DISCRETE-TIME SIGNALS AND SYSTEMS

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RESTON PUBLISHING COMPANY, INC.
A Prentice-Hall Company
Reston, Virginia

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Elements of Difference Equations

In this chapter we present an elementary discussion of linear difference equations with constant coefficients. Our motivation for doing so is that such difference equations will be used in Chapter 5 to describe and analyze discrete-time (DT) systems. Two methods for solving this class of difference equations will be included in this chapter, while a third method will be discussed in Chapter 3.

2.1 INTRODUCTORY REMARKS

The notion of linear difference equations with constant coefficients is best introduced by means of the simple resistive network that is shown in Fig. 2.1-1, where $V(n)$ denotes the voltage at the nth node, for $-2 \leq n \leq 3$. We wish to describe this network by means of a difference equation. To this end, we consider a typical section (below Fig. 2.1-1) of this network, where I_1, I_2, and I_3 denote currents leaving the node $n - 1$. Application of Kirchhoff's current law to node $n - 1$ leads to the equation

$$I_1 + I_2 + I_3 = 0$$

Substituting for I_1, I_2, and I_3 in the preceding equation, we obtain

$$\frac{V(n-1) - V(n)}{1} + \frac{V(n-1) - V(n-2)}{1} + \frac{V(n-1) - 0}{1} = 0$$

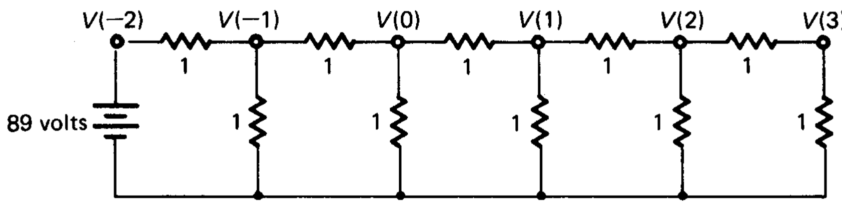
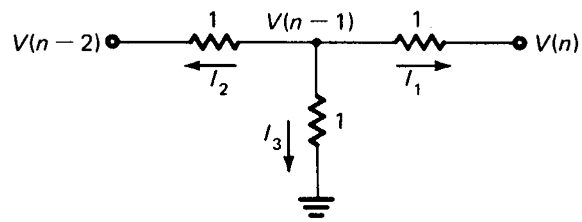


Fig. 2.1-1 Resistance network; each element value is 1 ohm.



which simplifies to yield

$$V(n) - 3V(n-1) + V(n-2) = 0, \qquad 0 \le n \le 3 \qquad (2.1\text{-}1)$$

Equation (2.1-1) is the desired difference equation that describes the network in Fig. 2.1-1 in terms of its node voltages. We observe that it is a *second-order* difference equation since the voltage at node n [i.e., $V(n)$] is expressed as a linear combination of the voltages at *two* previous node voltages $V(n-1)$ and $V(n-2)$.

2.2 SOLUTION OF DIFFERENCE EQUATIONS

A logical question that arises at this point is how one can solve (2.1-1) to obtain $V(n)$. Since (2.1-1) represents a second-order difference equation, we would require *two* known voltages, say $V(-2)$ and $V(-1)$, to obtain the rest. To illustrate,

$$V(-2) = 89 \text{ volts}$$

and

$$V(-1) = 34 \text{ volts}$$

Then a simple procedure for obtaining the remaining $V(n)$ for $0 \le n < 3$ would be a *recursive* method, since (2.1-1) implies that

$$V(n) = 3V(n-1) - V(n-2), \qquad 0 \le n \le 3 \qquad (2.2\text{-}1)$$

With $n = 0, 1, 2$, and 3, (2.2-1) yields the desired voltages to be as follows:

$$V(0) = 3V(-1) - V(-2) = 13 \text{ volts}$$
$$V(1) = 3V(0) - V(-1) = 5 \text{ volts}$$
$$V(2) = 3V(1) - V(0) = 2 \text{ volts}$$

and
$$V(3) = 3V(2) - V(1) = 1 \text{ volt}$$

We shall refer to the preceding scheme as the *recursive method* for solving difference equations. It is observed that, although this method yields each $V(n)$ in a simple recursive manner, it does not provide a *closed-form* solution, that is, a solution which yields $V(n)$ without having to first compute $V(0), V(1), \ldots, V(n-1)$. If a closed-form solution is desired, one can solve difference equations using the *method of undetermined coefficients*, which parallels the classical method of solving linear differential equations with constant coefficients.

