FOURIER ANALYSIS OF DISCRETE TIME SYSTEMS

FOURIER ANALYSIS: DISCRETE SYSTEMS

- The Fourier Analysis of Discrete Time Signals & Systems is quite similar to that of Continuous time Signals and Systems; yet there are <u>key differences</u> that we will point out along the way.
- We will also point out the relationship between the Continuous time representations and the corresponding discrete time versions as appropriate.
- Our goal, as will be demonstrated in Ch.9 discussions, will be to compute the Fourier Transform of Continuous time Signals on a digital Computer.
- Note: Because a discrete "time" Signal or System need not be derived from a Continuous System, the Fourier Analysis of Discrete Time Systems can stand on its own.

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- $x(n)$ is periodic with fundamental period, N, if

$$x(n) = x(n + kN), \quad \text{where } k \text{ is any integer.}$$

- The fundamental frequency of $x(n)$ is given by

$$\omega_1 = \frac{2\pi}{N} \text{ radians}$$

- In chapter 6, we observed that for any periodic signal $x(n)$, there were only N distinct harmonics, including DC (0 or 2π radians).

k^{th} harmonic $\boxed{\omega_k = \frac{2\pi k}{N}}$

- Unlike the continuous time case, the discrete time periodic signal can be always expressed as a finite, weighted sum of its harmonics.

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(continued)

We can write the discrete Fourier Series (DFS) of a periodic signal $x(n)$ with fund. period N:

$$\boxed{\begin{aligned} x(n) &= \sum_{k=0}^{N-1} c_k \cdot e^{j\frac{2\pi}{N}k \cdot n} \qquad (1) \\ c_k &= \frac{1}{N}\sum_{n=0}^{N-1} x(n) \cdot e^{-j\frac{2\pi}{N}nk} \qquad (2) \end{aligned}}$$

DFS coefficients $\to c_k$

Note that in (1), the variable of summation is "k", the frequency index; in (2), the variable of summation is the time index, "n". It is important to remember the time & frequency indexes in Discrete Time Systems, as they are easily confused.

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We can show that

$$\left\{ e^{j\frac{2\pi}{N}\cdot k n} \right\} \qquad k = 0, 1, 2, \ldots N-1$$

are orthogonal over one period, N samples.

<u>That is:</u>

Freq. variable

$$\sum_{n=0}^{N-1} \left[e^{j\frac{2\pi}{N}\cdot k\cdot n} \right] \left[e^{j\frac{2\pi}{N} l\cdot n} \right]^{*}$$

$$= \begin{cases} 0 & \text{if } k \neq l \\ N & \text{if } k = l + rN \end{cases}$$

integer

This can be written as:

$$\left[\sum_{n=0}^{N-1} \left[e^{j\frac{2\pi}{N}(k-l)n} \right] \right] = N \cdot \delta(k - l - rN)$$

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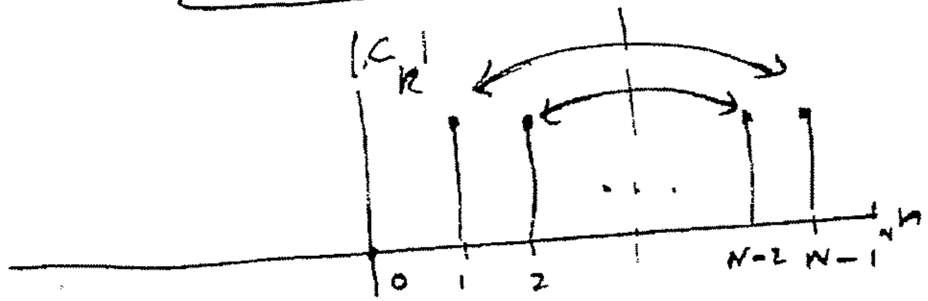
<u>Note</u>:

- Equations ① & ② could be computed over any <u>one</u> period, i.e. N samples.
- a_k is periodic with N, ie $a_k = a_{k+N}$.
- If $x(n)$ is real-valued, then

$$\boxed{c_k = c^*_{N-k}}$$

- 

- You can think of c_k, $k = 0, 1, 2, \cdots, \frac{N}{2} - 1$ as "+ve freq", and $k = (N-1), (N-2), \ldots, \frac{N}{2}$ as "−ve frequencies" as in cont. time EFS.

