Since $|\mathbb{A}| = 4 \neq 0$, a unique solution exists for x_1, x_2, and x_3. This solution is provided by Cramer's rule [Eq. (B.31)] as follows:

$$x_1 = \frac{1}{|\mathbb{A}|}\begin{vmatrix} 3 & 1 & 1 \\ 7 & 3 & -1 \\ 1 & 1 & 1 \end{vmatrix} = \frac{8}{4} = 2$$

$$x_2 = \frac{1}{|\mathbb{A}|}\begin{vmatrix} 2 & 3 & 1 \\ 1 & 7 & -1 \\ 1 & 1 & 1 \end{vmatrix} = \frac{4}{4} = 1$$

$$x_3 = \frac{1}{|\mathbb{A}|}\begin{vmatrix} 2 & 1 & 3 \\ 1 & 3 & 7 \\ 1 & 1 & 1 \end{vmatrix} = \frac{-8}{4} = -2$$

B.5 Partial Fraction Expansion

In the analysis of linear time-invariant systems, we encounter functions that are ratios of two polynomials in a certain variable, say x. Such functions are known as *rational functions*. A rational function $F(x)$ can be expressed as

$$F(x) = \frac{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}{x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0} \tag{B.32}$$

$$= \frac{P(x)}{Q(x)} \tag{B.33}$$

The function $F(x)$ is *improper* if $m \geq n$ and *proper* if $m < n$. An improper function can always be separated into the sum of a polynomial in x and a proper function. Consider, for example, the function

$$F(x) = \frac{2x^3 + 9x^2 + 11x + 2}{x^2 + 4x + 3} \tag{B.34a}$$

Because this is an improper function, we divide the numerator by the denominator until the remainder has a lower degree than the denominator.

$$\begin{array}{r} 2x + 1 \qquad\qquad\quad \\ x^2 + 4x + 3 \,\overline{)\, 2x^3 + 9x^2 + 11x + 2} \\ \underline{2x^3 + 8x^2 + 6x \quad\;\;} \\ x^2 + 5x + 2 \\ \underline{x^2 + 4x + 3} \\ x - 1 \end{array}$$

Therefore, $F(x)$ can be expressed as

$$F(x) = \frac{2x^3 + 9x^2 + 11x + 2}{x^2 + 4x + 3} = \underbrace{2x + 1}_{\text{polynomial in } x} + \underbrace{\frac{x - 1}{x^2 + 4x + 3}}_{\text{proper function}} \tag{B.34b}$$

A proper function can be further expanded into partial fractions. The remaining discussion in this section is concerned with various ways of doing this.

B.5-1 Method of Clearing Fractions

A rational function can be written as a sum of appropriate partial fractions with unknown coefficients, which are determined by clearing fractions and equating the coefficients of similar powers on the two sides. This procedure is demonstrated by the following example.

EXAMPLE B.8

Expand the following rational function $F(x)$ into partial fractions:

$$F(x) = \frac{x^3 + 3x^2 + 4x + 6}{(x+1)(x+2)(x+3)^2}$$

This function can be expressed as a sum of partial fractions with denominators $(x + 1)$, $(x + 2)$, $(x + 3)$, and $(x + 3)^2$, as follows:

$$F(x) = \frac{x^3 + 3x^2 + 4x + 6}{(x+1)(x+2)(x+3)^2} = \frac{k_1}{x+1} + \frac{k_2}{x+2} + \frac{k_3}{x+3} + \frac{k_4}{(x+3)^2}$$

