

DIFFERENCE EQUATIONS

ZERO-input & zero-state solutions

- * If we set $x(n) = 0$ and obtain the solution, we have the zero-input response. To obtain $y_h^{(n)}$, solve for $y_h^{(n)}$ using the initial conditions. It is also called the natural response.
- * If the initial conditions are set to zero, i.e. the system is relaxed) and we obtain the zero-state equation, it is called forced response or forced response, $y_{zs}^{(n)}$. The difference equation, $y_{zs}^{(n)}$, and solve for the responses $y_h^{(n)}$ and $y_p^{(n)}$, and obtain $y^{(n)} = y_h^{(n)} + y_p^{(n)}$ under initial conditions. You may have to with zero initial conditions. You may have to generate the first few values of $y_h^{(n)}$ from the diff. equation.
- * $\rightarrow y^{(n)} = y_{zs}^{(n)} + y_h^{(n)} + y_p^{(n)}.$

DIFFERENCE EQUATIONS

IMPULSE RESPONSE

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Impulse response is obtained by setting $x(n) = \delta(n)$,

- & zeroing out all initial conditions (i.e. relaxed system).
- & with $x(n) = \delta(n)$, $y_p(n) = 0$.
- & thus, $h(n) = y_u(n)$ which contains undetermined coefficients.
- & To determine the initial conditions, we generate the first few values (initial condition) of $h(n)$ recursively from the difference equation of $h(n)$ satisfied by $h(n)$. From the fact it is to be satisfied by $h(n)$, we solve for the resulting linear equations.
- & undetermined coefficients.
- & we illustrate this with an example.

DIFFERENCE EQUATIONS
IMPULSE RESPONSE
EXAMPLE

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Example (continued):

Find the impulse response of the causal LTI system

described by the LDE:

$$y(n) - \frac{1}{6}y(n-1) - \frac{1}{6}y(n-2) = 4x(n) - x(n-1)$$

$y(-1) = y(-2) = 0$ (relaxed system).

* we get $y(-1) = y(-2) = 0$.

* we find the difference equation:

$$h(n) - \frac{1}{6}h(n-1) - \frac{1}{6}h(n-2) = 4s(n) - s(n-1)$$

$$h(-1) = h(-2) = 0.$$

Characteristic Equation:

$$1 - \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2} = 0 \Rightarrow \alpha_1 = \frac{1}{2}, \alpha_2 = -\frac{1}{3}$$

$$h(n) = [k_1(\frac{1}{2})^n + k_2(-\frac{1}{3})^n] u(n)$$

DIFFERENCE EQUATIONS

Impulse Response Equations

Example (continued)

$h(n)$ contains coeffs k_1 & k_2 which need to be determined. Using $h(-1) = h(-2) = 0$ will yield bivial residual equation itself from the difference equation

$$\begin{aligned} n=0 \quad h(0) - \frac{1}{6} \cancel{h(-1)} - \frac{1}{6} \cancel{h(-2)} &= 4 \quad \Rightarrow h(0) = 4 \\ n=1 \quad 4 - \frac{1}{6} \cancel{h(0)} - \frac{1}{6} \cancel{h(-1)} &= -1 \quad \Rightarrow h(1) = -\frac{1}{3}. \end{aligned}$$