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Examples:

<u>Ex. (7.2.2)</u>:

$$x(n) = \cos\left(\frac{\pi n}{9}\right) + \sin\left(\frac{\pi n}{7} + \frac{1}{2}\right)$$

$\cos\left(\frac{\pi n}{9}\right) \to N_1,\ 2\pi f_1 \to N_1 = 18,\quad f_1 = \frac{1}{18}$

$\sin\left(\frac{\pi n}{7} + \frac{1}{2}\right) \to N_2,\ 2\pi f_2 \to f_2 = \frac{1}{14},\ N_2 = 14$

$x(n)$ is periodic with $N = LCM(N_1, N_2) = LCM(18, 14) = 126$ samples.

The fundamental freq. is: $e^{j\frac{2\pi}{126}n}$ or $e^{j\frac{\pi n}{63}}$.

Using Euler's identity, we have the DFS:

$$x(n) = \frac{1}{2}\left[e^{j\frac{\pi n}{9}} + e^{-j\frac{\pi n}{9}}\right] + \frac{1}{2j}\cdot\left[e^{j\left(\frac{\pi n}{7}+\frac{1}{2}\right)} - e^{-j\left(\frac{\pi n}{7}+\frac{1}{2}\right)}\right]$$

$$= \frac{1}{2}e^{j\frac{\pi n}{9}} + \frac{1}{2j}e^{j\frac{1}{2}}\cdot e^{j\frac{\pi n}{7}} + \frac{1}{2}\cdot e^{-j\frac{\pi n}{9}} - \frac{1}{2j}e^{-j\frac{1}{2}}\cdot e^{-j\frac{\pi n}{7}}$$

Harmonic #7

Harmonic #9

Harmonic # $(126-7) = 119$

Harmonic # $(126-9) = 117$

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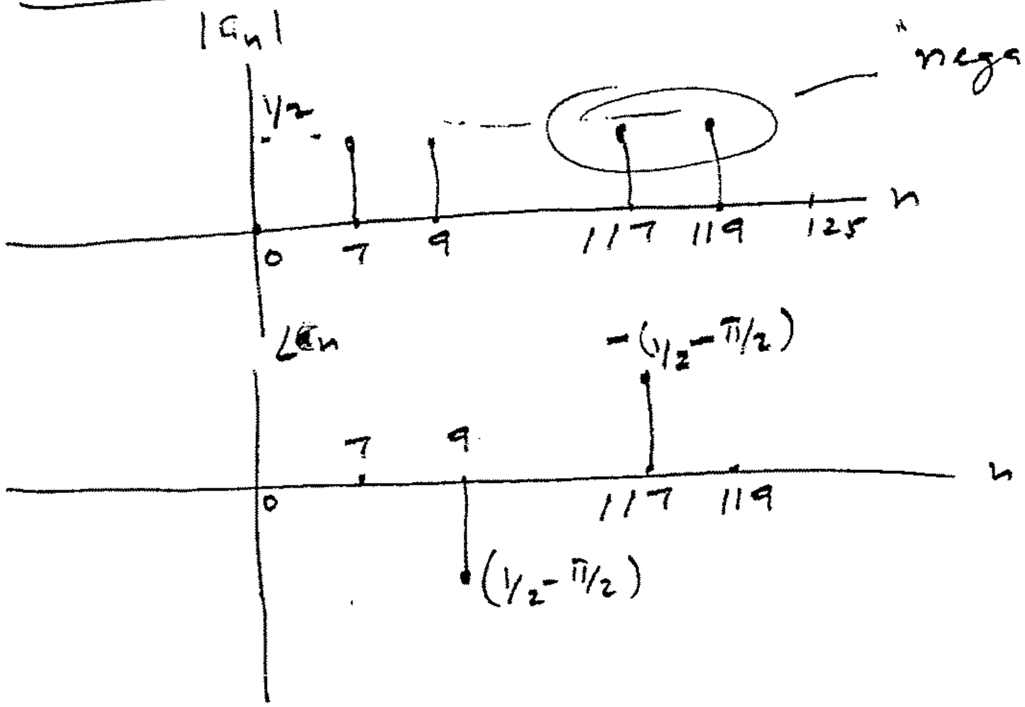
Example (continued)