To determine the unknowns k_1, k_2, k_3, and k_4 we clear fractions by multiplying both sides by $(x + 1)(x + 2)(x + 3)^2$ to obtain

$$\begin{aligned} x^3 + 3x^2 + 4x + 6 &= k_1(x^3 + 8x^2 + 21x + 18) + k_2(x^3 + 7x^2 + 15x + 9) \\ &\quad + k_3(x^3 + 6x^2 + 11x + 6) + k_4(x^2 + 3x + 2) \\ &= x^3(k_1 + k_2 + k_3) + x^2(8k_1 + 7k_2 + 6k_3 + k_4) \\ &\quad + x(21k_1 + 15k_2 + 11k_3 + 3k_4) + (18k_1 + 9k_2 + 6k_3 + 2k_4) \end{aligned}$$

Equating coefficients of similar powers on both sides yields

$$\begin{aligned} k_1 + k_2 + k_3 &= 1 \\ 8k_1 + 7k_2 + 6k_3 + k_4 &= 3 \\ 21k_1 + 15k_2 + 11k_3 + 3k_4 &= 4 \\ 18k_1 + 9k_2 + 6k_3 + 2k_4 &= 6 \end{aligned}$$

Solution of these four simultaneous equations yields

$$k_1 = 1, \qquad k_2 = -2, \qquad k_3 = 2, \qquad k_4 = -3$$

Therefore,

$$F(x) = \frac{1}{x+1} - \frac{2}{x+2} + \frac{2}{x+3} - \frac{3}{(x+3)^2}$$

Although this method is straightforward and applicable to all situations, it is not necessarily the most efficient. We now discuss other methods that can reduce numerical work considerably.

B.5-2 The Heaviside "Cover-Up" Method

Distinct Factors of $Q(x)$

We shall first consider the partial fraction expansion of $F(x) = P(x)/Q(x)$, in which all the factors of $Q(x)$ are distinct (not repeated). Consider the proper function

$$\begin{aligned} F(x) &= \frac{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}{x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0} \qquad m < n \\ &= \frac{P(x)}{(x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n)} \end{aligned} \tag{B.35a}$$

We can show that $F(x)$ in Eq. (B.35a) can be expressed as the sum of partial fractions

$$F(x) = \frac{k_1}{x-\lambda_1} + \frac{k_2}{x-\lambda_2} + \cdots + \frac{k_n}{x-\lambda_n} \tag{B.35b}$$

To determine the coefficient k_1, we multiply both sides of Eq. (B.35b) by $x - \lambda_1$ and then let $x = \lambda_1$. This yields

$$(x-\lambda_1)F(x)|_{x=\lambda_1} = k_1 + \frac{k_2(x-\lambda_1)}{(x-\lambda_2)} + \frac{k_3(x-\lambda_1)}{(x-\lambda_3)} + \cdots + \frac{k_n(x-\lambda_1)}{(x-\lambda_n)}\bigg|_{x=\lambda_1}$$

On the right-hand side, all the terms except k_1 vanish. Therefore,

$$k_1 = (x-\lambda_1)F(x)|_{x=\lambda_1} \tag{B.36}$$

Similarly, we can show that

$$k_r = (x-\lambda_r)F(x)|_{x=\lambda_r} \qquad r = 1, 2, \ldots, n \tag{B.37}$$

This procedure also goes under the name *method of residues*.

EXAMPLE B.9

Expand the following rational function $F(x)$ into partial fractions:

$$F(x) = \frac{2x^2 + 9x - 11}{(x+1)(x-2)(x+3)} = \frac{k_1}{x+1} + \frac{k_2}{x-2} + \frac{k_3}{x+3}$$

To determine k_1, we let $x = -1$ in $(x+1)F(x)$. Note that $(x+1)F(x)$ is obtained from $F(x)$ by omitting the term $(x+1)$ from its denominator. Therefore, to compute k_1 corresponding to the factor $(x+1)$, we cover up the term $(x+1)$ in the denominator of $F(x)$ and then substitute $x = -1$ in the remaining expression. [Mentally conceal the term $(x+1)$ in $F(x)$ with a finger and then let $x = -1$ in the remaining expression.] The steps in covering up the function

$$F(x) = \frac{2x^2 + 9x - 11}{(x+1)(x-2)(x+3)}$$

are as follows.

