Chapter 9

Filter Design

9.1 INTRODUCTION

This chapter considers the problem of designing a digital filter. The design process begins with the filter specifications, which may include constraints on the magnitude and/or phase of the frequency response, constraints on the unit sample response or step response of the filter, specification of the type of filter (e.g., FIR or IIR), and the filter order. Once the specifications have been defined, the next step is to find a set of filter coefficients that produce an acceptable filter. After the filter has been designed, the last step is to implement the system in hardware or software, quantizing the filter coefficients if necessary, and choosing an appropriate filter structure (Chap. 8).

9.2 FILTER SPECIFICATIONS

Before a filter can be designed, a set of filter specifications must be defined. For example, suppose that we would like to design a low-pass filter with a cutoff frequency ω_c. The frequency response of an ideal low-pass filter with linear phase and a cutoff frequency ω_c is

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\alpha\omega} & |\omega| \le \omega_c \\ 0 & \omega_c < |\omega| \le \pi \end{cases}$$

which has a unit sample response

$$h_d(n) = \frac{\sin(n-\alpha)\omega_c}{\pi(n-\alpha)}$$

Because this filter is unrealizable (noncausal and unstable), it is necessary to relax the ideal constraints on the frequency response and allow some deviation from the ideal response. The specifications for a low-pass filter will typically have the form

$$\begin{aligned} 1-\delta_p < |H(e^{j\omega})| \le 1+\delta_p \qquad & 0 \le |\omega| < \omega_p \\ |H(e^{j\omega})| \le \delta_s \qquad & \omega_s \le |\omega| < \pi \end{aligned}$$

as illustrated in Fig. 9-1. Thus, the specifications include the passband cutoff frequency, ω_p, the stopband cutoff frequency, ω_s, the passband deviation, δ_p, and the stopband deviation, δ_s. The passband and stopband deviations

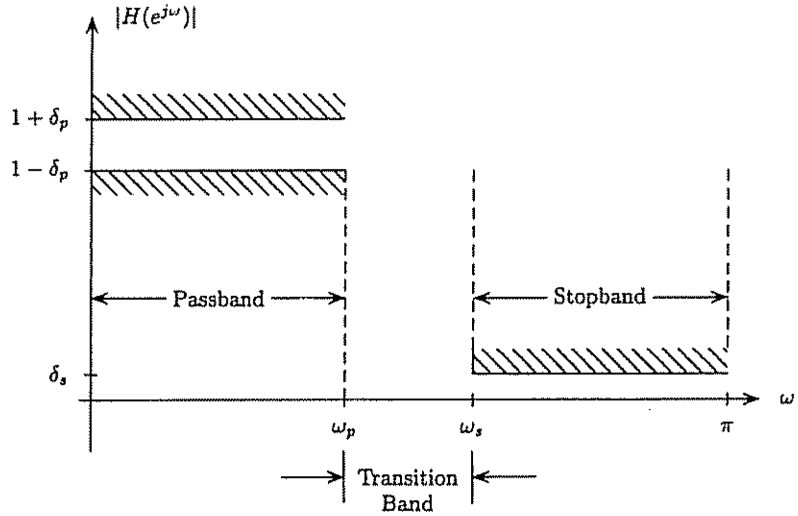

Fig. 9-1. Filter specifications for a low-pass filter.

are often given in decibels (dB) as follows:

$$\alpha_p = -20\log(1-\delta_p)$$

and

$$\alpha_s = -20\log(\delta_s)$$

The interval $[\omega_p, \omega_s]$ is called the *transition band*.

Once the filter specifications have been defined, the next step is to design a filter that meets these specifications.

9.3 FIR FILTER DESIGN

The frequency response of an Nth-order causal FIR filter is

$$H(e^{j\omega}) = \sum_{n=0}^{N} h(n)e^{-jn\omega}$$

and the design of an FIR filter involves finding the coefficients $h(n)$ that result in a frequency response that satisfies a given set of filter specifications. FIR filters have two important advantages over IIR filters. First, they are guaranteed to be stable, even after the filter coefficients have been quantized. Second, they may be easily constrained to have (generalized) linear phase. Because FIR filters are generally designed to have linear phase, in the following we consider the design of linear phase FIR filters.

9.3.1 Linear Phase FIR Design Using Windows

Let $h_d(n)$ be the unit sample response of an ideal frequency selective filter with linear phase,

$$H_d(e^{j\omega}) = A(e^{j\omega})e^{-j(\alpha\omega-\beta)}$$

Because $h_d(n)$ will generally be infinite in length, it is necessary to find an FIR approximation to $H_d(e^{j\omega})$. With the window design method, the filter is designed by windowing the unit sample response,

$$h(n) = h_d(n)w(n)$$

where $w(n)$ is a finite-length window that is equal to zero outside the interval $0 \le n \le N$ and is symmetric about its midpoint:

$$w(n) = w(N-n)$$

The effect of the window on the frequency response may be seen from the complex convolution theorem,

$$H(e^{j\omega}) = \frac{1}{2\pi}H_d(e^{j\omega}) * W(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\theta})W(e^{j(\omega-\theta)})\,d\theta$$

Thus, the ideal frequency response is *smoothed* by the discrete-time Fourier transform of the window, $W(e^{j\omega})$.