Method of Undetermined Coefficients

We illustrate this method via examples. Suppose we seek the general solution of the second-order difference equation

$$y(n) - \frac{5}{6}y(n-1) + \frac{1}{6}y(n-2) = 5^{-n}, \qquad n \geq 0 \qquad (2.2\text{-}2)$$

with initial conditions $y(-2) = 25$ and $y(-1) = 6$.

In (2.2-2), $y(n)$ may be interpreted as the response (output) of a DT system to the input (forcing) function 5^{-n} for $n \geq 0$, where n is a time index. It is apparent that (2.2-2) is a second-order difference equation since it expresses the output $y(n)$ at time n as a linear combination of *two* previous outputs $y(n-1)$ and $y(n-2)$.

The general (or closed-form) solution $y(n)$ of (2.2-2) is obtained in three steps that are similar to those used for solving second-order differential equations. They are as follows:

1. Obtain the *complementary solution* $y_c(n)$ in terms of two arbitrary constants c_1 and c_2.
2. Obtain the *particular solution* $y_p(n)$, and write

$$y(n) = y_c(n) + y_p(n) = f(c_1,c_2) + y_p(n) \qquad (2.2\text{-}3)$$

where $y_c(n) = f(c_1, c_2)$ implies that $y_c(n)$ is a function of c_1 and c_2.

3. Solve for c_1 and c_2 in (2.2-3) using two given initial conditions.

In what follows, we elaborate on the preceding steps.

STEP 1. We assume that the complementary solution $y_c(n)$ has the form

$$y_c(n) = c_1 a_1^n + c_2 a_2^n \tag{2.2-4}$$

where the a_i are real constants.

Next substitute $y(n) = a^n$ in the homogeneous equation to get

$$a^n - \frac{5}{6} a^{n-1} + \frac{1}{6} a^{n-2} = 0 \tag{2.2-5}$$

Dividing both sides of (2.2-5) by a^{n-2}, we obtain

$$a^2 - \frac{5}{6} a + \frac{1}{6} = 0$$

or

$$\left(a - \frac{1}{2}\right)\left(a - \frac{1}{3}\right) = 0$$

which yields the *characteristic roots*

$$a_1 = \frac{1}{2} \quad \text{and} \quad a_2 = \frac{1}{3}$$

Thus the complementary solution is

$$y_c(n) = c_1 2^{-n} + c_2 3^{-n}$$

where c_1 and c_2 are arbitrary constants.

STEP 2. The particular solution $y_p(n)$ is assumed to be

$$y_p(n) = c_3 5^{-n}$$

since the forcing function is 5^{-n}; see (2.2-2).

Substitution of $y(n) = y_p(n) = c_3 5^{-n}$ in (2.2-2) leads to

$$c_3[5^{-n} - \left(\frac{5}{6}\right)5^{-(n-1)} + \left(\frac{1}{6}\right)5^{-(n-2)}] = 5^{-n}$$

Dividing both sides of this equation by 5^{-n}, we obtain

$$c_3\left[1 - \left(\frac{5}{6}\right)5 + \left(\frac{1}{6}\right)5^2\right] = 1$$

which implies that $c_3 = 1$. Thus

$$\begin{aligned} y(n) &= y_c(n) + y_p(n) \\ &= c_1 2^{-n} + c_2 3^{-n} + 5^{-n} \end{aligned} \tag{2.2-6}$$