Step 1. Cover up (conceal) the factor $(x+1)$ from $F(x)$:

$$\frac{2x^2 + 9x - 11}{(x+1)(x-2)(x+3)}$$

Step 2. Substitute $x = -1$ in the remaining expression to obtain k_1:

$$k_1 = \frac{2 - 9 - 11}{(-1-2)(-1+3)} = \frac{-18}{-6} = 3$$

Similarly, to compute k_2, we cover up the factor $(x-2)$ in $F(x)$ and let $x = 2$ in the remaining function, as follows:

$$k_2 = \left.\frac{2x^2 + 9x - 11}{(x+1)(x-2)(x+3)}\right|_{x=2} = \frac{8 + 18 - 11}{(2+1)(2+3)} = \frac{15}{15} = 1$$

and

$$k_3 = \left.\frac{2x^2 + 9x - 11}{(x+1)(x-2)(x+3)}\right|_{x=-3} = \frac{18 - 27 - 11}{(-3+1)(-3-2)} = \frac{-20}{10} = -2$$

Therefore,

$$F(x) = \frac{2x^2 + 9x - 11}{(x+1)(x-2)(x+3)} = \frac{3}{x+1} + \frac{1}{x-2} - \frac{2}{x+3}$$

COMPLEX FACTORS OF $Q(x)$

The procedure just given works regardless of whether the factors of $Q(x)$ are real or complex. Consider, for example,

$$\begin{aligned} F(x) &= \frac{4x^2 + 2x + 18}{(x+1)(x^2 + 4x + 13)} \\ &= \frac{4x^2 + 2x + 18}{(x+1)(x+2-j3)(x+2+j3)} \\ &= \frac{k_1}{x+1} + \frac{k_2}{x+2-j3} + \frac{k_3}{x+2+j3} \end{aligned} \tag{B.38}$$

where

$$k_1 = \left[\frac{4x^2 + 2x + 18}{(x+1)(x^2 + 4x + 13)}\right]_{x=-1} = 2$$

Similarly,

$$k_2 = \left[\frac{4x^2 + 2x + 18}{(x+1)(x+2-j3)(x+2+j3)}\right]_{x=-2+j3} = 1 + j2 = \sqrt{5}e^{j63.43^\circ}$$

$$k_3 = \left[\frac{4x^2 + 2x + 18}{(x+1)(x+2-j3)(x+2+j3)}\right]_{x=-2-j3} = 1 - j2 = \sqrt{5}e^{-j63.43^\circ}$$

Therefore,

$$F(x) = \frac{2}{x+1} + \frac{\sqrt{5}e^{j63.43^\circ}}{x+2-j3} + \frac{\sqrt{5}e^{-j63.43^\circ}}{x+2+j3} \tag{B.39}$$

The coefficients k_2 and k_3 corresponding to the complex conjugate factors are also conjugates of each other. This is generally true when the coefficients of a rational function are real. In such a case, we need to compute only one of the coefficients.

QUADRATIC FACTORS

Often we are required to combine the two terms arising from complex conjugate factors into one quadratic factor. For example, $F(x)$ in Eq. (B.38) can be expressed as

$$F(x) = \frac{4x^2 + 2x + 18}{(x+1)(x^2 + 4x + 13)} = \frac{k_1}{x+1} + \frac{c_1 x + c_2}{x^2 + 4x + 13}$$

The coefficient k_1 is found by the Heaviside method to be 2. Therefore,

$$\frac{4x^2 + 2x + 18}{(x+1)(x^2 + 4x + 13)} = \frac{2}{x+1} + \frac{c_1 x + c_2}{x^2 + 4x + 13} \tag{B.40}$$

The values of c_1 and c_2 are determined by clearing fractions and equating the coefficients of similar powers of x on both sides of the resulting equation. Clearing fractions on both sides of