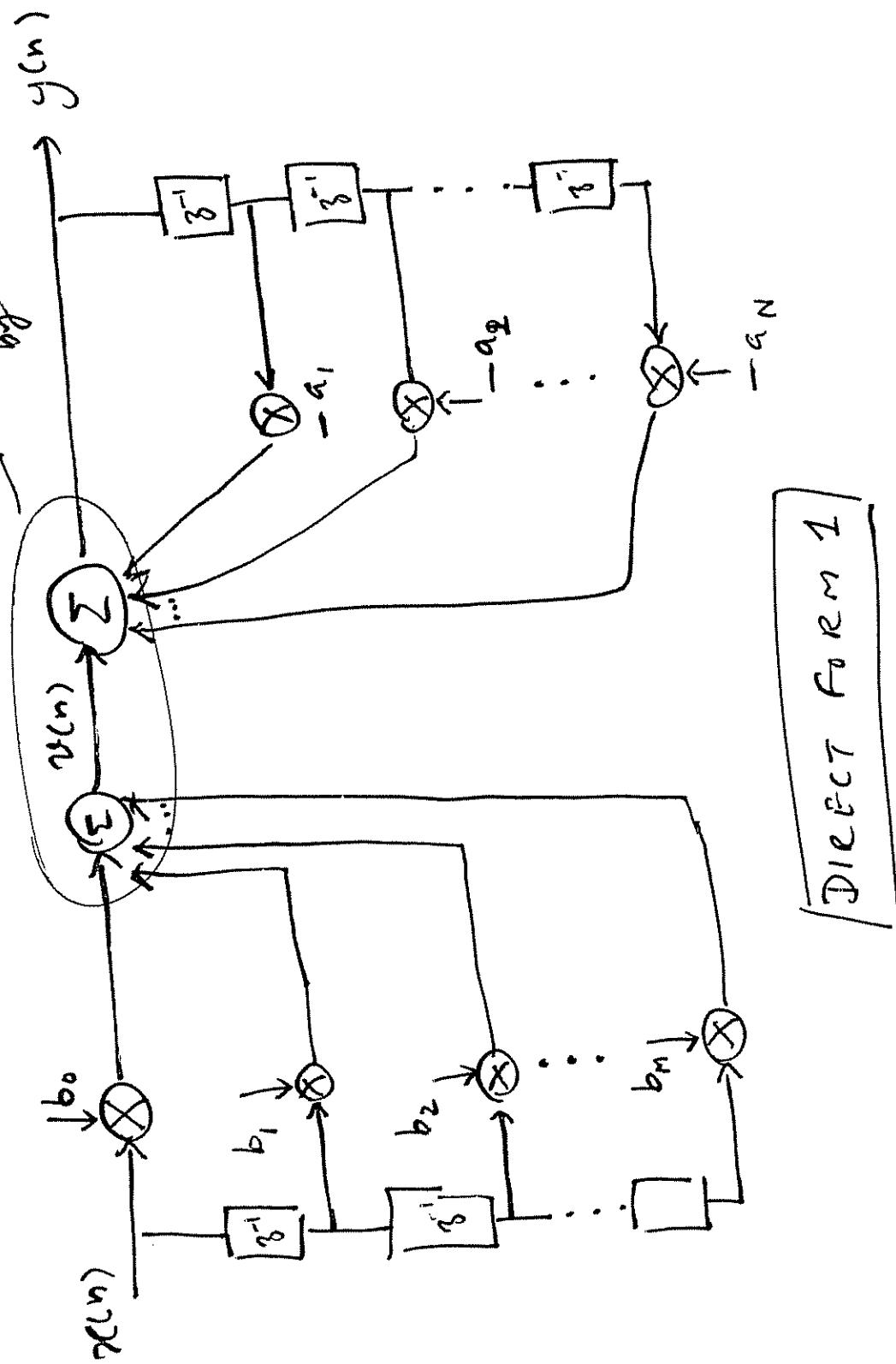
$$\begin{aligned} n=1 \quad h(1) - \frac{1}{6} h(0) + \frac{1}{3} h(-1) &= \\ &\quad \text{But } h(0) = k_1 + k_2 \\ h(1) = k_1 \left(\frac{1}{2}\right) + k_2 \left(-\frac{1}{3}\right) & \end{aligned}$$

$$\left. \begin{aligned} k_1 + k_2 &= 4 \\ \frac{k_1}{2} - \frac{k_2}{3} &= -1/3 \end{aligned} \right\} \Rightarrow \boxed{\begin{array}{l} k_1 = 1 \cdot 2 \\ k_2 = 2 \cdot 8 \end{array}} \quad \boxed{h(n) = \left[1 \cdot 2 \left(\frac{1}{2}\right)^n + 2 \cdot 8 \left(-\frac{1}{3}\right)^n \right] u(n)}$$

