

Periodic Extension

A signal can be used to generate a periodic signal by repeating it at multiples of a chosen fundamental period, and adding all the replicas to the original.

Let $x(n)$ be a sequence, and N , any positive integer. Then

$$x_p(n) = \sum_{k=-\infty}^{\infty} x(n-kN) \quad \begin{matrix} \text{is periodic with a} \\ \text{fundamental period of} \\ N \text{ samples} \end{matrix}$$

$x_p(n)$ is known as the N -point periodic extension of $x(n)$. Often the notation

$x((n))_N$ is used to denote the periodic extension, and may also be interpreted as " $x(n)$ modulus N ". Thus, it represents one period of the periodic signal, which describes it for $-\infty < n < \infty$.

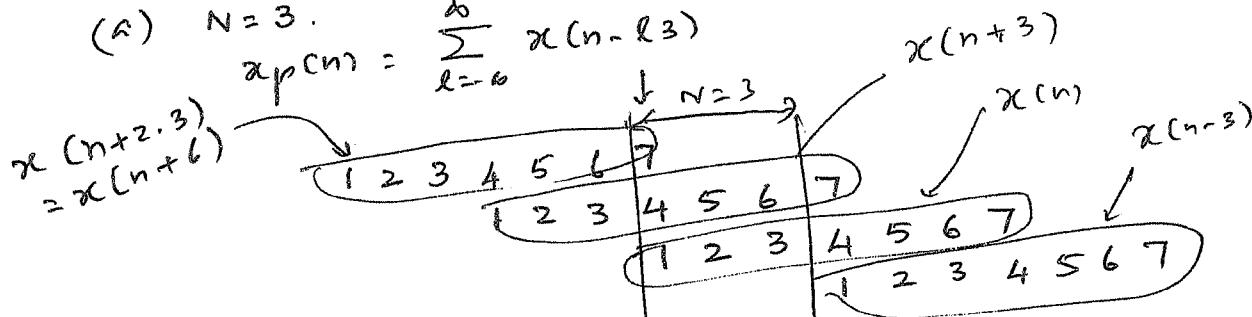
Note that if $x((n-n_0))_N = x((n-n_0+N))_N$, so that the inner argument falls between 0 and N (both inclusive). [Why?].

Example:

$$1. \quad x(n) = \{ \underset{1}{\uparrow} \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \}$$

$$(A) \quad N=3.$$

$$x_p(n) = \sum_{k=-\infty}^{\infty} x(n-k3)$$



Add

$$1 \ 2 \ 7 \ 9$$

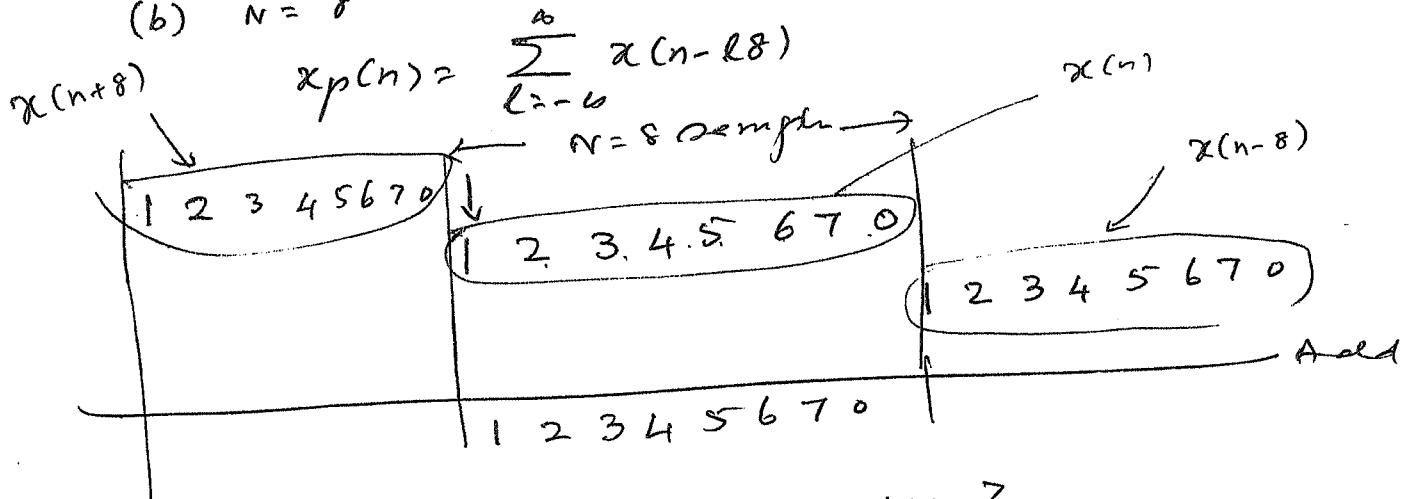
$$\Rightarrow \boxed{x((n))_3 = \{1 \ 2 \ 7 \ 9\}}.$$

