# Upsampling, Downsampling, Interpolation, Decimation & Rate change

## Upsampling & Interpolation

$$x(n) \longrightarrow \boxed{\uparrow L} \longrightarrow y_1(n) = \begin{cases} x\left(\frac{n}{L}\right), & n = \pm kL, \quad k = 0, 1, 2 \ldots \\ 0, & \text{otherwise} \end{cases} \qquad \text{—(1)}$$

upsampling operation

Note that $y_1(n)$ inserts $(L-1)$ zeros between the samples of $x(n)$, expanding the "duration". Sometimes, we call it an Expander.

$$y_1(n) = \sum_{k=-\infty}^{\infty} x(k)\, \delta(n - kL) \qquad [\text{why?}] \qquad \text{—(2)}$$

$$Y_1(\omega) = \sum_{n=-\infty}^{\infty} y_1(n)\, e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \left[ \sum_{k=-\infty}^{\infty} x(k)\, \delta(n-kL) \right] e^{-j\omega n}$$

$$= \sum_{k=-\infty}^{\infty} x(k) \left[ \sum_{n=-\infty}^{\infty} \delta(n-kL)\, e^{-j\omega n} \right]$$

$$= \sum_{k=-\infty}^{\infty} x(k)\, e^{-j\omega L k} = X(\omega L)$$

$$\Longrightarrow \boxed{Y_1(\omega) = X(\omega L)} \qquad \text{—(3)}.$$

Thus the spectrum, $Y_1(\omega)$ is an $L$-fold compressed version of the spectrum $X(\omega)$. This implies that there are $L$ replications of the frequency-scaled $X(\omega)$ in $[0, 2\pi)$.

If $x(n)$ is the sampled version of $x(t)$ [satisfying the sampling theorem] at a sampling rate $F_s = \frac{1}{T}$ Hz, and we desire the DT sequence $x_i(n)$, which is obtained by sampling at $F_s' = LF_s = \frac{L}{T} = \frac{1}{T/L}$, we can obtain $x_i(n)$ from $y_1(n)$ by further DT processing.

$x_i(n)$ is the <u>interpolated</u> DT sequence from $x(n)$.

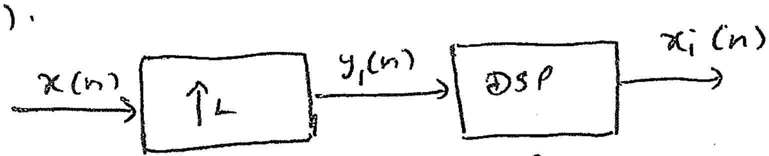



Fig 1: Interpolation operation.

