LINEAR PHASE SEQUENCES

- Linear Phase

  A sequence $h(n)$ has linear phase if

  $$H(w) = \underbrace{|H(w)|}_{\text{mag}}\, e^{-jw\alpha} \Rightarrow \underbrace{\theta_H(w)}_{\text{Phase}} = -w\alpha.$$

  Such a sequence has a constant group delay

  $$\tau_g = -\frac{d\theta_H(w)}{dw} = \alpha.$$

- Generalized Linear Phase

  $$H(w) = A(w)\, e^{-j(w\alpha-\beta)}$$

  where $A(w)$ is the (bipolar) amplitude spectrum.

  $$\theta_H(w) = \begin{cases} -w\alpha+\beta & \text{if } A(w) > 0 \\ -w\alpha+\beta+\pi & \text{if } A(w) < 0. \end{cases}$$

  $$\tau_g = -\frac{d\theta_H(w)}{dw} = \alpha \quad (\text{constant}).$$

- often, generalized ~~pha~~ linear phase ~~se~~quences are also called linear phase sequences because they have constant group delay.

Linear Phase Sequence (cont'd)

- Causal IIR filters cannot have linear phase (generalized)
- Causal FIR filters can have linear phase (generalized)

<u>Sufficient condition:</u>

$$\boxed{\begin{array}{l} h(n) = h(N-n) \\ \text{or} \quad h(n) = -h(N-n) \\ \quad \text{for } 0 \le n \le N. \end{array}}$$

"Even symmetric" about midpoint
"Antisymmetric"

N — order of the filter
$(N+1)$ — Length of the filter

- Four Types of (generalized) Linear Phase FIR Sequences

Type 1: Even symmetric, Even order (N) / odd length ($N+1$)
2: Even symmetric, odd order (N) / Even length ($N+1$)
3: Antisymmetric, Even order (N) / odd Length ($N+1$)
4: Antisymmetric, odd order (N) / Even Length ($N+1$)

Linear Phase Sequence (Cont'd)

- Type 1: $\boxed{N - \text{even order}, \quad \text{Even Symmetric}}$

$$H(\omega) = h(0) + h(1)e^{-j\omega} + h(2)e^{-j2\omega} + \cdots + h\left(\tfrac{N}{2}\right)e^{-j\omega N/2} + \cdots + h(1)e^{-j\omega(N-1)} + h(0)e^{-j\omega N}$$

$$H(\omega) = \left[h\left(\tfrac{N}{2}\right) + \sum_{k=0}^{\left(\frac{N}{2}-1\right)} h(k)\left[e^{j\omega\left(\frac{N}{2}-k\right)} + e^{-j\omega\left(\frac{N}{2}-k\right)}\right]\right] e^{-j\omega N/2}$$

$$= \underbrace{\left[h\left(\tfrac{N}{2}\right) + \sum_{k=0}^{(N/2-1)} 2h(k)\cos\left(\tfrac{N}{2}-k\right)\omega\right]}_{A(\omega)\ -\ \text{Amplitude spectrum}} e^{-j\omega\frac{N}{2}}$$

$\boxed{\text{Group Delay} = \frac{N}{2} \text{ samples}}$

$$\boxed{\theta_H(\omega) = \begin{cases} -\omega\frac{N}{2}, & A(\omega) > 0 \\ -\omega\frac{N}{2} + \pi, & A(\omega) < 0 \end{cases}}$$

- Type 3: $\boxed{N - \text{even order} \quad \text{Anti Symmetric}}$

$$h(n) = -h(N-n) \qquad 0 \le n \le N$$

$$\Longrightarrow \boxed{h\left(\tfrac{N}{2}\right) = 0}$$

$$H(\omega) = \left[\sum_{k=0}^{(N/2-1)} 2j\, h(k)\sin\left(\tfrac{N}{2}-k\right)\right] e^{-j\omega\frac{N}{2}}$$

$$= \underbrace{\left[\sum_{k=0}^{(N/2-1)} 2h(k)\sin\left(\tfrac{N}{2}-k\right)\right]}_{A(\omega)} e^{-j\left(\omega\frac{N}{2} - \pi/2\right)}$$

$$\theta_H(\omega) = \begin{cases} -\omega\frac{N}{2} + \pi/2, & A(\omega) > 0 \\ -\omega\frac{N}{2} + \frac{3\pi}{2}, & A(\omega) < 0 \end{cases}$$

$\boxed{\text{Group Delay} = N/2 \text{ samples}}$

$\cancel{\theta_H(\omega) = -\omega\frac{N}{2} - \pi/2}$

Linear Phase Sequences (Cont'd)