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Example 1 (continued)

$$(b) N = 8$$



$$\Rightarrow x((n))_8 = \{1, 2, 3, 4, 5, 6, 7, 0\}$$

Example 2: Let $x(n) = d^n u(n)$ with $|d| < 1$.

Let n be any positive integer.

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN) = \sum_{l=-\infty}^{\infty} d^{n-lN} u(n-lN)$$

To obtain $x((n))_N$, we note that $0 \leq n \leq N-1$,

and for this choice of n ,

$$u(n-lN) = \begin{cases} 1, & l \leq 0 \\ 0, & l > 0 \end{cases} \quad \text{with } 0 \leq n \leq N-1$$

$$\Rightarrow x_p(n) = \sum_{l=-\infty}^0 d^{n-lN} = \frac{d^n}{1-d^N}.$$

$$\Rightarrow \boxed{x((n))_N = \frac{d^n}{1-d^N}}.$$

PERIODIC CONVOLUTION

Let $x_p(n)$ and $h_p(n)$ be periodic signals with a common period of N samples. Convolution in the usual sense does not converge, and therefore does not exist.

We define the periodic convolution of $x_p(n)$ and $h_p(n)$ as follows:

$$\begin{aligned} y_p(n) &= \sum_{k=0}^{N-1} x_p(k) h_p(n-k) \\ &= \sum_{k=0}^{N-1} h_p(k) x_p(n-k), \quad 0 \leq n \leq N-1 \\ &= x_p(n) \circledcirc h_p(n) = h_p(n) \circledcirc x_p(n). \end{aligned} \quad (1)$$

Where \circledcirc refers to the periodic convolution with the period = N samples.

Note: 1. $y_p(n)$ is periodic with N -samples

2. To compute $y_p(n)$ for any n , $n=0, 1, 2, \dots, (N-1)$, the same # of terms (same # of MACs) are used.

3. To compute negative values of argument of $x_p(n)$ or $h_p(n)$, we invoke the periodicity of the signals. (i.e. $x_p(n-n_0) = x_p(n-n_0+LN)$ where $(n-n_0) < 0$, and "L" is such that $(n-n_0+LN)$ is in the range, $[0, N-1]$. [Example: $x_p(-4) = x_p(-4+10) = x_p(6)$ with $N=10$ samples].

Alternate form of (1):

$$\begin{aligned} y_p(n) &= \sum_{k=0}^{N-1} x(k) h((n-k))_N \\ &= \sum_{k=0}^{N-1} h(k) x((n-k))_N \quad 0 \leq n \leq N-1 \end{aligned} \quad (2)$$

(Contd.)

PERIODIC CONVOLUTION (Cont'd)

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where $x(n)$ and $h(n)$ are one-periodic
exten~~sion~~ of $x_p(n)$ and $h_p(n)$ respectively.

Computing the Periodic Convolution:

Method 1. Circulant Matrix Method:

$$\begin{bmatrix} x(0) & x(n-1) & \dots & x(1) \\ x(1) & x(0) & & x(2) \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & & \vdots \\ x(n-1) & x(n-2) & & x(0) \end{bmatrix} \begin{bmatrix} h(0) \\ h(1) \\ \vdots \\ h(n-1) \end{bmatrix} = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(n-1) \end{bmatrix}$$

$y(n)$ is one-period ($n=0, 1, 2 \dots n-1$) of $y_p(n)$.