IMPLEMENTATION OF LCCDE

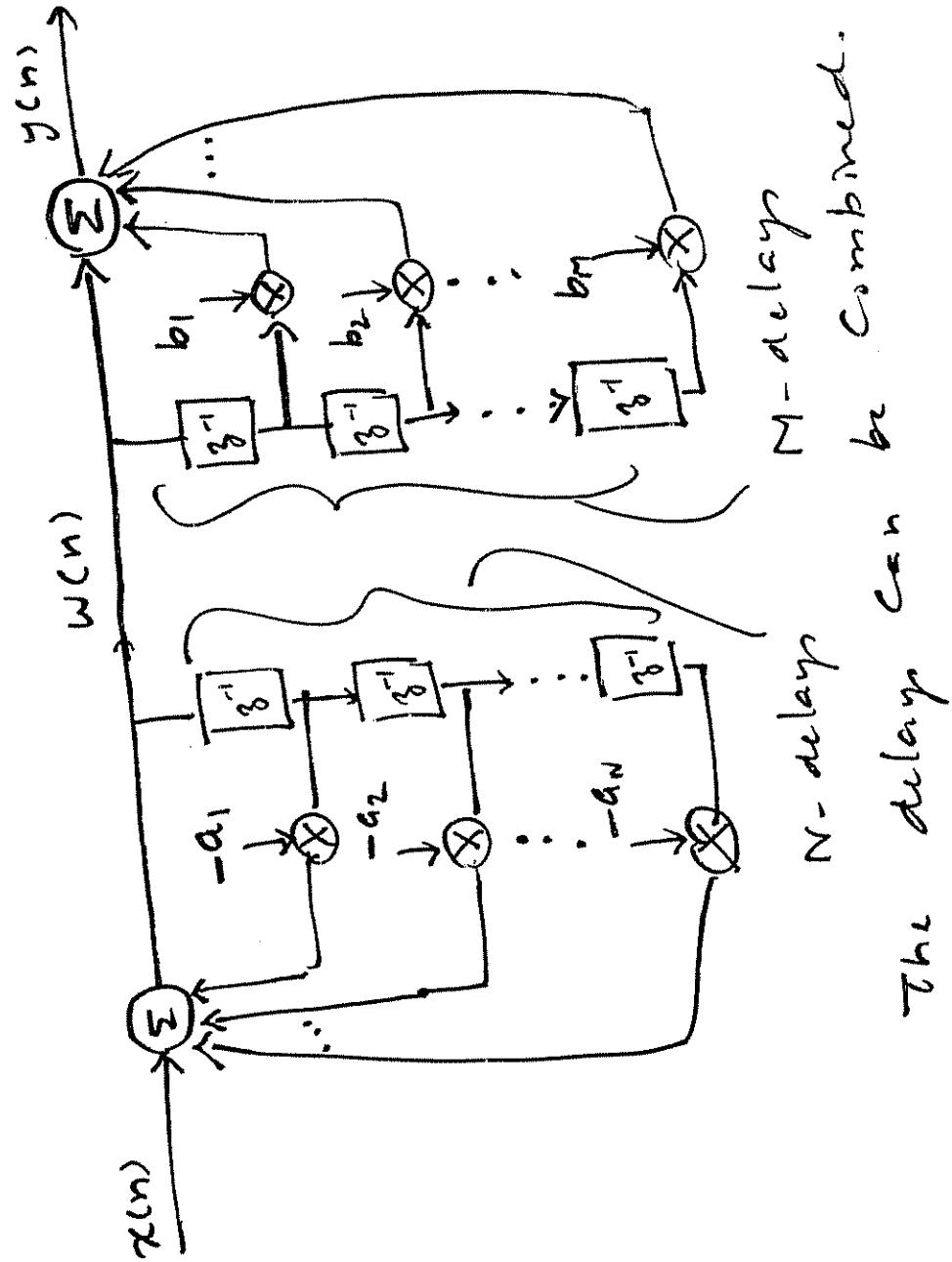
$$y(n) = - \sum_{k=1}^N a_k y(n-k) + \sum_{k=0}^n b_k x(n-k)$$