Ex. 7.2.2

$$c_7 = \frac{1}{2} \qquad c_9 = \frac{1}{2j} e^{j1/2} = \frac{1}{2} \cdot e^{j\left(\frac{1}{2} - \frac{\pi}{2}\right)}$$

$$c_{119} = \frac{1}{2} \qquad c_{117} = -\frac{1}{2j} e^{-j1/2} = \frac{1}{2} \cdot e^{-j\left(\frac{1}{2} - \frac{\pi}{2}\right)}$$



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Example 7.2.4

Let $x(n)$ be the <u>periodic extension</u> (what is this?) of the sequence

$$\{2, -1, 1, 2\}.$$

The fundamental period is 4, and the fundamental frequency is $e^{j\frac{2\pi}{4}n} = e^{j\pi/2\, n}$.

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn}$$

$$c_0 = \frac{1}{4}\left[2 - \cancel{1} + \cancel{1} + 2\right] = 1.$$

$$c_1 = \frac{1}{4}\left[2 \cdot 1 + (-1)\cdot e^{-j\frac{\pi}{2}\cdot 1} + (1)\cdot e^{-j\frac{\pi}{2}\cdot \cancel{2}} + (2)\cdot e^{-j\frac{\pi}{2}\cdot 3}\right]$$

$$= \frac{1}{4}\left[2 + j + (-1) + (2)\cdot(+j)\right] = \frac{1}{4}\left[1 + 3j\right]$$

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Example 7.2.4 (Continued)

$$c_2 = \frac{1}{4}\Big(2 \cdot 1 + (-1) \cdot e^{-j\frac{\pi}{\cancel{2}}\cdot\cancel{2}\cdot 1} + (1) \cdot e^{-j\frac{\pi}{\cancel{2}}\cdot\cancel{2}\cdot 2} + (2) \cdot e^{-j\frac{\pi}{\cancel{2}}\cdot\cancel{2}\cdot 3} \Big]$$

$$= \frac{1}{4}\left[2 + 1 + 1 - 2\right] = \frac{1}{2}.$$

$$c_3 = a_{4-1} = a_1^* = \left\{\frac{1}{4}[1+3j]\right\}^* = \frac{1}{4}[1-3j]$$

Example 7.2.5

A periodic sequence $x(n)$ has the following Fourier coefficients.

$$c_k = \frac{1}{6}\sin\frac{k\pi}{6} + \frac{1}{12}\cos\frac{k\pi}{2}, \quad \boxed{0 \le k \le 11}$$

Note that $\boxed{N = 12}$. We want to find $x(n)$.

ANSWER: {0, j, 0, 1/2, 0, 0, 0, 0, 0, 1/2, 0, -j }

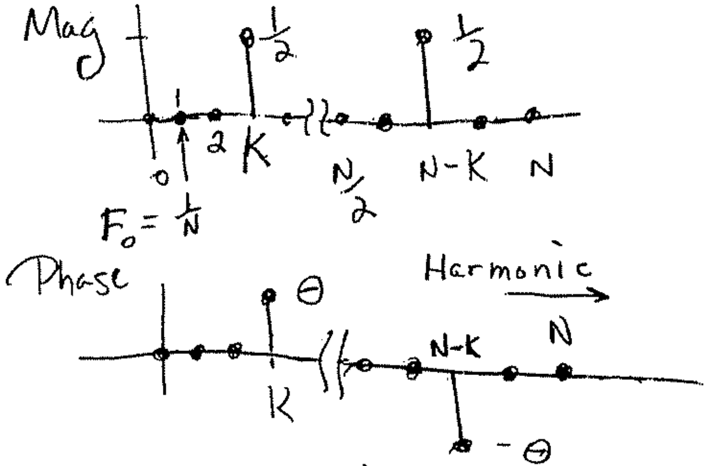
FA 10

(Courtesy of Dr. Pete Bernardin)