Eq. (B.40) yields

$$\begin{aligned} 4x^2 + 2x + 18 &= 2(x^2 + 4x + 13) + (c_1 x + c_2)(x + 1) \\ &= (2 + c_1)x^2 + (8 + c_1 + c_2)x + (26 + c_2) \end{aligned} \tag{B.41}$$

Equating terms of similar powers yields $c_1 = 2$, $c_2 = -8$, and

$$\frac{4x^2 + 2x + 18}{(x + 1)(x^2 + 4x + 13)} = \frac{2}{x + 1} + \frac{2x - 8}{x^2 + 4x + 13} \tag{B.42}$$

SHORTCUTS

The values of c_1 and c_2 in Eq. (B.40) can also be determined by using shortcuts. After computing $k_1 = 2$ by the Heaviside method as before, we let $x = 0$ on both sides of Eq. (B.40) to eliminate c_1. This gives us

$$\frac{18}{13} = 2 + \frac{c_2}{13}$$

Therefore,

$$c_2 = -8$$

To determine c_1, we multiply both sides of Eq. (B.40) by x and then let $x \to \infty$. Remember that when $x \to \infty$, only the terms of the highest power are significant. Therefore,

$$4 = 2 + c_1$$

and

$$c_1 = 2$$

In the procedure discussed here, we let $x = 0$ to determine c_2 and then multiply both sides by x and let $x \to \infty$ to determine c_1. However, nothing is sacred about these values ($x = 0$ or $x = \infty$). We use them because they reduce the number of computations involved. We could just as well use other convenient values for x, such as $x = 1$. Consider the case

$$\begin{aligned} F(x) &= \frac{2x^2 + 4x + 5}{x(x^2 + 2x + 5)} \\ &= \frac{k}{x} + \frac{c_1 x + c_2}{x^2 + 2x + 5} \end{aligned}$$

We find $k = 1$ by the Heaviside method in the usual manner. As a result,

$$\frac{2x^2 + 4x + 5}{x(x^2 + 2x + 5)} = \frac{1}{x} + \frac{c_1 x + c_2}{x^2 + 2x + 5} \tag{B.43}$$

To determine c_1 and c_2, if we try letting $x = 0$ in Eq. (B.43), we obtain ∞ on both sides. So let us choose $x = 1$. This yields

$$\frac{11}{8} = 1 + \frac{c_1 + c_2}{8}$$

or

$$c_1 + c_2 = 3$$

We can now choose some other value for x, such as $x = 2$, to obtain one more relationship to use in determining c_1 and c_2. In this case, however, a simple method is to multiply both sides of Eq. (B.43) by x and then let $x \to \infty$. This yields

$$2 = 1 + c_1$$

so that

$$c_1 = 1 \qquad \text{and} \qquad c_2 = 2$$

Therefore,

$$F(x) = \frac{1}{x} + \frac{x+2}{x^2 + 2x + 5}$$

B.5-3 Repeated Factors of $Q(x)$

If a function $F(x)$ has a repeated factor in its denominator, it has the form

$$F(x) = \frac{P(x)}{(x-\lambda)^r (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_j)} \tag{B.44}$$

Its partial fraction expansion is given by

$$\begin{aligned} F(x) = {} & \frac{a_0}{(x-\lambda)^r} + \frac{a_1}{(x-\lambda)^{r-1}} + \cdots + \frac{a_{r-1}}{(x-\lambda)} \\ & + \frac{k_1}{x-\alpha_1} + \frac{k_2}{x-\alpha_2} + \cdots + \frac{k_j}{x-\alpha_j} \end{aligned} \tag{B.45}$$

