii) "shifted" values of $y(n)$. off the present, and (iii) past values of $y(n)$.

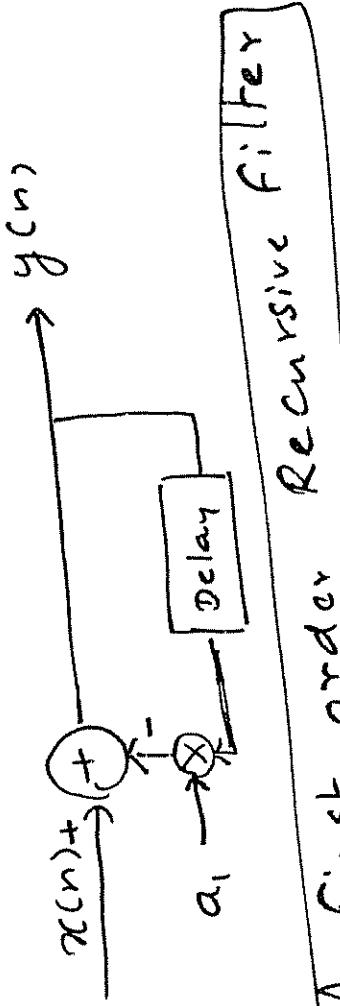


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Such a system is known as a Recursive System.

Example: $y(n) - a_1 y(n-1) = x(n)$



A first order Recursive filter

Recursive filters/systems generally have an impulse response that lasts for all time (although it may become quite small after a reasonable amount of time). Or, the impulse response of an "infinite" amount of time. \Rightarrow ITR

ITR means infinite Impulse Response System

DISCRETE TIME SYSTEMS

DIFFERENCE EQUATIONS

A special form of (1) is :

$$y(n) = \sum_{k=0}^N b_k x(n-k)$$

The output is a weighted sum of current and time shifted input values. It does not depend on the past values of the output, $y(m)$.

Such a system is called nonrecursive system. Such a response of nonrecursive system to an impulse response of n ; if does not last only for finite time. These systems are known as Finite Impulse Response (FIR) systems.

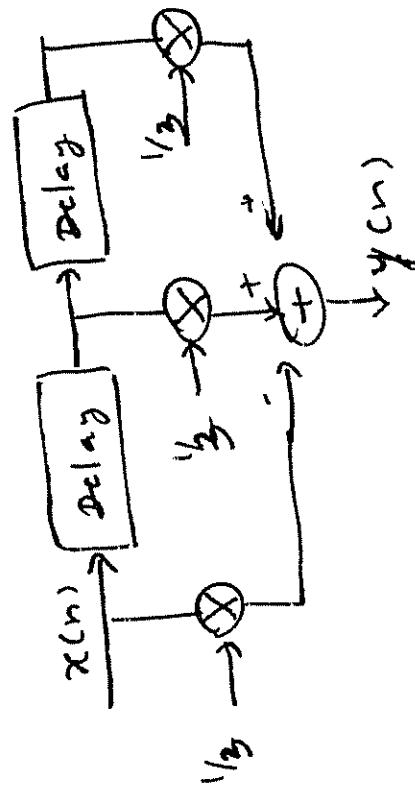
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DIFFERENCE EQUATIONS

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Example:

$$y(n) = \frac{1}{3}x(n) + \frac{1}{3}x(n-1) + \frac{1}{3}x(n-2)$$



FIR FILTER: MOVING AVERAGE FILTER

In general:

Non Recursive \longleftrightarrow FIR
Recursive \longleftrightarrow IIR
 Exceptions: IIR is recursive, but recursive filters may have FIR b(m).

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DIFFERENCE EQUATIONS

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Homogeneous Solution

"Solution" to the homogeneous equation.

We develop the

— (2)

$$\sum_{k=0}^N a_k y(n-k) = 0.$$

Let $y_h^{(n)} = A \cdot d^n$.

This must satisfy (2):

— (3)

$$\sum_{k=0}^N a_k \cdot A \cdot d^{(n-k)} = 0$$

$$a_0 + a_1 d + a_2 d^2 + \dots + a_N d^N = 0 \quad — (4)$$

or $a_0 + a_1 d + a_2 d^2 + \dots + a_N d^N = 0 \quad —$
known as the characteristic Equation,

(4) is which than N-roots.

DISCRETE TIME SYSTEMS

DIFFERENCE EQUATIONS

Homogeneous Solution (continuer).

Let us denote the n roots by $d_1, d_2, d_3, \dots, d_N$.
Let us denote the n roots by distinct or real.
(not all of them may be distinct)

Real & Distinct Roots:

$$y_h(n) = A_1 d_1^n + A_2 d_2^n + \dots + A_N d_N^n$$

$$y_h(n) = A_{P_i} d_i^{n_{P_i}} + A_{P_i+1} d_{P_i+1}^{n_{P_i+1}} + \dots + A_N d_N^n$$

Real, but P_i roots repeated:

$$y_h(n) = A_{P_i} d_i^{n_{P_i}} + A_{P_i+1} d_{P_i+1}^{n_{P_i+1}} + \dots + A_{P_i-1} d_i^{n_{P_i-1}}$$

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DIF. EQU.