Discrete Fourier Series Repesentation of Cosine

$$x(n) = \cos\left[\frac{2\pi K n}{N} + \theta\right]$$

$$0 \le K < N/2$$



$$x(n) = \cos\left[\frac{2\pi K n}{N} + \theta\right] = \frac{1}{2}\left[e^{j(2\pi K n/N + \theta)} + e^{-j(2\pi K n/N + \theta)}\right]$$

$$= \frac{1}{2}\left[e^{j(2\pi n K/N + \theta)} + e^{+j(2\pi n(-K) n/N - \theta)}\right]$$

periodic in N

$$= \frac{1}{2}\left\{ \text{''} \qquad \text{''} + e^{+j[2\pi n(-K+N) - \theta]} \right\}$$

C_K, $\measuredangle C_K$, C_{N-K}, $\measuredangle C_{N-K}$

$$= \frac{1}{2} e^{j(2\pi F_0 n K + \theta)} + \frac{1}{2} e^{j[2\pi F_0 n (N-K) - \theta]}$$

$$x(n) = \sin\left[\frac{2\pi K n}{N} + \theta\right] = \cos\left[\frac{2\pi K n}{N} + \theta - \frac{\pi}{2}\right]$$

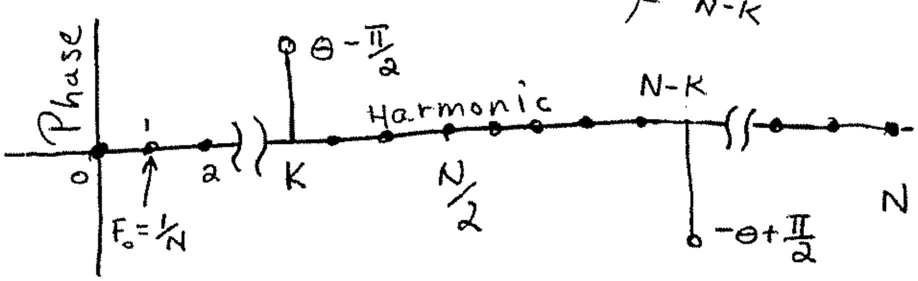
$$0 \le K < N/2$$

$\measuredangle C_K$

$$= \frac{1}{2} e^{j(2\pi F_0 n K + \theta - \pi/2)} + \frac{1}{2} e^{j[2\pi F_0 n (N-K) - \theta + \frac{\pi}{2}]}$$

$\measuredangle C_{N-K}$

Same Magnitude Plot as cos( )



FA-11

(Courtesy of Dr. Pete Bernardin)

Discrete Fourier Series Repesentation of Sine

$$x(n) = \sin\left[\frac{2\pi K n}{N} + \theta\right] = \frac{1}{2j}\left[e^{j(2\pi K n/N + \theta)} - e^{-j(2\pi K n/N + \theta)}\right]$$

