GROUP DELAY EXPRESSIONS (SCHAUM'S OUTLINES)

Assuming that this relationship is satisfied, find the output of the system when $a = \frac{1}{2}$ and

$$x(n) = \left(\tfrac{1}{2}\right)^n u(n)$$

The frequency response of the LSI system described by this difference equation is

$$H(e^{j\omega}) = \frac{b + e^{-j\omega}}{1 - ae^{-j\omega}}$$

The squared magnitude is

$$|H(e^{j\omega})|^2 = \frac{(b + e^{-j\omega})(b + e^{j\omega})}{(1 - ae^{-j\omega})(1 - ae^{j\omega})} = \frac{1 + b^2 + 2b\cos\omega}{1 + a^2 - 2a\cos\omega}$$

Therefore, it follows that $|H(e^{j\omega})|^2 = 1$ if and only if $b = -a$.
With $a = \frac{1}{2}$ and $b = -\frac{1}{2}$, if $x(n) = (\frac{1}{2})^n u(n)$, $Y(e^{j\omega})$ is given by

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) = \frac{-\frac{1}{2} + e^{-j\omega}}{1 - \frac{1}{2}e^{-j\omega}} \cdot \frac{1}{1 - \frac{1}{2}e^{-j\omega}} = \frac{-\frac{1}{2} + e^{-j\omega}}{\left(1 - \frac{1}{2}e^{-j\omega}\right)^2}$$

Using the DTFT pair

$$(n+1)a^n u(n) \overset{DTFT}{\Longleftrightarrow} \frac{1}{(1 - ae^{-j\omega})^2}$$

given in Table 2-1, and using the linearity and delay properties of the DTFT, we have

$$y(n) = -\tfrac{1}{2}(n+1)\left(\tfrac{1}{2}\right)^n u(n) + n\left(\tfrac{1}{2}\right)^{n-1} u(n-1)$$

What we observe from this example is that although $|H(e^{j\omega})| = 1$, the nonlinear phase has a significant effect on the values of the input sequence.

2.18 Show that the group delay of a linear shift-invariant system with a frequency response $H(e^{j\omega})$ may be expressed as

$$\tau_h(\omega) = \frac{H_R(e^{j\omega})G_R(e^{j\omega}) + H_I(e^{j\omega})G_I(e^{j\omega})}{|H(e^{j\omega})|^2}$$

where $H_R(e^{j\omega})$ and $H_I(e^{j\omega})$ are the real and imaginary parts of $H(e^{j\omega})$, respectively, and $G_R(e^{j\omega})$ and $G_I(e^{j\omega})$ are the real and imaginary parts of the DTFT of $nh(n)$.

In terms of magnitude and phase, the frequency response is

$$H(e^{j\omega}) = |H(e^{j\omega})|e^{j\phi_h(\omega)}$$

Note that if we take the logarithm of $H(e^{j\omega})$, we have an explicit expression for the phase

$$\ln H(e^{j\omega}) = \ln|H(e^{j\omega})| + j\phi_h(\omega)$$

Differentiating with respect to ω, we have

$$\frac{d}{d\omega}\ln H(e^{j\omega}) = \frac{1}{H(e^{j\omega})}\frac{d}{d\omega}H(e^{j\omega}) = \frac{d}{d\omega}\ln|H(e^{j\omega})| + j\frac{d}{d\omega}\phi_h(\omega)$$

Equating the imaginary parts of both sides of this equation yields

$$\frac{d}{d\omega}\phi_h(\omega) = \text{Im}\left\{\frac{1}{H(e^{j\omega})}\frac{d}{d\omega}H(e^{j\omega})\right\}$$

If we define

$$\frac{d}{d\omega}H(e^{j\omega}) = H_R'(e^{j\omega}) + jH_I'(e^{j\omega})$$

where $H_R'(e^{j\omega})$ is the derivative of the real part of $H(e^{j\omega})$ and $H_I'(e^{j\omega})$ is the derivative of the imaginary part, the group delay may be written as

$$\tau_h(\omega) = -\frac{d}{d\omega}\phi_h(\omega) = -\text{Im}\left\{\frac{H_R'(e^{j\omega}) + jH_I'(e^{j\omega})}{H(e^{j\omega})}\right\}$$