can be replaced by



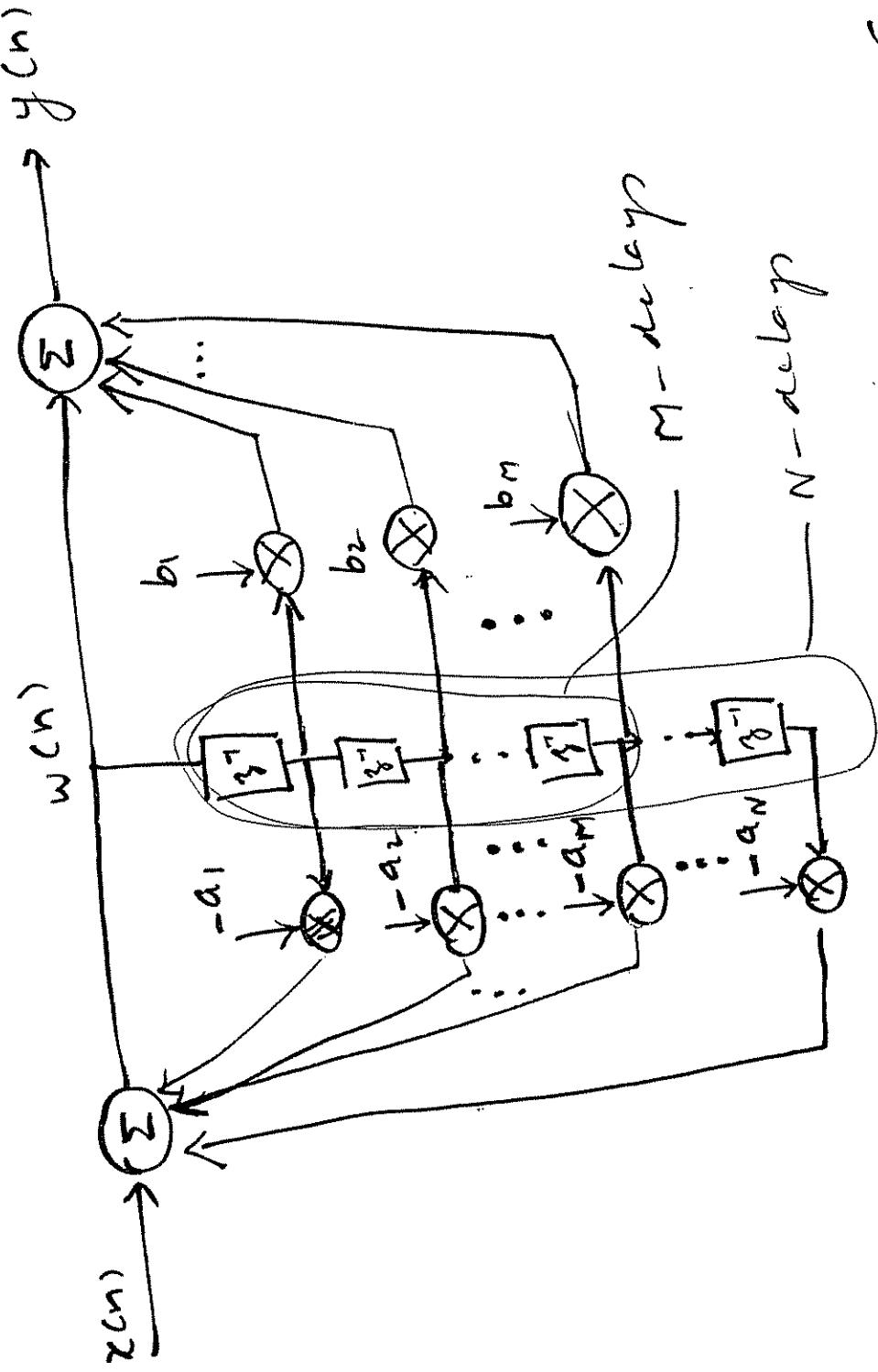
IMPLEMENTATION OF LCCDE DIRECT FORM 2 (CANONICAL FORM)