Method 2: Linear Convolution Method

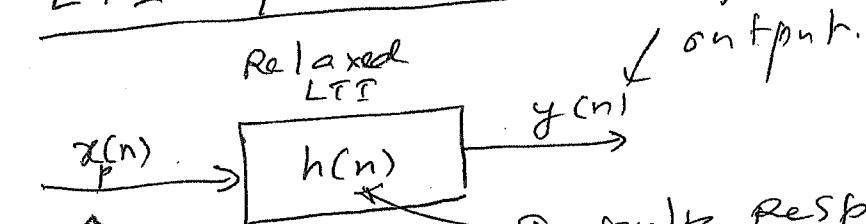
Let $y_l^{(n)} = x(n) * h(n)$ be the linear convolution of $x(n)$ & $h(n)$. $y_l^{(n)}$ is $(2n+1)$ points long.

$y_p(n) = y_l^{(n)}|_N$, the n -point periodic extension of $y_l^{(n)}$.

$$\begin{aligned} y(0) &= y_l^{(0)} + y_l^{(n)} \\ y(1) &= y_l^{(1)} + y_l^{(n+1)} \\ &\vdots \\ y(n-2) &= y_l^{(2n-2)} \\ y(n-1) &= y_l^{(n)} \end{aligned}$$

NOTE: Other methods exist, and you may refer to the text book Chapter on DFT and search for "Circular Convolution".

LTI System: Periodic Input Signal



$x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN)$, where $x(n)$ is one period of $x_p(n)$.

$$\begin{aligned} y(n) &= x_p(n) * h(n) = x_p(n) \left[\sum_{l=-\infty}^{\infty} x(n-lN) * h(n) \right] \\ &= \sum_{l=-\infty}^{\infty} [x(n-lN) * h(n)]. \end{aligned} \quad \text{--- (1)}$$

Note that: $y(n+q_N) = y(n) \Rightarrow y(n)$ is periodic with N samples

$$x(n-lN) * h(n) = y_l$$

$$\text{where } y_l = x(n) * h(n).$$

$$\Rightarrow y(n) = \sum_{l=-\infty}^{\infty} y_l (n-lN). \quad \text{--- (2)}$$

$y(n)$ is the N-point periodic extension of y_l , the convolution of 1-period of $x_p(n)$ and the impulse response, $h(n)$.

We can also write (1) as:

$$y(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} x(k-lN) h(n-k). \quad \text{--- (3)}$$

Letting $z = k-lN$ in the above equation, we get,

$$y(n) = \sum_{z=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} x(z) \cdot h(n-z-lN)$$

$$= \sum_{z=-\infty}^{\infty} x(z) \cdot h((n-z)_N) \quad \text{--- (4)}$$

$$= \sum_{z=0}^{N-1} x(z) \cdot h((n-z)_N) \quad \text{--- (4) (why?)}$$

(4) $\Rightarrow y(n)$ is given by the periodic convolution of $x_p(n)$ and $h(n)_N$, N-point periodic extension of $h(n)$.

Summary:

To determine the output of a relaxed LTI with a periodic input with periodically N samples, any of the following methods may be used.

Method 1: (Eqn. ②)

- * Compute the linear convolution of one period of the periodic input with the impulse response of the LTI System. Let the output be $y_o(n)$.
- * The output $y(n) = y_o(n)_N$ which is the N -point periodic extension of $y_o(n)$.

Method 2: (Eqn. ③)

- * Determine $h((n))_N$, the N -point periodic extension of $h(n)$.
- * Perform a periodic convolution of $x_p(n)$ and $h((n))_N$ to obtain $y(n)$.

Method 3:

- * Perform the linear convolution of one period of the periodic input with $h((n))_N$. Let the output be $y_e(n)$, $n=0, 1, 2, \dots, 2^N-1$.
- * Then $y(n) = y_e(n)_N$. i.e., N -point periodic extension of $y_e(n)$.