The coefficients $k_1, k_2, \ldots, k_j$ corresponding to the unrepeated factors in this equation are determined by the Heaviside method, as before [Eq. (B.37)]. To find the coefficients $a_0, a_1, a_2, \ldots, a_{r-1}$, we multiply both sides of Eq. (B.45) by $(x-\lambda)^r$. This gives us

$$\begin{aligned} (x-\lambda)^r F(x) = {} & a_0 + a_1(x-\lambda) + a_2(x-\lambda)^2 + \cdots + a_{r-1}(x-\lambda)^{r-1} \\ & + k_1\frac{(x-\lambda)^r}{x-\alpha_1} + k_2\frac{(x-\lambda)^r}{x-\alpha_2} + \cdots + k_n\frac{(x-\lambda)^r}{x-\alpha_n} \end{aligned} \tag{B.46}$$

If we let $x = \lambda$ on both sides of Eq. (B.46), we obtain

$$(x-\lambda)^r F(x)\big|_{x=\lambda} = a_0 \tag{B.47a}$$

Therefore, a_0 is obtained by concealing the factor $(x-\lambda)^r$ in $F(x)$ and letting $x = \lambda$ in the remaining expression (the Heaviside "cover-up" method). If we take the derivative (with respect to x) of both sides of Eq. (B.46), the right-hand side is a_1+ terms containing a factor $(x-\lambda)$ in their numerators. Letting $x = \lambda$ on both sides of this equation, we obtain

$$\frac{d}{dx}\left[(x-\lambda)^r F(x)\right]\bigg|_{x=\lambda} = a_1$$

Thus, a_1 is obtained by concealing the factor $(x-\lambda)^r$ in $F(x)$, taking the derivative of the remaining expression, and then letting $x=\lambda$. Continuing in this manner, we find

$$a_j = \frac{1}{j!}\frac{d^j}{dx^j}\left[(x-\lambda)^r F(x)\right]\Bigg|_{x=\lambda} \tag{B.47b}$$

Observe that $(x-\lambda)^r F(x)$ is obtained from $F(x)$ by omitting the factor $(x-\lambda)^r$ from its denominator. Therefore, the coefficient a_j is obtained by concealing the factor $(x-\lambda)^r$ in $F(x)$, taking the jth derivative of the remaining expression, and then letting $x=\lambda$ (while dividing by $j!$).

EXAMPLE B.10

Expand $F(x)$ into partial fractions if

$$F(x) = \frac{4x^3+16x^2+23x+13}{(x+1)^3(x+2)}$$

The partial fractions are

$$F(x) = \frac{a_0}{(x+1)^3} + \frac{a_1}{(x+1)^2} + \frac{a_2}{x+1} + \frac{k}{x+2}$$

The coefficient k is obtained by concealing the factor $(x+2)$ in $F(x)$ and then substituting $x=-2$ in the remaining expression:

$$k = \frac{4x^3+16x^2+23x+13}{(x+1)^3\,(x+2)}\Bigg|_{x=-2} = 1$$

To find a_0, we conceal the factor $(x+1)^3$ in $F(x)$ and let $x=-1$ in the remaining expression:

$$a_0 = \frac{4x^3+16x^2+23x+13}{(x+1)^3\,(x+2)}\Bigg|_{x=-1} = 2$$

To find a_1, we conceal the factor $(x+1)^3$ in $F(x)$, take the derivative of the remaining expression, and then let $x=-1$:

$$a_1 = \frac{d}{dx}\left[\frac{4x^3+16x^2+23x+13}{(x+1)^3\,(x+2)}\right]\Bigg|_{x=-1} = 1$$

Similarly,

$$a_2 = \frac{1}{2!}\frac{d^2}{dx^2}\left[\frac{4x^3+16x^2+23x+13}{(x+1)^3\,(x+2)}\right]\Bigg|_{x=-1} = 3$$

Therefore,

$$F(x) = \frac{2}{(x+1)^3} + \frac{1}{(x+1)^2} + \frac{3}{x+1} + \frac{1}{x+2}$$

B.5-4 Mixture of the Heaviside "Cover-Up" and Clearing Fractions

For multiple roots, especially of higher order, the Heaviside expansion method, which requires repeated differentiation, can become cumbersome. For a function that contains several repeated and unrepeated roots, a hybrid of the two procedures proves to be the best. The simpler coefficients are determined by the Heaviside method, and the remaining coefficients are found by clearing fractions or shortcuts, thus incorporating the best of the two methods. We demonstrate this procedure by solving Example B.10 once again by this method.