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Solutions (continuous)

Homogeneous solutions $\{a_i\}$ are real, we may have:

If the coefficients $\{a_i\}$ are real, we may have:

Complex conjugate root pairs:

Complex conjugate pairs:

$$d_1 = r_1 e^{j\theta}$$

$$\text{Roots: } d_2 = d_1^* = r_1 e^{-j\theta} \text{ to } d_1, d_2, \text{ we will have}$$

Then, Corresponding $y_h(n) = [A_1 \cos n\theta + A_2 \sin n\theta]. r_1^n$. And

$$y_h(n) = [A_1 \cos n\theta + A_2 \sin n\theta + \dots + A_{2k} \cos kn\theta + A_{2k+1} \sin kn\theta]$$

Repeated complex roots shall be handled the same way as repeated real roots.

DISCRETE TIME SYSTEMS

DIFFERENCE EQUATIONS

Homogeneous Solution (cont'd)

Example:

$$y(n) - \frac{1}{6} y(n-1) - \frac{1}{6} y(n-2) = 0$$

The characteristic equation is:

$$1 - \frac{1}{6} \lambda - \frac{1}{6} \lambda^2 = 0$$

$$\text{or } \lambda^2 - \frac{1}{6} \lambda - 1 = 0 \Rightarrow$$

$$\boxed{\lambda_1 = \frac{1}{2}, \quad \lambda_2 = -\frac{1}{3}}$$

The homogeneous solution is:

$y_h(n) = A_1 \left(\frac{1}{2}\right)^n + A_2 \left(-\frac{1}{3}\right)^n$

We solve for A_1 & A_2 from the initial conditions using the values of $y(n)$

DISCRETE TIME SYSTEMS

DIFF. EQUNS.

PARTICULAR SOLUTION:

$$\sum_{k=0}^N a_k y(m-k) = \sum_{k=0}^M b_k x(m-k)$$

The RHS is known and is 'a weighted sum of $x(n)$ and its delayed versions. The particular solution $y_p(n)$ has the same form as the "forcing function" (i.e. the RHS of the equation "form of $y_p(n)$ ".

- Forcing function $\leftrightarrow c_1$ (constant)
- Constant $\leftrightarrow c_1 \beta^n$
- β^n (β is not a root of ch. eqn.) $\leftrightarrow c_1 \sin(n\theta) + c_2 \cos(n\theta)$
- $\sin(n\theta + \phi)$ $\leftrightarrow c_0 + c_1 n + \dots + c_p n^p$
- n^p

$$c_0 + c_1 n + \dots + c_p n^p$$

DISCRETE TIME SYSTEMS

DIFFERENCE EQUATIONS

Particular Solution (continued)

Example: Combination of previous example)

$$\underline{y(n) - \frac{1}{6}y(n-1) - \frac{1}{6}y(n-2) = x(n)}$$

$x(n) = 4u(n)$
 $y_p(n) = c_1$ (because $x(n)$ is constant)
 we assume that $y_p(n) = c_1$ into the difference equation, we
 substituting $y_p(n)$ into the
 $n > 0$

get: $c_1 - \frac{1}{6} - \frac{1}{6} = 4$

$$\Rightarrow \boxed{c_1 = 6} \quad \text{or } y_p(n) = 6.$$

(Read Examples 6.5.4 & 6.5.5)

DISCRETE TIME SYSTEMS

DIFF. EQUNS.

Total Solution:

Total solution, $y(n)$, is sum of homogeneous
and particular solutions.

$$\boxed{y(n) = y_h(n) + y_p(n)}$$

and particular coefficients are determined by initial conditions.

But $y_h(n)$ contains undetermined coefficients.
These are determined by using the
 $x(n) = u(n)$

Example: (continued)

$$y(n) - \frac{1}{6}y(n-1) - \frac{1}{6}y(n-2) = 12.$$

$$y(-1) = 0,$$

$$y(n) = y_h(n) + y_p(n)$$

$$= A_1 \cdot \left(\frac{1}{2}\right)^n + A_2 \left(-\frac{1}{3}\right)^n + 6.$$

DISCRETE TIME SYSTEMS

DIFF. EQUNS.

Total Solution (continued)

Example (continued)

$$y(n) = A_1 \left(\frac{1}{2}\right)^n + A_2 \left(-\frac{1}{3}\right)^n + 6$$

$$y(n) =$$

$$A_1 \cdot \left(\frac{1}{2}\right)^{-1} + A_2 \left(-\frac{1}{3}\right)^{-1} + 6$$

$$\Rightarrow \boxed{2A_1 - 3A_2 = -6}$$

$$-2 \quad -2 \quad = 12$$

$$n = -2$$

$$\boxed{4A_1 + 9A_2 = 6}$$

$$A_2 = 1 \cdot 2$$

$$A_1 = -1 \cdot 2$$

Thus,

$$y(n) = -1 \cdot 2 \cdot \left(\frac{1}{2}\right)^n + 1 \cdot 2 \cdot \left(-\frac{1}{3}\right)^n + 6$$

$$n > 0$$