# Z- TRANSFORM OF A DAMPED SINUSOIDAL SEQUENCE

Let

$$x(n) = [r^n \cos(\omega_0 n + \theta)]\, u(n).$$

To have a decaying sinusoid, we require that $r < 1$.

$$x(n) = \left[\frac{1}{2} r^n e^{j(\omega_0 n + \theta)} + \frac{1}{2} r^n e^{-j(\omega_0 n + \theta)}\right] u(n)$$

$$= \left[\frac{1}{2}(r e^{j\omega_0})^n \cdot e^{j\theta} + \frac{1}{2}(r e^{-j\omega_0})^n e^{-j\theta}\right] u(n)$$

$$X(z) = \frac{\frac{1}{2} e^{j\theta}}{1 - p z^{-1}} + \frac{\frac{1}{2} e^{-j\theta}}{1 - p^* z^{-1}}, \quad |z| > r \qquad — (1)$$

where $p = r e^{j\omega_0}$ and $p^* = r e^{-j\omega_0}$

From (1),

$$X(z) = \frac{\frac{1}{2}(e^{j\theta} + e^{-j\theta}) - \frac{1}{2} z^{-1}(p e^{-j\theta} + p^* e^{j\theta})}{(1 - p z^{-1})(1 - p^* z^{-1})} \qquad — (2)$$

# Z- Transform of a Damped Sinusoidal Sequence

$$X(z) = \frac{\cos\theta - \frac{1}{\cancel{2}} z^{-1} \cancel{2} . \mathrm{Re}\{p e^{-j\theta}\}}{[1 - z^{-1}(p+p^*) + |p|^2 z^{-2}]} \quad -(3)$$

$$\mathrm{Re}\{p e^{-j\theta}\} = \mathrm{Re}\left\{r e^{j(\omega_0-\theta)}\right\} = r\cos(\omega_0-\theta)$$

$$p+p^* = 2r\cos\omega_0$$

$$|p|^2 = r^2$$

$$\Longrightarrow \boxed{X(z) = \frac{[\cos\theta - [r\cos(\omega_0-\theta)]z^{-1}]}{1-(2r\cos\omega_0)z^{-1} + r^2 z^{-2}}} \quad -(4)$$

(4) is the general form from which the special cases of damped cosine and damped sine sequences can be handled.

<u>Case (i)</u> : $\boxed{\theta = 0} \Longrightarrow x(n) = [r^n \cos\omega_0 n]\,u(n)$

using $\theta = 0$ in (4) we get

$$[r^n \cos\omega_0 n]\,u(n) \longleftrightarrow \frac{1-(r\cos\omega_0)z^{-1}}{1-(2r\cos\omega_0)z^{-1}+r^2 z^{-2}}.$$

# Z- Transform of a Damped Sinusoidal Sequence

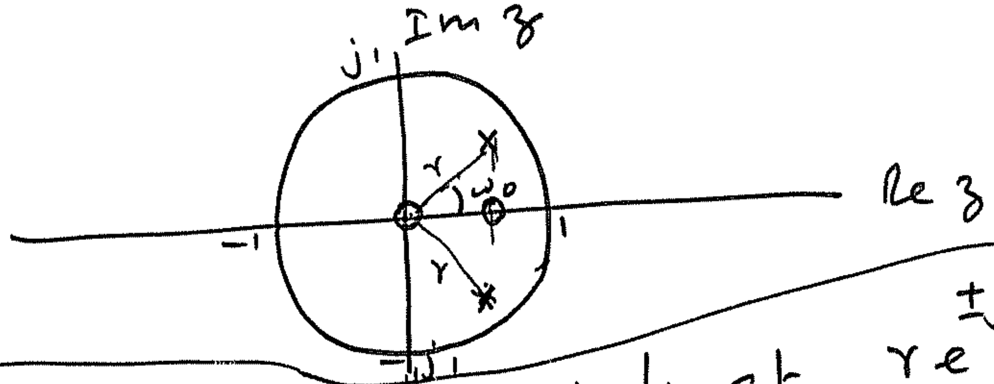
<u>Case (ii)</u> $\theta = -\frac{\pi}{2} \implies x(n) = [r^n \cos(\omega_0 n - \frac{\pi}{2})]\, u(n)$
$$= [r^n \sin \omega_0 n]\, u(n)$$

using $\theta = -\pi/2$ in ④, we have the z-transform pair:

$$[r^n \sin \omega_0 n]\, u(n) \longleftrightarrow \frac{[r \sin \omega_0]\, z^{-1}}{1 - (2r\cos\omega_0) z^{-1} + r^2 z^{-2}}.$$