In Example B.10, coefficients k and a_0 are relatively simple to determine by the Heaviside expansion method. These values were found to be $k_1 = 1$ and $a_0 = 2$. Therefore,

$$\frac{4x^3 + 16x^2 + 23x + 13}{(x+1)^3(x+2)} = \frac{2}{(x+1)^3} + \frac{a_1}{(x+1)^2} + \frac{a_2}{x+1} + \frac{1}{x+2}$$

We now multiply both sides of this equation by $(x+1)^3(x+2)$ to clear the fractions. This yields

$$\begin{aligned} 4x^3 &+ 16x^2 + 23x + 13 \\ &= 2(x+2) + a_1(x+1)(x+2) + a_2(x+1)^2(x+2) + (x+1)^3 \\ &= (1+a_2)x^3 + (a_1 + 4a_2 + 3)x^2 + (5 + 3a_1 + 5a_2)x + (4 + 2a_1 + 2a_2 + 1) \end{aligned}$$

Equating coefficients of the third and second powers of x on both sides, we obtain

$$\left.\begin{aligned} 1 + a_2 &= 4 \\ a_1 + 4a_2 + 3 &= 16 \end{aligned}\right\} \Longrightarrow \begin{aligned} a_1 &= 1 \\ a_2 &= 3 \end{aligned}$$

We may stop here if we wish because the two desired coefficients, a_1 and a_2, are now determined. However, equating the coefficients of the two remaining powers of x yields a convenient check on the answer. Equating the coefficients of the x^1 and x^0 terms, we obtain

$$\begin{aligned} 23 &= 5 + 3a_1 + 5a_2 \\ 13 &= 4 + 2a_1 + 2a_2 + 1 \end{aligned}$$

These equations are satisfied by the values $a_1 = 1$ and $a_2 = 3$, found earlier, providing an additional check for our answers. Therefore,

$$F(x) = \frac{2}{(x+1)^3} + \frac{1}{(x+1)^2} + \frac{3}{x+1} + \frac{1}{x+2}$$

which agrees with the earlier result.

A Mixture of the Heaviside "Cover-Up" and Shortcuts

In Example B.10, after determining the coefficients $a_0 = 2$ and $k = 1$ by the Heaviside method as before, we have

$$\frac{4x^3 + 16x^2 + 23x + 13}{(x+1)^3(x+2)} = \frac{2}{(x+1)^3} + \frac{a_1}{(x+1)^2} + \frac{a_2}{x+1} + \frac{1}{x+2}$$

There are only two unknown coefficients, a_1 and a_2. If we multiply both sides of this equation by x and then let $x \to \infty$, we can eliminate a_1. This yields

$$4 = a_2 + 1 \quad \Longrightarrow \quad a_2 = 3$$

Therefore,

$$\frac{4x^3 + 16x^2 + 23x + 13}{(x+1)^3(x+2)} = \frac{2}{(x+1)^3} + \frac{a_1}{(x+1)^2} + \frac{3}{x+1} + \frac{1}{x+2}$$

There is now only one unknown a_1, which can be readily found by setting x equal to any convenient value, say $x = 0$. This yields

$$\frac{13}{2} = 2 + a_1 + 3 + \frac{1}{2} \quad \Longrightarrow \quad a_1 = 1$$

which agrees with our earlier answer.

There are other possible shortcuts. For example, we can compute a_0 (coefficient of the highest power of the repeated root), subtract this term from both sides, and then repeat the procedure.