$0 \le K < N/2$; $F_0 = 1/N$ Fundamental Frequency

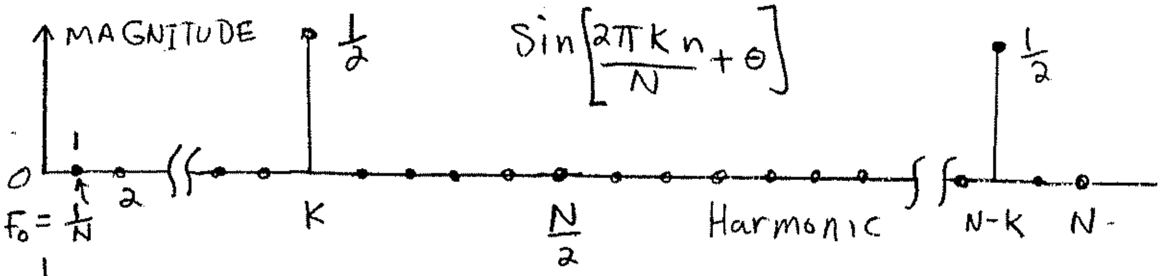
$$= \frac{-j}{2}\left\{e^{j(2\pi K n/N + \theta)} - e^{+j[\underbrace{2\pi(-K) n/N}_{\text{Periodic in } N} - \theta]}\right\}$$

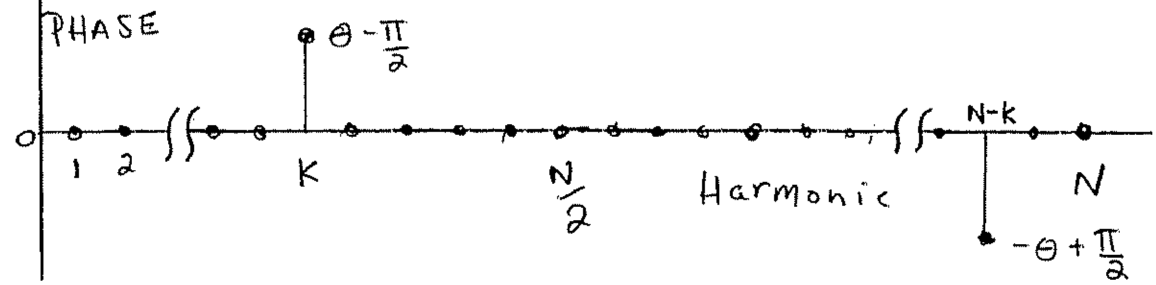
$$= \frac{e^{-j\pi/2}}{2}\left\{e^{j(2\pi K n/N + \theta)} - e^{+j[2\pi(N-K)n/N - \theta]}\right\}$$

$$= \frac{1}{2} e^{j(2\pi F_0 n K + \theta - \frac{\pi}{2})} + \frac{1}{2} \underbrace{e^{-j\frac{\pi}{2}} e^{-j\pi}}_{\text{period in } 2\pi} e^{j[2\pi F_0 (N-K) n - \theta]}$$

$$= \frac{1}{2} e^{j(2\pi F_0 n K + \theta - \frac{\pi}{2})} + \frac{1}{2} e^{j(2\pi - \frac{3\pi}{2})} e^{j[\;"\qquad"\;]}$$

$$= \underbrace{\frac{1}{2}}_{C_K} e^{j(2\pi F_0 n K + \underbrace{\theta - \frac{\pi}{2}}_{\measuredangle C_K})} + \underbrace{\frac{1}{2}}_{C_{N-K}} e^{j[2\pi F_0 (N-K) n \underbrace{- \theta + \frac{\pi}{2}}_{\measuredangle C_{N-K}}]}$$





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FA12

USEFUL IDENTITIES FOR DISCRETE SINUSOIDS

1) $$\sin\left(\frac{2\pi n K}{N} + \theta\right) = \frac{1}{2} e^{j\left(\frac{2\pi n K}{N} + \theta - \frac{\pi}{2}\right)} + \frac{1}{2} e^{j\left(\frac{2\pi (N-K) n}{N} - \theta + \frac{\pi}{2}\right)}$$

since $\cos\phi = \sin(\phi + \frac{\pi}{2})$

2) $$\cos\left(\frac{2\pi n K}{N} + \theta\right) = \frac{1}{2} e^{j\left(\frac{2\pi n K}{N} + \theta\right)} + \frac{1}{2} e^{j\left(\frac{2\pi (N-K) n}{N} - \theta\right)}$$
$$= \frac{1}{2}\delta(k - K) e^{j\theta} + \frac{1}{2}\delta(k - N + K) e^{-j\theta}$$

1) & 2) are true for $n = 0, 1, 2, \ldots, N-1$ & $K = 0, 1, 2, \ldots \frac{N}{2} - 1$

Let $\omega_K = \frac{2\pi K}{N}$; The following are useful identities:

3) $\cos(\omega_K n + \theta) = \cos[\omega_K(n \pm mN) + \theta] = \cos[\omega_K n + (\theta \pm 2m\pi)]$

for m any integer.