Type 2: N – odd order, Even symmetric

$\Longrightarrow \frac{N}{2}$ – non-integer

$$H(\omega) = \Bigg[\underbrace{\sum_{k=0}^{(N-1)/2} 2h(k)\cos\left(\left(\tfrac{N}{2}-k\right)\omega\right)}_{A(\omega)}\Bigg] e^{-j\omega \frac{N}{2}}$$

$$\boxed{\theta_H(\omega) = \begin{cases} -\omega \frac{N}{2} & A(\omega) > 0 \\ -\omega\frac{N}{2} + \pi & A(\omega) < 0. \end{cases} \quad \text{Group Delay} = \frac{N}{2} \text{ samples}}$$

Type 4! N – ~~Odd~~ odd order, ~~Anti~~ Anti-symmetric $N/2$ – non-integer

$$H(\omega) = \Bigg[\underbrace{\sum_{k=0}^{(N-1)/2} 2h(k)\sin\left(\left(\tfrac{N}{2}-k\right)\omega\right)}_{A(\omega)}\Bigg] e^{-j\left(\omega\frac{N}{2} - \frac{\pi}{2}\right)}$$

$$\boxed{\theta_H(\omega) = \begin{cases} -\omega\frac{N}{2} + \pi/2, & A(\omega) > 0 \\ -\omega\frac{N}{2} + 3\pi/2, & A(\omega) < 0 \end{cases}}$$

$$\boxed{\text{Group Delay} = \frac{N}{2} \text{ samples}}$$

Linear Phase Sequences (Cont'd)

Zero Locations:

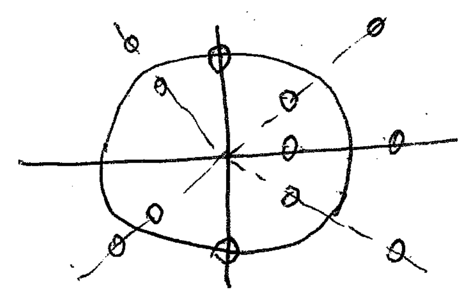
$h(n) = \pm h(N-n), 0 \le n \le N \Rightarrow \boxed{H(z) = z^{-N} H(z^{-1})}$

This implies that the zeros of $H(z)$ are also the zeros of $H(z^{-1})$ and vice versa.

$\Rightarrow$ If z_0 is a zero of $H(z)$, then $\frac{1}{z_0}$ is a ~~root~~ zero of $H(z^{-1})$

$\Rightarrow$ $\boxed{\text{Both } z_0 \text{ and } \frac{1}{z_0} \text{ are zeros of } H(z)}$

$\Rightarrow$
- If $z_0 = re^{j\omega_0}$, $\frac{1}{z_0} = \frac{1}{r} e^{-j\omega_0} \Rightarrow$ Conjugate reciprocal symmetry
- If $z_0 = \alpha$, $\frac{1}{z_0} = \frac{1}{\alpha}$, α real.
- If $z_0 = j\alpha$, α real $\rightarrow$ $\frac{1}{z_0} = \frac{-j}{\alpha}$.



Linear Phase Sequences (Cont'd)

Zero Locations — Constraints on the 4 types of Linear Phase Sequences

Type 1: N - even, Even symmetric

$h(n) = h(N-n), \quad 0 \le n \le N \implies H(z) = z^{-N} H(z^{-1})$

$z = 1$ (DC):

$H(1) = \underbrace{\cancel{(1)^{-N}}}_{1} H(1) \implies$ There may or may not be a zero at $z = \phi$.

$z = -1$ (Half Sampling Frequency)

$H(-1) = \underbrace{\cancel{(-1)^{-N}}}_{1} H(-1) \implies$ $\boxed{\text{There may or may not be a zero at } z = -1}$

Type 2: N - odd, Even symmetric

$H(z) = z^{-N} H(z^{-1})$

$z = 1$

$H(1) = \cancel{(1)^{-N}} H(1) \longrightarrow$ $\boxed{\text{May or May not have a zero at } z = 1}$

$z = -1$

$H(-1) = (-1)^{-N} H(-1)$

$\implies H(-1) = -H(-1) \implies H(-1) = 0 \implies$ $\boxed{\text{Definite zero at } z = -1}$

Linear Phase Sequences (Cont'd)

Zero Location — Constraints. (Cont'd).