STEP 3. Since the initial conditions are

$$y(-2) = 25 \quad \text{and} \quad y(-1) = 6$$

(2.2-6) yields the simultaneous equations

$$4c_1 + 9c_2 = 0 \tag{2.2-7}$$

and

$$2c_1 + 3c_2 = 1$$

Solving (2.2-7) for c_1 and c_2, we obtain

$$c_1 = \frac{3}{2} \quad \text{and} \quad c_2 = -\frac{2}{3}$$

Thus the desired general solution is given by (2.2-6) to be

$$y(n) = \frac{3}{2}(2^{-n}) - \frac{2}{3}(3^{-n}) + 5^{-n}, \qquad n \geq 0 \tag{2.2-8}$$

As mentioned earlier, $y(n)$ can be interpreted as the output of a DT system when it is subjected to the exponential input (forcing function) 5^{-n}, which is the right-hand side of the given difference equation in (2.2-2).

RULES FOR CHOOSING PARTICULAR SOLUTIONS. As is the case with the solution of differential equations, there are a set of rules one must follow to form appropriate particular solutions while solving difference equations, as summarized in Table 2.2-1. For example, the form of the particular solution related to the difference equation in (2.2-2) was $c_3 5^{-n}$, which agrees with line 3 of Table 2.2-1. We will illustrate the use of this table by means of more examples.

Table 2.2-1 Rules for Choosing Particular Solutions

Terms in forcing function	Choice of particular solution†
1. A constant	c; c is a constant
2. $b_1 n^k$; b_1 is a constant	$c_0 + c_1 n + c_2 n^2 + \cdots + c_k n^k$; the c_i are constants
3. $b_2 d^{\pm n}$; b_2 and d are constants	Proportional to $d^{\pm n}$
4. $b_3 \cos(n\omega)$ } b_3 and b_4 are constants	$c_1 \sin(n\omega) + c_2 \cos(n\omega)$
5. $b_4 \sin(n\omega)$ } b_3 and b_4 are constants	

†If a term in any of the particular solutions in this column is a part of the complementary solution, it is necessary to modify the corresponding choice by multiplying it by n before using it. If such a term appears r times in the complementary solution, the corresponding choice must be multiplied by n^r.

Example 2.2-1: Solve the second-order difference equation

$$y(n) - \frac{3}{2}y(n-1) + \frac{1}{2}y(n-2) = 1 + 3^{-n}, \qquad n \geq 0 \qquad (2.2\text{-}9)$$

with the initial conditions $y(-2) = 0$ and $y(-1) = 2$.

Solution: The solution consists of three steps.

STEP 1. Assume the complementary solution as $y_c(n) = c_1 a_1^n + c_2 a_2^n$. Substituting $y(n) = a^n$ in the homogeneous counterpart of (2.2-9), we obtain the characteristic equation

$$a^2 - \frac{3}{2}a + \frac{1}{2} = 0$$

the roots of which are $a_1 = \dfrac{1}{2}$ and $a_2 = 1$.
Thus

$$y_c(n) = c_1 2^{-n} + c_2 1^n = c_1 2^{-n} + c_2 \qquad (2.2\text{-}10)$$

STEP 2. To choose an appropriate particular solution, we refer to Table 2.2-1. From the given forcing function and lines 1 and 3 of Table 2.2-1, it follows that a choice for the particular solution is $c_3 + c_4 3^{-n}$. However, we observe that this choice for the particular solution and $y_c(n)$ in (2.2-10) have common terms, each of which is a constant; that is, c_3 and c_2, respectively. Thus in accordance with the footnote of Table 2.2-1, we modify the choice $c_3 + c_4 3^{-n}$ to obtain

$$y_p(n) = c_3 n + c_4 3^{-n} \tag{2.2-11}$$

Next, substitution of $y_p(n)$ in (2.2-11) into (2.2-9) leads to

$$c_3 n + c_4 3^{-n} - \frac{3}{2} c_3 n + \frac{3}{2} c_3 - \frac{9}{2} c_4 3^{-n} + \frac{1}{2} c_3 n - c_3 + \frac{9}{2} c_4 3^{-n} = 3^{-n} + 1 \tag{2.2-12}$$