By using the commutative and associative property of convolution, direct form 1 can be rearranged as below.



IMPLEMENTATION OF LCCDE
DIRECT FORM 2 (CANONICAL)

Assuming $N > M$, and sharing the delay we get:



Storage ("delays") are minimized \Rightarrow Canonical

DISCRETE SYSTEMS - LT

BIBO STABILITY

- Impulse response must be absolutely summable.
- $\sum_{k=-\infty}^{\infty} |h(k)| < \infty$ [Fouier transform exists]
- in the absolute convergent sense.

$$h(n) = \sum_{k=1}^N a_k \alpha^k$$

a_k are roots of characteristic equation

$$\Rightarrow |a_k| < 1 \quad k=1, 2 \dots N$$

FREQUENCY ANALYSIS
OF LTI SYSTEMS

$$y(n) = \sum_{k=-\infty}^{\infty} h(k) x(n-k)$$

Let $x(n) = e^{j\omega n}$, i.e. complex exponential ("dimensionic")

$$\begin{aligned} y(n) &= \sum_{k=-\infty}^{\infty} h(k) e^{j\omega(n-k)} \\ &= e^{j\omega n} \cdot \underbrace{\sum_{k=-\infty}^{\infty} h(k) e^{-jk\omega}}_{H(\omega)} \end{aligned}$$

Eigen function.
(?)

Fourier transform
of $h(n)$.

Sundaray applied Complex Exponential

$$\begin{aligned} x(n) &= e^{j\omega n} \cdot u(n) \\ y(n) &= \sum_{k=0}^{\infty} h(k) e^{j\omega(n-k)} \end{aligned}$$

Free Analysis of LTI System

$$y(n) = \sum_{k=-\infty}^{\infty} h(k) \cdot e^{j\omega(n-k)} - \sum_{k=-\infty}^{-1} h(k) \cdot e^{j\omega(n-k)}$$

$$= H(\omega) \cdot e^{j\omega n} - e^{j\omega n} \left[\sum_{k=-\infty}^{-1} h(k) e^{-jk\omega} \right]$$

↓
 Steady state
 ↑ Transient
 (For a stable system,
 this will "die down"
 as $n \rightarrow \infty$)

CORRELATION

Energy Signals

$$\begin{aligned}
 r_{xy}(l) &= x(l) * y(-l) && \rightarrow \text{Cross correlation} \\
 &= \sum_{n=-\infty}^{\infty} x(n)y(n-l) \\
 r_{xx}(l) &= x(l) * x(-l) && \leftarrow \text{Auto-correlation}
 \end{aligned}$$

Power Signals

$$\begin{aligned}
 r_{xy}(l) &= \lim_{M \rightarrow \infty} \frac{1}{2^{M+1}} \sum_{n=-M}^M x(n)y(n-l) \\
 r_{xx}(l) &= \lim_{M \rightarrow \infty} \frac{1}{2^{M+1}} \sum_{n=-M}^M x(n)x(n-l)
 \end{aligned}$$

CORRELATION (Continued)

Periodic Signals:

$$\sum_{n=0}^{N-1} x(n) y(n-\ell)$$

$$r_{xy}(\ell) = \frac{1}{N}$$

$$\sum_{n=0}^{N-1} x(n) x(n-\ell)$$

$$r_{xx}(\ell) = \frac{1}{N}$$

\downarrow
Periodic

Reading Assignment: Section 2.6