4) $\cos(\omega_K n + \theta) = \sin(\omega_K n + \theta + \frac{\pi}{2}) = \sin(\omega_K n + \theta - \frac{3\pi}{2})$

For complex exponentials the identities are similar:

5) $\exp[j(\omega_K n + \theta)] = \exp\{j[\omega_K(n \pm mN) + \theta]\} = \exp\{j[\omega_K n + (\theta \pm 2m\pi)]\}$

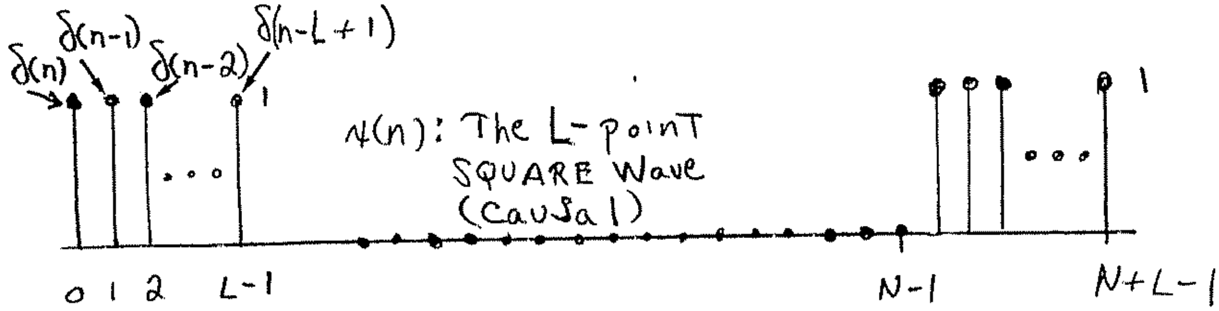
for $n = 0, 1, 2, \ldots, N-1$; $\omega_K = \frac{2\pi K}{N}$ $K = 0, 1, 2, \ldots N-1$

ALL of the above 1)-5) are true whether using discrete frequencies $\omega_K = \frac{2\pi K}{N}$, or continuous frequencies $\Omega = 2\pi f / F_s$ where $0 \leq \Omega \leq 2\pi$

(Courtesy of Dr. Pete Bernardin)

FA13

Discrete Fourier Series Representation of L-Point Square Wave



$$c_K = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}nK} = \frac{1}{N}\sum_{n=0}^{L-1}\left(e^{-j\frac{2\pi K}{N}}\right)^n$$

$$= \frac{1}{N}\,\frac{1-\left(e^{-j\frac{2\pi}{N}K}\right)^L}{1-e^{-j\frac{2\pi}{N}K}} = \frac{1}{N}\left[\frac{e^{j\frac{\pi K}{N}L} - e^{-j\frac{\pi K}{N}L}}{e^{j\frac{\pi K}{N}} - e^{-j\frac{\pi K}{N}}}\right]\frac{e^{-j\frac{\pi K}{N}L}}{e^{-j\frac{\pi K}{N}}}$$

$$c_K = \frac{1}{N}\,\frac{\sin\left(\frac{\pi L K}{N}\right)}{\sin\left(\frac{\pi K}{N}\right)}\, e^{-j\frac{\pi K}{N}(L-1)}$$