Multiplying the numerator and denominator by $H^*(e^{j\omega}) = H_R(e^{j\omega}) - jH_I(e^{j\omega})$ yields

$$\tau_h(\omega) = -\text{Im}\left\{\frac{\left[H_R'(e^{j\omega}) + jH_I'(e^{j\omega})\right]\left[H_R(e^{j\omega}) - jH_I(e^{j\omega})\right]}{|H(e^{j\omega})|^2}\right\}$$

$$= \frac{H_I(e^{j\omega})H_R'(e^{j\omega}) - H_R(e^{j\omega})H_I'(e^{j\omega})}{|H(e^{j\omega})|^2}$$

Finally, recall that if $H(e^{j\omega})$ is the DTFT of $h(n)$, the DTFT of $g(n) = nh(n)$ is

$$G(e^{j\omega}) = G_R(e^{j\omega}) + jG_I(e^{j\omega}) = j\frac{d}{d\omega}H(e^{j\omega}) = -H_I'(e^{j\omega}) + jH_R'(e^{j\omega})$$

where $G_R(e^{j\omega})$ is the real part of the DTFT of $nh(n)$, and $G_I(e^{j\omega})$ is the imaginary part. Therefore, $H_R'(e^{j\omega}) = G_I(e^{j\omega})$ and $H_I'(e^{j\omega}) = -G_R(e^{j\omega})$. Expressed in terms of $G_R(e^{j\omega})$ and $G_I(e^{j\omega})$, the group delay becomes

$$\tau_h(\omega) = \frac{H_R(e^{j\omega})G_R(e^{j\omega}) + H_I(e^{j\omega})G_I(e^{j\omega})}{|H(e^{j\omega})|^2}$$

Note that this expression for the group delay is convenient for digital evaluation, because it only requires computing the DTFT of $h(n)$ and $nh(n)$, and no derivatives.

2.19 Find the group delay for each of the following systems, where α is a real number:

(*a*) $H_1(e^{j\omega}) = 1 - \alpha e^{-j\omega}$

(*b*) $H_2(e^{j\omega}) = \dfrac{1}{1 - \alpha e^{-j\omega}}$

(*c*) $H_3(e^{j\omega}) = \dfrac{1}{1 - 2\alpha\cos\theta e^{-j\omega} + \alpha^2 e^{-j2\omega}}$

(*a*) For the first system, the frequency response is

$$H_1(e^{j\omega}) = 1 - \alpha\cos\omega + j\alpha\sin\omega$$

Therefore, the phase is

$$\phi_1(\omega) = \tan^{-1}\frac{\alpha\sin\omega}{1 - \alpha\cos\omega}$$

Because

$$\frac{d}{dx}\tan^{-1}u = \frac{1}{1+u^2}\frac{du}{dx}$$

the group delay is

$$\tau_1(\omega) = -\frac{d}{d\omega}\phi_1(\omega) = -\frac{1}{1 + \left(\frac{\alpha\sin\omega}{1-\alpha\cos\omega}\right)^2}\frac{d}{d\omega}\left(\frac{\alpha\sin\omega}{1 - \alpha\cos\omega}\right)$$

Therefore,

$$\tau_1(\omega) = -\frac{1}{1 + \left(\frac{\alpha\sin\omega}{1-\alpha\cos\omega}\right)^2}\frac{(1 - \alpha\cos\omega)\alpha\cos\omega - (\alpha\sin\omega)^2}{(1 - \alpha\cos\omega)^2}$$

which, after simplification, becomes

$$\tau_1(\omega) = -\frac{(1-\alpha\cos\omega)\alpha\cos\omega - (\alpha\sin\omega)^2}{(1-\alpha\cos\omega)^2 + (\alpha\sin\omega)^2} = \frac{\alpha^2 - \alpha\cos\omega}{1+\alpha^2 - 2\alpha\cos\omega}$$