Type 3: N – Even order, Antisymmetric

$h(n) = -h(N-n), \; 0 \le n \le N \Rightarrow H(z) = -z^{-N} H(z^{-1})$

$z = 1$

$H(1) = -(1)^{N} H(1) \Rightarrow H(1) = -H(1) \Rightarrow H(1) = 0$

$\Rightarrow \boxed{z = 1 \text{ is a definite zero of } H(z)}$

$z = -1$

$H(-1) = -(-1)^{N} H(-1) \Rightarrow H(-1) = -H(-1) \Rightarrow H(-1) = 0$

$\Rightarrow \boxed{z = -1 \text{ is a definite zero of } H(z)}$

Type 4 N – odd order, Antisymmetric

$z = 1$

$H(1) = -(+1)^{-N} \cdot H(1) \Rightarrow H(1) = -H(1) \Rightarrow H(1) = 0$

$\Rightarrow \boxed{z = 1 \text{ is a definite zero of } H(z)}$

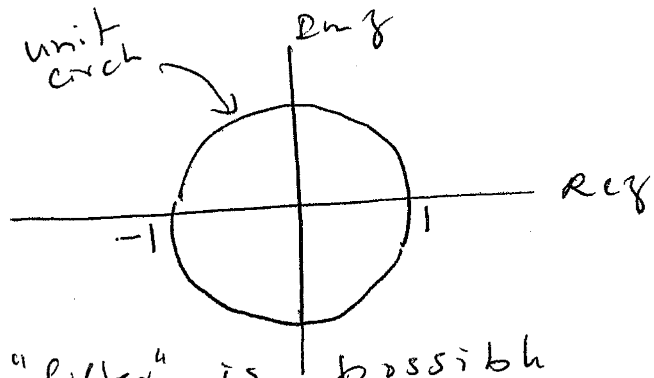
$z = -1$

$H(-1) = -(-1)^{-N} H(-1) \Rightarrow H(-1) = H(-1)$

$\Rightarrow \boxed{z = -1 \text{ may or may not be a zero of } H(z)}$

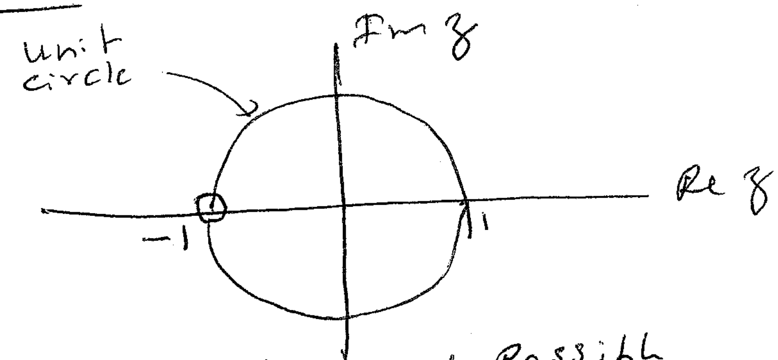
Linear Phase Sequences (Cont'd)
(Definite Zero Location)

Type I: N-even, symmetric



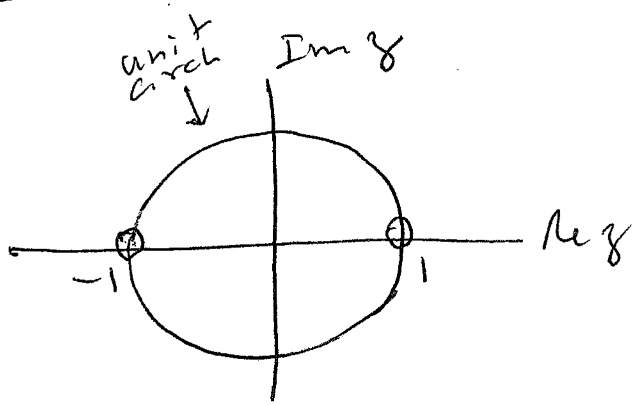
Any "filter" is possible

Type 2 N-odd, symmetric



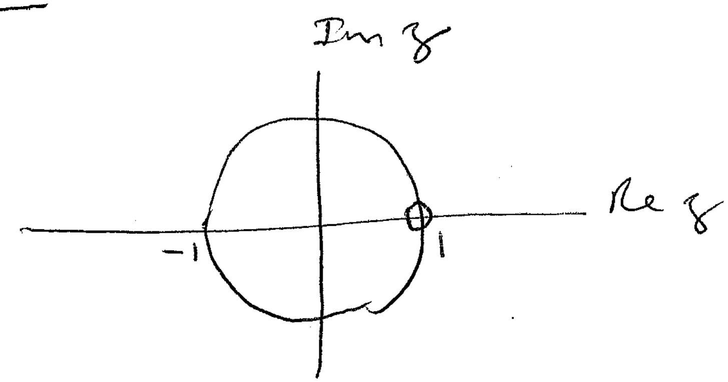
High Pass Filter } not Possible
Band Stop Filter }

Type 3 N-even, Antisymmetric



Low Pass }
High Pass } not possible
Band stop }

Type 4 N-odd, Antisymmetric



Low Pass } Not Possible.
Bandstop }