From (2.2-12) it follows that

$$\frac{1}{2} c_3 = 1$$

and

$$c_4\left[1 - \frac{9}{2} + \frac{9}{2}\right] 3^{-n} = 3^{-n}$$

which results in

$$c_3 = 2 \quad \text{and} \quad c_4 = 1$$

Thus, combining (2.2-10) and (2.2-11), we get

$$y(n) = c_1 2^{-n} + c_2 + 2n + 3^{-n} \tag{2.2-13}$$

STEP 3. To evaluate c_1 and c_2 in (2.2-13), the given initial conditions are used; that is, $y(-2) = 0$ and $y(-1) = 2$. This leads to the simultaneous equations

$$4c_1 + c_2 = -5$$

$$2c_1 + c_2 = 1$$

Solving, we obtain $c_1 = -3$ and $c_2 = 7$, which yields the desired solution as

$$y(n) = (-3)2^{-n} + 7 + 2n + 3^{-n}, \qquad n \geq 0$$

Example 2.2-2: Find the general solution of the first-order difference equation

$$y(n) - 0.9y(n-1) = 0.5 + (0.9)^{n-1}, \qquad n \geq 0 \qquad (2.2\text{-}14)$$

with $y(-1) = 5$.

Solution:

STEP 1. Substituting $y(n) = a^n$ in the homogeneous equation

$$y(n) - 0.9y(n-1) = 0$$

we obtain

$$y_c(n) = c_1(0.9)^n \qquad (2.2\text{-}15)$$

since we are dealing with a first-order difference equation.

STEP 2. From the forcing function in (2.2-14), the complementary solution in (2.2-15), and lines 1 and 3 of Table 2.2-1, it follows that

$$y_p(n) = c_2 n(0.9)^n + c_3$$

Substitution of $y(n) = y_p(n)$ in (2.2-14) results in

$$c_3 + c_2 n(0.9)^n - 0.9c_2(n-1)(0.9)^{n-1} - 0.9c_3 = 0.5 + (0.9)^{n-1}$$

which leads to

$$0.1c_3 = 0.5$$

and

$$(0.9)^n c_2 = (0.9)^{n-1}$$

Thus we have

$$c_3 = 5 \quad \text{and} \quad c_2 = \frac{10}{9}$$

which implies that

$$y_p(n) = \frac{10}{9} n(0.9)^n + 5 \qquad (2.2\text{-}16)$$

Combining (2.2-15) and (2.2-16), we get

$$y(n) = c_1(0.9)^n + \frac{10}{9} n(0.9)^n + 5 \qquad (2.2\text{-}17)$$

STEP 3. From (2.2-17) and the initial condition $y(-1) = 5$, it follows that $c_1 = \frac{10}{9}$. Hence the desired solution can be written as

$$y(n) = (n + 1)(0.9)^{n-1} + 5, \qquad n \geq 0$$

Example 2.2-3: Find the general solution of the second-order difference equation

$$y(n) - 1.8y(n - 1) + 0.81y(n - 2) = 2^{-n}, \qquad n \geq 0 \qquad (2.2\text{-}18)$$

Leave the answer in terms of unknown constants, which one can evaluate if the initial conditions are given.

Solution:

STEP 1. With $y(n) = a^n$ substituted into the homogeneous counterpart of (2.2-18), we obtain

$$a^2 - 1.8a + 0.81 = 0$$

which results in the *repeated roots*

$$a_1 = a_2 = 0.9$$

Thus, as in the case of differential equations, we consider the complementary solution to be

$$y_c(n) = c_1(0.9)^n + c_2 n(0.9)^n \qquad (2.2\text{-}19)$$

STEP 2. From the given forcing function in (2.2-18), $y_c(n)$ in (2.2-19), and line 3 of Table 2.2-1, it is clear that

$$y_p(n) = c_3 2^{-n} \qquad (2.2\text{-}20)$$

Substitution of (2.2-20) in (2.2-18) leads to

$$c_3[1 - (3)(6) + (3)(24)]2^{-n} = 2^{-n}$$

which yields $c_3 = \frac{25}{16}$. Thus the desired solution is given by (2.2-19) and (2.2-20) to be

$$y(n) = c_1(0.9)^n + c_2 n(0.9)^n + \left(\frac{25}{16}\right)2^{-n}$$

where c_1 and c_2 can be evaluated if two initial conditions are specified.