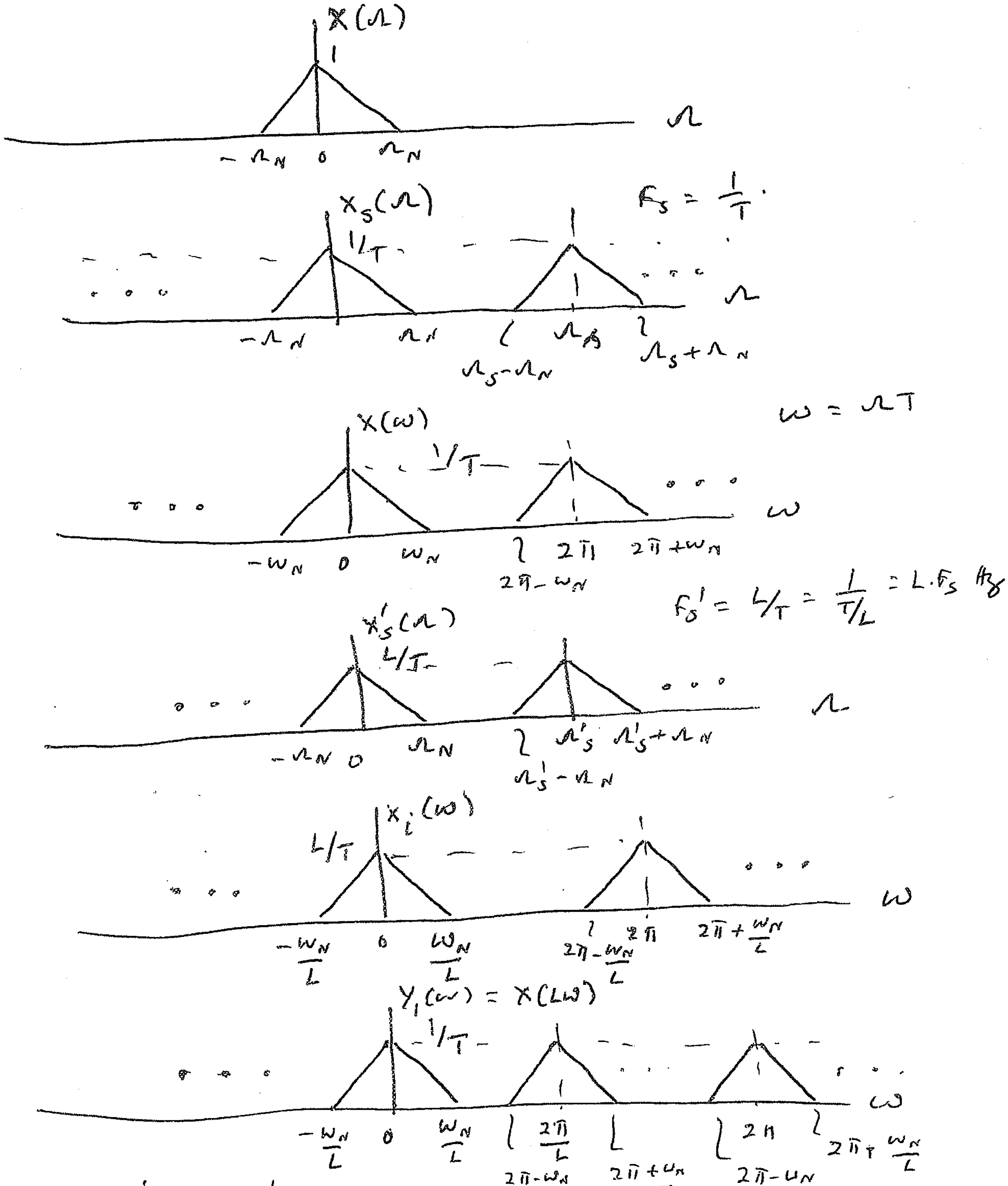



Fig 2: upsampling spectrum

From the figures on the previous page, we see that to obtain $X_i(\omega)$ from $Y_1(\omega)$, we simply need to reject the "unwanted images" in $Y_1(\omega)$, and boost the gain by a factor of $L$.

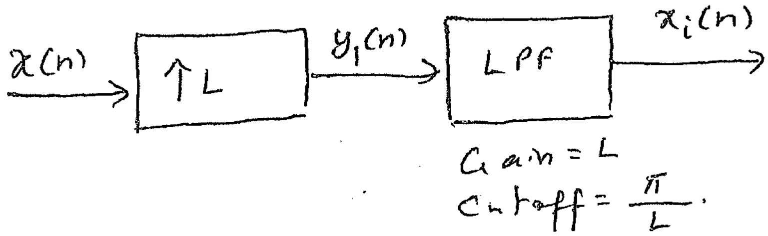



Fig 3: Interpolation operation.

## Downsampling & Decimation

Consider the sequence $y_2(n)$ defined below:

$$y_2(n) = \begin{cases} x(n), & n = kM, \quad k = \pm 0, 1, 2 \ldots \\ 0, & \text{otherwise.} \end{cases} \qquad — (4)$$

Note that

$$y_2(n) = x(n) \cdot w(n)$$

where $w(n) = \sum_{k=-\infty}^{\infty} \delta(n-kM)$.

But $w(n)$ is a periodic sequence whose DFS form is given by

$$w(n) = \sum_{k=0}^{M-1} \frac{1}{M} e^{j\frac{2\pi}{M}kn} \qquad — (5)$$

$$\Longrightarrow \quad y_2(n) = \frac{1}{M} \sum_{k=0}^{M-1} x(n) e^{j\frac{2\pi}{M}kn} \qquad — (6)$$

$$Y_2(\omega) = \sum_{n=-\infty}^{\infty} y_2(n) e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left[ \frac{1}{M} \cdot \sum_{k=0}^{M-1} x(n) e^{j\frac{2\pi}{M}kn} \right] e^{-j\omega n}$$

$$= \frac{1}{M} \sum_{k=0}^{M-1} x(n) e^{-j(\omega - \frac{2\pi}{M}k)n} = \frac{1}{M} \sum_{k=0}^{M-1} X(\omega - \frac{2\pi}{M}k)$$

or

$$\boxed{Y_2(\omega) = \frac{1}{M} \sum_{k=0}^{M-1} X(\omega - \frac{2\pi}{M}k)} \qquad — (7)$$