Another way to solve this problem is to use the expression for the group delay derived in Prob. 2.18. With

$$H_1(e^{j\omega}) = 1 - \alpha\cos\omega + j\alpha\sin\omega$$

we see that

$$H_R(e^{j\omega}) = 1 - \alpha\cos\omega \qquad H_I(e^{j\omega}) = \alpha\sin\omega$$

Because the unit sample response is

$$h(n) = \delta(n) - \alpha\delta(n-1)$$

then

$$g(n) = nh(n) = -\alpha\delta(n-1)$$

and

$$G(e^{j\omega}) = -\alpha e^{-j\omega} = -\alpha\cos\omega + j\alpha\sin\omega$$

Therefore, the group delay is

$$\begin{aligned}\tau_1(\omega) &= \frac{H_R(e^{j\omega})G_R(e^{j\omega}) + H_I(e^{j\omega})G_I(e^{j\omega})}{|H(e^{j\omega})|^2} \\ &= \frac{-\alpha\cos\omega(1-\alpha\cos\omega) + (\alpha\sin\omega)^2}{(1-\alpha\cos\omega)^2 + (\alpha\sin\omega)^2} = \frac{\alpha^2 - \alpha\cos\omega}{1+\alpha^2-2\alpha\cos\omega}\end{aligned}$$

which is the same as before.

(*b*) Having found the group delay for $H_1(e^{j\omega}) = 1 - \alpha e^{-j\omega}$, we may easily derive the group delay for $H_2(e^{j\omega})$, which is the inverse of $H_1(e^{j\omega})$:

$$H_2(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega}} = \frac{1}{H_1(e^{j\omega})}$$

Specifically, because

$$H_2(e^{j\omega}) = \frac{1}{H_2(e^{j\omega})}$$

$\phi_2(\omega) = -\phi_1(\omega)$ and, therefore,

$$\tau_2(\omega) = -\,\tau_1(\omega) = -\frac{\alpha^2 - \alpha\cos\omega}{1+\alpha^2-2\alpha\cos\omega}$$

(*c*) For the last system, $H_3(e^{j\omega})$ may be factored as follows:

$$H_3(e^{j\omega}) = \frac{1}{1-2\alpha\cos\theta e^{-j\omega} + \alpha^2 e^{-j2\omega}} = \frac{1}{1-\alpha e^{j\theta}e^{-j\omega}}\,\frac{1}{1-\alpha e^{-j\theta}e^{-j\omega}}$$

The group delay of $H_3(e^{j\omega})$ is thus the sum of the group delays of these two factors. Furthermore, the group delay of each factor may be found straightforwardly by differentiating the phase. However, the group delay of these terms may also be found from $\tau_2(\omega)$ in part (*b*) if we use the modulation property of the DTFT. Specifically, recall that if $X(e^{j\omega})$ is the DTFT of $x(n)$, the DTFT of $e^{jn\theta}x(n)$ is

$$e^{jn\theta}x(n) \overset{DTFT}{\Longleftrightarrow} X(e^{j(\omega-\theta)}) = |X(e^{j(\omega-\theta)})|e^{j\phi(\omega-\theta)}$$

Therefore, if the group delay of $x(n)$ is $\tau(\omega)$, the group delay of $e^{jn\theta}x(n)$ will be $\tau(\omega-\theta)$. In part (*b*), we found that the group delay of $H(e^{j\omega}) = 1/(1-\alpha e^{-j\omega})$ is

$$\tau(\omega) = -\frac{\alpha^2 - \alpha\cos\omega}{1+\alpha^2-2\alpha\cos\omega}$$