Example 2.2-4: Find the particular solution for the first-order difference equation

$$y(n) - 0.5y(n-1) = \sin\left(\frac{n\pi}{2}\right), \qquad n \geq 0 \tag{2.2-21}$$

Solution: Since the forcing function is sinusoidal, we refer to line 5 of Table 2.2-1 and choose a particular solution of the form

$$y_p(n) = c_1 \sin\left(\frac{n\pi}{2}\right) + c_2 \cos\left(\frac{n\pi}{2}\right) \tag{2.2-22}$$

Substitution of $y(n) = y_p(n)$ in (2.2-21) leads to

$$\begin{aligned} c_1 \sin\left(\frac{n\pi}{2}\right) + c_2 \cos\left(\frac{n\pi}{2}\right) - 0.5c_1 \sin\left[\frac{(n-1)\pi}{2}\right] \\ - 0.5c_2 \cos\left[\frac{(n-1)\pi}{2}\right] = \sin\left(\frac{n\pi}{2}\right) \end{aligned} \tag{2.2-23}$$

We now use the following identities:

$$\begin{aligned} \sin\left[\frac{(n-1)\pi}{2}\right] &= \sin\left(\frac{n\pi}{2} - \frac{\pi}{2}\right) = -\cos\left(\frac{n\pi}{2}\right) \\ \cos\left[\frac{(n-1)\pi}{2}\right] &= \cos\left(\frac{n\pi}{2} - \frac{\pi}{2}\right) = \sin\left(\frac{n\pi}{2}\right) \end{aligned} \tag{2.2-24}$$

Substituting (2.2-24) in (2.2-23), we obtain

$$(c_1 - 0.5c_2) \sin\left(\frac{n\pi}{2}\right) + (0.5c_1 + c_2) \cos\left(\frac{n\pi}{2}\right) = \sin\left(\frac{n\pi}{2}\right)$$

which yields the simultaneous equations

$$\begin{aligned} c_1 - 0.5c_2 &= 1 \\ 0.5c_1 + c_2 &= 0 \end{aligned} \tag{2.2-25}$$

The solution of (2.2-25) yields $c_1 = \frac{4}{5}$ and $c_2 = -\frac{2}{5}$. Hence the desired result is given by (2.2-22) to be

$$y_p(n) = \frac{4}{5}\sin\left(\frac{n\pi}{2}\right) - \frac{2}{5}\cos\left(\frac{n\pi}{2}\right), \qquad n \geq 0$$

Example 2.2-5: In Example 2.2-3, suppose the forcing function is $(0.9)^n$ instead of 2^{-n}, $n \geq 0$. What would be the appropriate choice for the particular solution?

Solution: The complementary solution is given by (2.2-19) to be

$$y_c(n) = c_1(0.9)^n + c_2n(0.9)^n$$

Since the forcing function is $(0.9)^n$, line 3 of Table 2.2-1 implies that a choice for the particular solution is $c_3(0.9)^n$. However, since this choice and the preceding $y_c(n)$ have a term in common, we must modify our choice according to the footnote of Table 2.2-1 to obtain $c_3n(0.9)^n$. But this choice again has a term in common with $y_c(n)$. Thus we apply to the footnote of Table 2.2-1 once again to obtain

$$y_p(n) = c_3n^2(0.9)^n \tag{2.2-26}$$

which has no more terms in common with $y_c(n)$.

Hence $y_p(n)$ in (2.2-26) is the appropriate choice for the particular solution for the difference equation in (2.2-18) when the forcing function is $(0.9)^n$; that is,

$$y(n) - 1.8y(n-1) + 0.81y(n-2) = (0.9)^n, \qquad n \geq 0$$

2.3 SUMMARY

Our treatment of linear difference equations with constant coefficients in this chapter was confined to first- and second-order difference equations. Higher-order difference equations of this type will be considered in Chapters 3 and 5. Although our interest in such difference equations is restricted to DT systems as they relate to electrical engineering, they have a variety of applications in diverse areas such as economics, psychology, and sociology. The interested reader may refer to [3] for more details.