$\boxed{\text{Pole-zero Configurations}}$

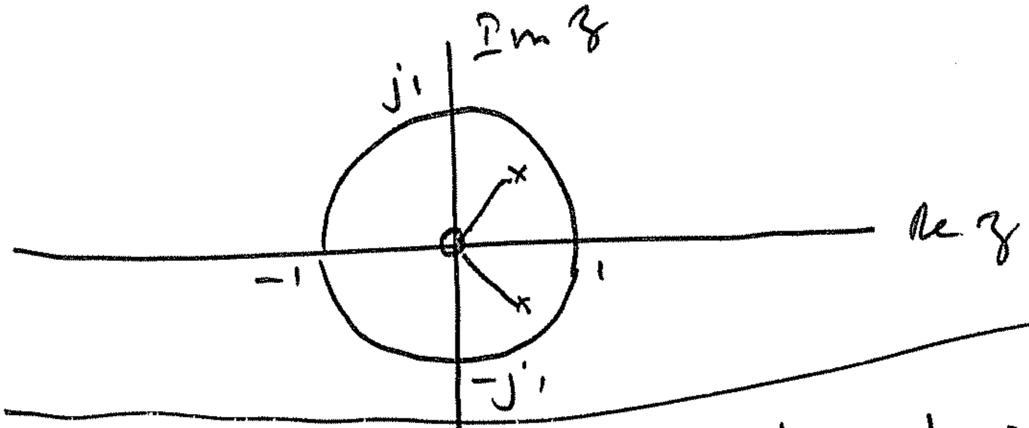
(i) $x(n) = [r^n \cos \omega_0 n]\, u(n)$ [Damped Cosine]




Complex conjugate poles at $re^{\pm j\omega_0}$
Real zeros at $z = r\cos\omega_0$, $z = 0$

# Z-Transform of a Damped Sinusoidal Sequence

(ii) $x(n) = [r^n \sin \omega_0 n]\, u(n)$ Damped Sine Sequence

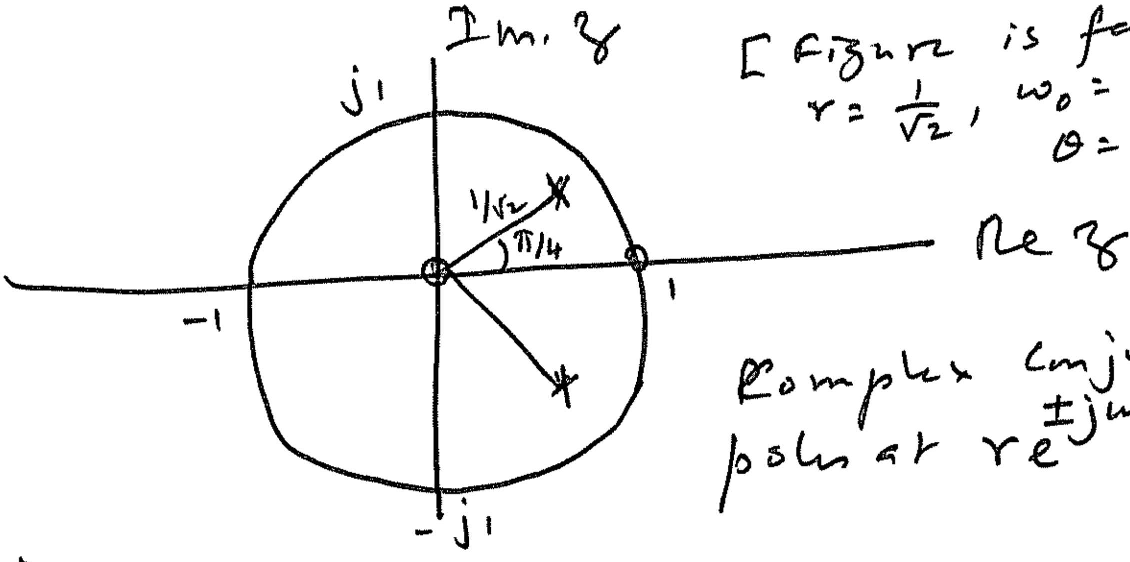



Complex conjugate poles at $re^{\pm j\omega_0}$

zeros at $z = 0$, $z = \infty$.

(iii) $x(n) = [r^n \cos(\omega_0 n + \theta)]\, u(n)$

General Case




[Figure is for $r = \frac{1}{\sqrt{2}}$, $\omega_0 = \pi/4$, $\theta = \pi/4$]

Complex conjugate poles at $re^{\pm j\omega_0}$

Zeros at $z = 0$

and $z = \left[\dfrac{r\cos(\omega_0 - \theta)}{\cos\theta}\right]$ [Depends on $\theta$.