"Zero-Phase" term — Linear Phase term

Phase of "Zero Phase" term $= \text{PHASE}\left[\frac{\sin\left(\frac{\pi L K}{N}\right)}{\sin\left(\frac{\pi K}{N}\right)}\right] = \begin{cases} \pi; & \text{if } \frac{\sin\left(\frac{\pi L K}{N}\right)}{\sin\frac{\pi K}{N}} < 0 \\ 0; & \text{if } \frac{\sin\left(\frac{\pi L K}{N}\right)}{\sin\left(\frac{\pi K}{N}\right)} \geq 0 \end{cases}$

$$\measuredangle c_K = \underbrace{-\frac{\pi(L-1)K}{N}}_{\text{Linear Phase}} + \underbrace{\text{PHASE}\left[\frac{\sin\left(\frac{\pi L K}{N}\right)}{\sin\left(\frac{\pi K}{N}\right)}\right]}_{\text{Zero Phase term}}$$

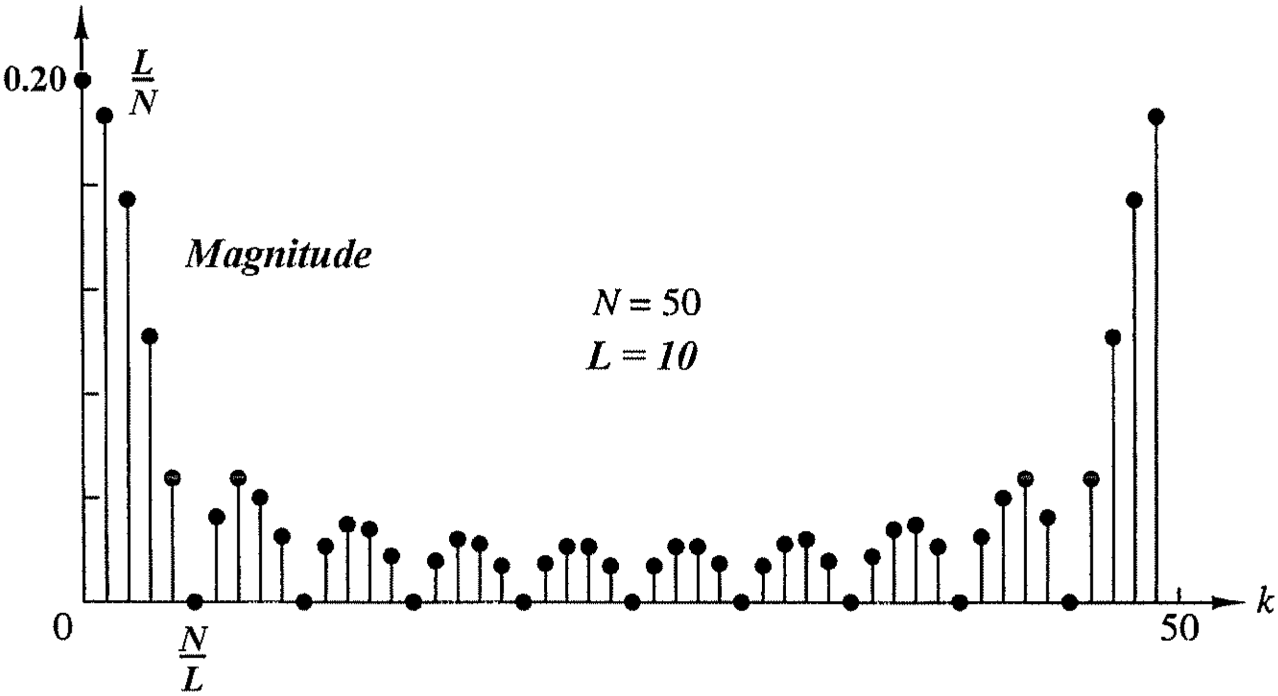
$$|c_K| = \frac{1}{N}\left|\frac{\sin\left(\frac{\pi L K}{N}\right)}{\sin\left(\frac{\pi K}{N}\right)}\right|$$