There are many different types of windows that may be used in the window design method, a few of which are listed in Table 9-1.

How well the frequency response of a filter designed with the window design method approximates a desired response, $H_d(e^{j\omega})$, is determined by two factors (see Fig. 9-2):

1. The width of the main lobe of $W(e^{j\omega})$.
2. The peak side-lobe amplitude of $W(e^{j\omega})$.

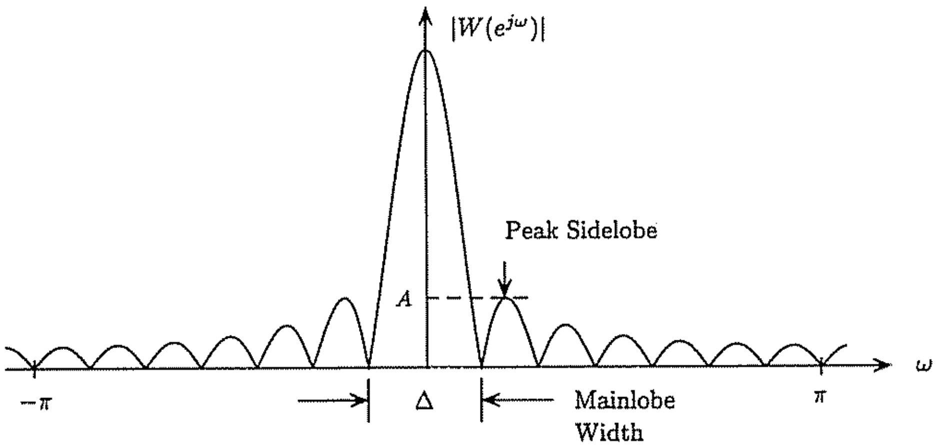

Fig. 9-2. The DTFT of a typical window, which is characterized by the width of its main lobe, Δ, and the peak amplitude of its side lobes, A, relative to the amplitude of $W(e^{j\omega})$ at $\omega = 0$.

Ideally, the main-lobe width should be narrow, and the side-lobe amplitude should be small. However, for a fixed-length window, these cannot be minimized independently. Some general properties of windows are as follows:

1. As the length N of the window increases, the width of the main lobe decreases, which results in a decrease in the transition width between passbands and stopbands. This relationship is given approximately by

$$N\Delta f = c \tag{9.1}$$

   where Δf is the transition width, and c is a parameter that depends on the window.
2. The peak side-lobe amplitude of the window is determined by the shape of the window, and it is essentially independent of the window length.
3. If the window shape is changed to decrease the side-lobe amplitude, the width of the main lobe will generally increase.

Listed in Table 9.2 are the side-lobe amplitudes of several windows along with the approximate transition width and stopband attenuation that results when the given window is used to design an Nth-order low-pass filter.

Table 9-1 Some Common Windows

Rectangular	$w(n) = \begin{cases} 1 & 0 \le n \le N \\ 0 & \text{else} \end{cases}$
Hanning[1]	$w(n) = \begin{cases} 0.5 - 0.5\cos\left(\frac{2\pi n}{N}\right) & 0 \le n \le N \\ 0 & \text{else} \end{cases}$
Hamming	$w(n) = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{N}\right) & 0 \le n \le N \\ 0 & \text{else} \end{cases}$
Blackman	$w(n) = \begin{cases} 0.42 - 0.5\cos\left(\frac{2\pi n}{N}\right) + 0.08\cos\left(\frac{4\pi n}{N}\right) & 0 \le n \le N \\ 0 & \text{else} \end{cases}$

[1] In the literature, this window is also called a Hann window or a von Hann window.

Table 9-2 The Peak Side-Lobe Amplitude of Some Common Windows and the Approximate Transition Width and Stopband Attenuation of an Nth-Order Low-Pass Filter Designed Using the Given Window.