Note that

$$y_2(n) = \begin{cases} y_3\left(\frac{n}{M}\right), & n = \pm kM, \; k = 0, 1, 2 \ldots \\ 0, & \text{otherwise.} \end{cases} \quad (8)$$

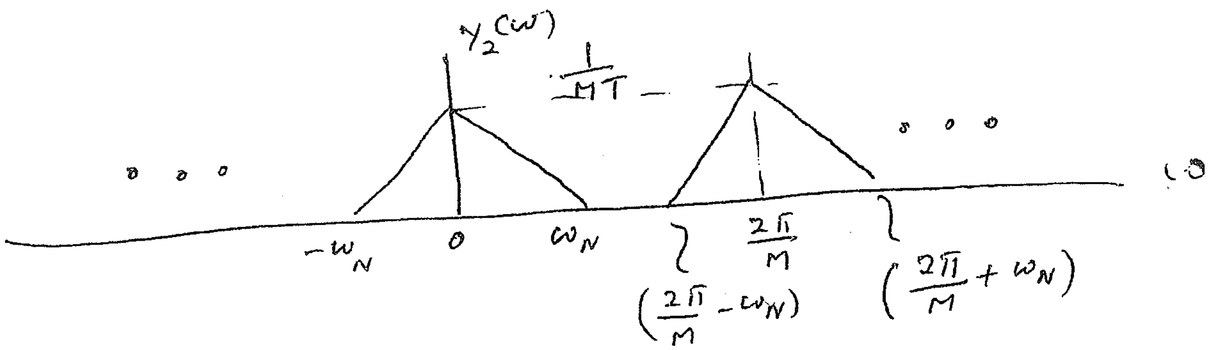
In other words, $y_2(n)$ is the "M-expanded" sequence of $y_3(n)$. Thus, we write:

$$Y_2(\omega) = Y_3(M\omega)$$

$$\Rightarrow \boxed{Y_3(\omega) = Y_2\left(\frac{\omega}{M}\right)} \quad (9)$$

$$\Rightarrow Y_3(\omega) = \frac{1}{M}\sum_{k=1}^{M-1} X\left(\frac{\omega}{M} - \frac{2\pi}{M}k\right)$$

$$\boxed{Y_3(\omega) = \frac{1}{M}\sum_{k=1}^{M-1} X\left(\frac{\omega - 2\pi k}{M}\right)} \quad (10)$$



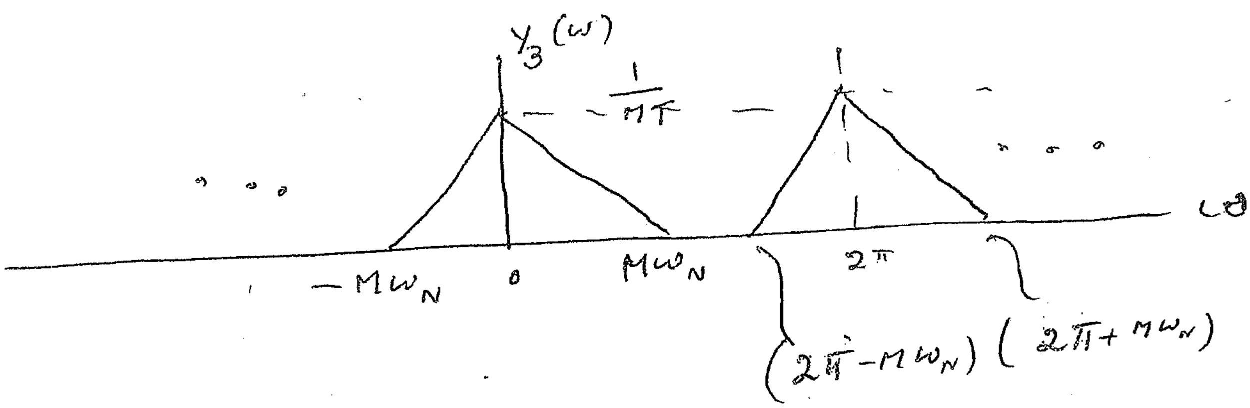




Fig. 4: Downsampling Spectrum

In order to ensure that the downsampled signal is not aliased, we require that

$$2\pi - M\omega_N \geq M\omega_N \implies \boxed{\omega_N \leq \frac{\pi}{M}}$$

We implement the <u>decimation</u> scheme as below

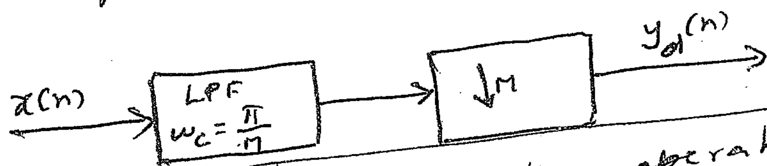



Figure 5: Decimation operation

In the above scheme, the LPF ensures that any possible aliasing is eliminated in the subsequent downsampling by a factor of M. In the absence of any aliasing, $y_d(n) = y_3(n)$. Otherwise $y_d(n)$ only consists of the non-aliased portion of $y_3(n)$.