Thus, it follows from the modulation property that the group delay of $H(e^{j\omega}) = 1/(1-\alpha e^{-j(\omega-\theta)})$ is

$$\tau(\omega) = -\frac{\alpha^2 - \alpha\cos(\omega-\theta)}{1+\alpha^2 - 2\alpha\cos(\omega-\theta)}$$

and that the group delay of $H(e^{j\omega}) = 1/(1-\alpha e^{-j(\omega+\theta)})$ is

$$\tau(\omega) = -\frac{\alpha^2 - \alpha\cos(\omega+\theta)}{1+\alpha^2 - 2\alpha\cos(\omega+\theta)}$$

Therefore, the group delay of $H_3(e^{j\omega})$ is the sum of these:

$$\tau_3(\omega) = -\frac{\alpha^2 - \alpha\cos(\omega-\theta)}{1+\alpha^2 - 2\alpha\cos(\omega-\theta)} - \frac{\alpha^2 - \alpha\cos(\omega+\theta)}{1+\alpha^2 - 2\alpha\cos(\omega+\theta)}$$

2.20 Find the DTFT of each of the following sequences:

(*a*) $x_1(n) = \left(\frac{1}{2}\right)^n u(n+3)$

(*b*) $x_2(n) = \alpha^n \sin(n\omega_0)\, u(n)$

(*c*) $x_3(n) = \begin{cases} \left(\frac{1}{2}\right)^n & n = 0, 2, 4, \ldots \\ 0 & \text{otherwise} \end{cases}$

(*a*) For the first sequence, the DTFT may be evaluated directly as follows:

$$X_1(e^{j\omega}) = \sum_{n=-3}^{\infty} \left(\tfrac{1}{2}\right)^n e^{-jn\omega} = \sum_{n=-3}^{\infty} \left(\tfrac{1}{2}e^{-j\omega}\right)^n$$

$$= \left(\tfrac{1}{2}e^{-j\omega}\right)^{-3} \sum_{n=0}^{\infty} \left(\tfrac{1}{2}e^{-j\omega}\right)^n = \frac{8e^{3j\omega}}{1-\frac{1}{2}e^{-j\omega}}$$

(*b*) The best way to find the DTFT of $x_2(n)$ is to express the sinusoid as a sum of two complex exponentials as follows:

$$x_2(n) = \frac{1}{2j}[\alpha^n e^{jn\omega_0} - \alpha^n e^{-jn\omega_0}]u(n)$$

The DTFT of the first term is

$$\frac{1}{2j}\sum_{n=0}^{\infty} \alpha^n e^{jn\omega_0} e^{-jn\omega} = \frac{1}{2j}\sum_{n=0}^{\infty}\left(\alpha e^{-j(\omega-\omega_0)}\right)^n = \frac{1}{2j}\frac{1}{1-\alpha e^{-j(\omega-\omega_0)}}$$

Similarly, for the second term we have

$$-\frac{1}{2j}\sum_{n=0}^{\infty} \alpha^n e^{-jn\omega_0} e^{-jn\omega} = -\frac{1}{2j}\frac{1}{1-\alpha e^{-j(\omega+\omega_0)}}$$

Therefore,

$$X_2(e^{j\omega}) = \frac{1}{2j}\left[\frac{1}{1-\alpha e^{-j(\omega-\omega_0)}} - \frac{1}{1-\alpha e^{-j(\omega+\omega_0)}}\right] = \frac{(\alpha\sin\omega_0)e^{-j\omega}}{1-(2\alpha\cos\omega_0)e^{-j\omega}+\alpha^2 e^{-2j\omega}}$$

(*c*) Finally, for $x_3(n)$, we have

$$X_3(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_3(n)e^{-jn\omega} = \sum_{n=0,2,4,\ldots}^{\infty} \left(\tfrac{1}{2}\right)^n e^{-jn\omega}$$

Therefore,

$$X_3(e^{j\omega}) = \sum_{n=0}^{\infty}\left(\tfrac{1}{2}\right)^{2n} e^{-2jn\omega} = \sum_{n=0}^{\infty}\left(\tfrac{1}{4}e^{-2j\omega}\right)^n = \frac{1}{1-\frac{1}{4}e^{-2j\omega}}$$