Window	Side-Lobe Amplitude (dB)	Transition Width (Δf)	Stopband Attenuation (dB)
Rectangular	−13	$0.9/N$	−21
Hanning	−31	$3.1/N$	−44
Hamming	−41	$3.3/N$	−53
Blackman	−57	$5.5/N$	−74

EXAMPLE 9.3.1 Suppose that we would like to design an FIR linear phase low-pass filter according to the following specifications:

$$0.99 \le |H(e^{j\omega})| \le 1.01 \qquad 0 \le |\omega| \le 0.19\pi$$
$$|H(e^{j\omega})| \le 0.01 \qquad 0.21\pi \le |\omega| \le \pi$$

For a stopband attenuation of $20\log(0.01) = -40$ dB, we may use a Hanning window. Although we could also use a Hamming or a Blackman window, these windows would overdesign the filter and produce a larger stopband attenuation at the expense of an increase in the transition width. Because the specification calls for a transition width of $\Delta\omega = \omega_s - \omega_p = 0.02\pi$, or $\Delta f = 0.01$, with

$$N\Delta f = 3.1$$

for a Hanning window (see Table 9.2), an estimate of the required filter order is

$$N = \frac{3.1}{\Delta f} = 310$$

The last step is to find the unit sample response of the ideal low-pass filter that is to be windowed. With a cutoff frequency of $\omega_c = (\omega_s + \omega_p)/2 = 0.2\pi$, and a delay of $\alpha = N/2 = 155$, the unit sample response is

$$h_d(n) = \frac{\sin[0.2\pi(n-155)]}{(n-155)\pi}$$

In addition to the windows listed in Table 9-1, Kaiser developed a *family* of windows that are defined by

$$w(n) = \frac{I_0[\beta(1-[(n-\alpha)/\alpha]^2)^{1/2}]}{I_0(\beta)} \qquad 0 \le n \le N$$

where $\alpha = N/2$, and $I_0(\cdot)$ is a zeroth-order modified Bessel function of the first kind, which may be easily generated using the power series expansion

$$I_0(x) = 1 + \sum_{k=1}^{\infty}\left[\frac{(x/2)^k}{k!}\right]^2$$

The parameter β determines the shape of the window and thus controls the trade-off between main-lobe width and side-lobe amplitude. A *Kaiser window* is nearly optimum in the sense of having the most energy in its main lobe for a given side-lobe amplitude. Table 9-3 illustrates the effect of changing the parameter β.

There are two empirically derived relationships for the Kaiser window that facilitate the use of these windows to design FIR filters. The first relates the stopband ripple of a low-pass filter, $\alpha_s = -20\log(\delta_s)$, to the parameter β,

$$\beta = \begin{cases} 0.1102(\alpha_s - 8.7) & \alpha_s > 50 \\ 0.5842(\alpha_s - 21)^{0.4} + 0.07886(\alpha_s - 21) & 21 \le \alpha_s \le 50 \\ 0.0 & \alpha_s < 21 \end{cases}$$

Table 9-3 Characteristics of the Kaiser Window as a Function of β

Parameter β	Side Lobe (dB)	Transition Width ($N\Delta f$)	Stopband Attenuation (dB)
2.0	−19	1.5	−29
3.0	−24	2.0	−37
4.0	−30	2.6	−45
5.0	−37	3.2	−54
6.0	−44	3.8	−63
7.0	−51	4.5	−72
8.0	−59	5.1	−81
9.0	−67	5.7	−90
10.0	−74	6.4	−99

The second relates N to the transition width Δf and the stopband attenuation α_s,

$$N = \frac{\alpha_s - 7.95}{14.36\Delta f} \qquad \alpha_s \geq 21 \tag{9.2}$$

Note that if $\alpha_s < 21$ dB, a rectangular window may be used ($\beta = 0$), and $N = 0.9/\Delta f$.

EXAMPLE 9.3.2 Suppose that we would like to design a low-pass filter with a cutoff frequency $\omega_c = \pi/4$, a transition width $\Delta\omega = 0.02\pi$, and a stopband ripple $\delta_s = 0.01$. Because $\alpha_s = -20\log(0.01) = -40$, the Kaiser window parameter is

$$\beta = 0.5842(40 - 21)^{0.4} + 0.07886(40 - 21) = 3.4$$

With $\Delta f = \Delta\omega/2\pi = 0.01$, we have

$$N = \frac{40 - 7.95}{14.36 \cdot (0.01)} = 224$$

Therefore,

$$h(n) = h_d(n)w(n)$$

where

$$h_d(n) = \frac{\sin[(n - 112)\pi/4]}{(n - 112)\pi}$$

is the unit sample response of the ideal low-pass filter.

Although it is simple to design a filter using the window design method, there are some limitations with this method. First, it is necessary to find a closed-form expression for $h_d(n)$ (or it must be approximated using a very long DFT). Second, for a frequency selective filter, the transition widths between frequency bands, and the ripples within these bands, will be approximately the same. As a result, the window design method requires that the filter be designed to the tightest tolerances in all of the bands by selecting the smallest transition width and the smallest ripple. Finally, window design filters are not, in general, *optimum* in the sense that they do not have the smallest possible ripple for a given filter order and a given set of cutoff frequencies.

9.3.2 *Frequency Sampling Filter Design*

Another method for FIR filter design is the frequency sampling approach. In this approach, the desired frequency response, $H_d(e^{j\omega})$, is first uniformly sampled at N equally spaced points between 0 and 2π:

$$H(k) = H_d\left(e^{j2\pi k/N}\right) \qquad k = 0, 1, \ldots, N-1$$