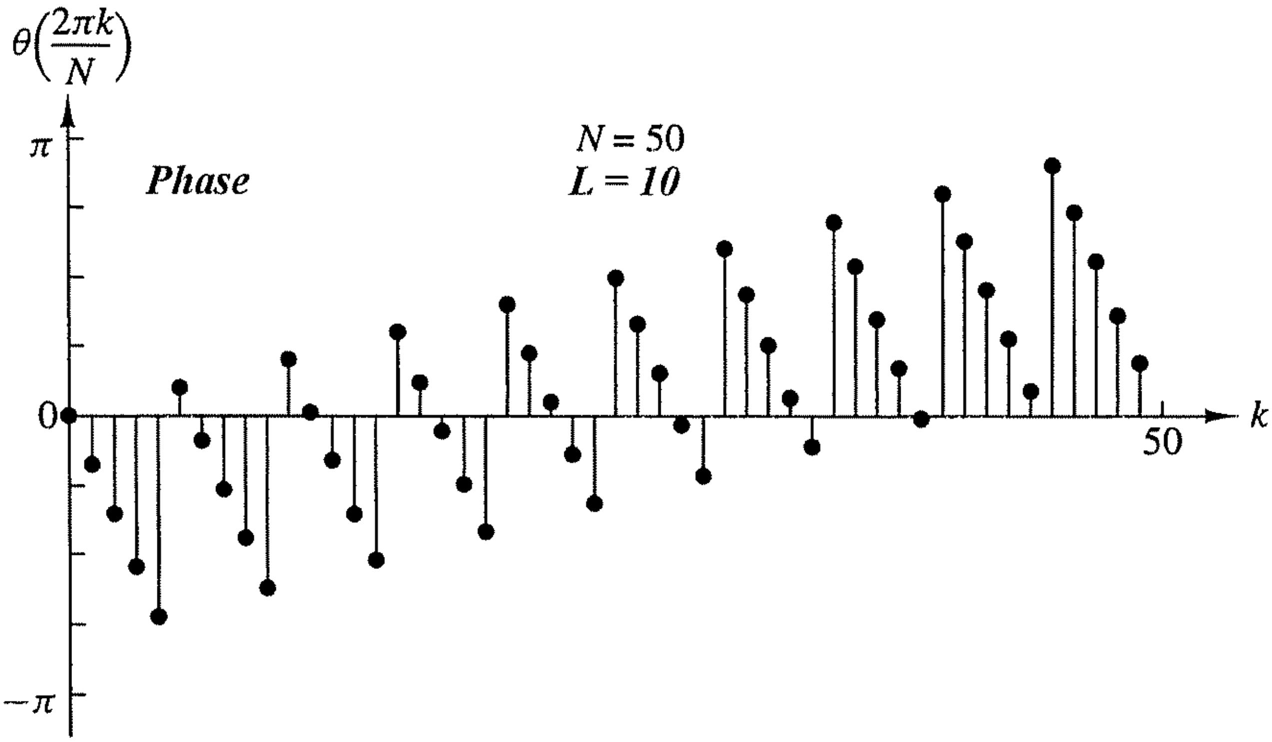
NOTE: $|C_0| = \frac{L}{N}$ from L'Hospital's Rule

(Courtesy of Dr. Pete Bernardin)

FA14

Discrete Fourier Series Representation of the L-Point Square Wave





POWER SPECTRAL DENSITY OF PERIODIC SIGNALS

The power of a periodic signal $x(n)$ with a fundamental period N is given by

$$P_x = \frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2.$$

writing $x(n) = \sum_{k=0}^{N-1} c_k e^{j\frac{2\pi k}{N} \cdot n}$

and substituting in the expression for P_x, we can show that

$$P_x = \sum_{k=0}^{N-1} |c_k|^2.$$

Thus,

$$P_x = \frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2 = \sum_{k=0}^{N-1} |c_k|^2$$

(Parseval's theorem)

$|c_k|^2$ is the power of the k-th harmonic component of the periodic signal $x(n)$. The description of power as a function of frequency can be described by the plot of $|c_k|^2$ vs k, and is called the Power Spectral Density of the periodic signal.