Figure 6 will be equivalent of Fig. 5.

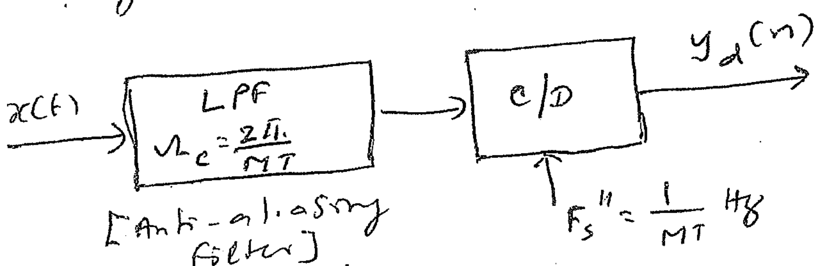
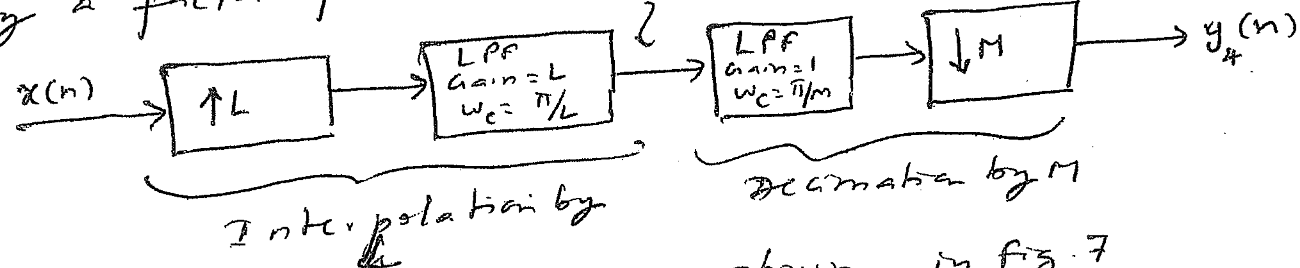



Figure 6: Sampling of $x(t)$ to produce the decimated signal, $y_d(t)$

## Rate change

$$y_4(n) = x\left(\frac{Mn}{L}\right)$$

We first interpolate by a factor of L, and then decimate by a factor of M.




This can be simplified as shown in fig. 7

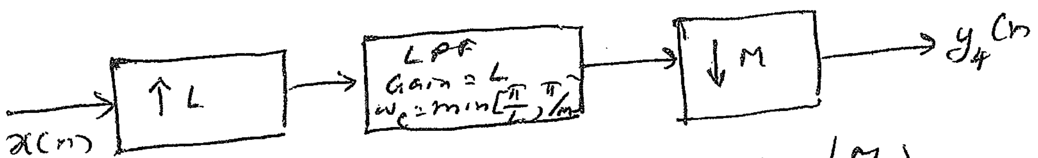



Fig 7: Rate change by a factor $\left(\frac{M}{L}\right)$

Note that #7 digitally implements the following sampling of the CT signal $x(t)$.

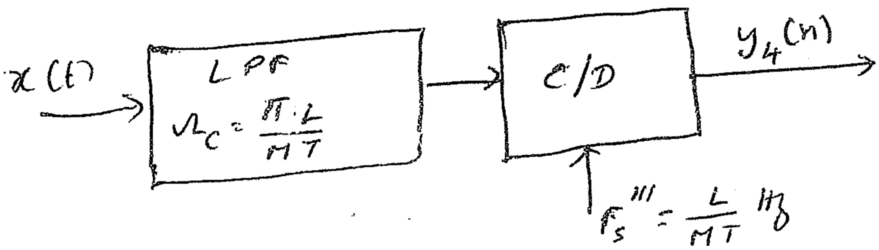



Fig. 8: sampling of the CT signal $x(t)$.