B.5-5 Improper $F(x)$ with $m = n$

A general method of handling an improper function is indicated in the beginning of this section. However, for a special case of the numerator and denominator polynomials of $F(x)$ being of the same degree ($m = n$), the procedure is the same as that for a proper function. We can show that for

$$\begin{aligned} F(x) &= \frac{b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0}{x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0} \\ &= b_n + \frac{k_1}{x-\lambda_1} + \frac{k_2}{x-\lambda_2} + \cdots + \frac{k_n}{x-\lambda_n} \end{aligned}$$

the coefficients $k_1, k_2, \ldots, k_n$ are computed as if $F(x)$ were proper. Thus,

$$k_r = (x - \lambda_r)F(x)|_{x=\lambda_r}$$

For quadratic or repeated factors, the appropriate procedures discussed in Sections B.5-2 or B.5-3 should be used as if $F(x)$ were proper. In other words, when $m = n$, the only difference between the proper and improper case is the appearance of an extra constant b_n in the latter. Otherwise the procedure remains the same. The proof is left as an exercise for the reader.

EXAMPLE B.11

Expand $F(x)$ into partial fractions if

$$F(x) = \frac{3x^2 + 9x - 20}{x^2 + x - 6} = \frac{3x^2 + 9x - 20}{(x-2)(x+3)}$$

Here $m = n = 2$ with $b_n = b_2 = 3$. Therefore,

$$F(x) = \frac{3x^2 + x - 20}{(x-2)(x+3)} = 3 + \frac{k_1}{x-2} + \frac{k_2}{x+3}$$

in which

$$k_1 = \left.\frac{3x^2 + 9x - 20}{(x-2)(x+3)}\right|_{x=2} = \frac{12 + 18 - 20}{(2+3)} = \frac{10}{5} = 2$$

and

$$k_2 = \left.\frac{3x^2 + 9x - 20}{(x-2)(x+3)}\right|_{x=-3} = \frac{27 - 27 - 20}{(-3-2)} = \frac{-20}{-5} = 4$$

Therefore,

$$F(x) = \frac{3x^2 + 9x - 20}{(x-2)(x+3)} = 3 + \frac{2}{x-2} + \frac{4}{x+3}$$

B.5-6 Modified Partial Fractions

In finding the inverse z-transforms (Chapter 5), we require partial fractions of the form $kx/(x - \lambda_i)^r$ rather than $k/(x - \lambda_i)^r$. This can be achieved by expanding $F(x)/x$ into partial fractions. Consider, for example,

$$F(x) = \frac{5x^2 + 20x + 18}{(x+2)(x+3)^2}$$

Dividing both sides by x yields

$$\frac{F(x)}{x} = \frac{5x^2 + 20x + 18}{x(x+2)(x+3)^2}$$

Expansion of the right-hand side into partial fractions as usual yields

$$\frac{F(x)}{x} = \frac{5x^2 + 20x + 18}{x(x+2)(x+3)^2} = \frac{a_1}{x} + \frac{a_2}{x+2} + \frac{a_3}{(x+3)} + \frac{a_4}{(x+3)^2}$$

Using the procedure discussed earlier, we find $a_1 = 1$, $a_2 = 1$, $a_3 = -2$, and $a_4 = 1$. Therefore,

$$\frac{F(x)}{x} = \frac{1}{x} + \frac{1}{x+2} - \frac{2}{x+3} + \frac{1}{(x+3)^2}$$

Now multiplying both sides by x yields

$$F(x) = 1 + \frac{x}{x+2} - \frac{2x}{x+3} + \frac{x}{(x+3)^2}$$

This expresses $F(x)$ as the sum of partial fractions having the form $kx/(x-\lambda_i)^r$.

B.6 Vectors and Matrices

An entity specified by n numbers in a certain order (ordered n-tuple) is an n-dimensional *vector*. Thus, an ordered n-tuple $(x_1, x_2, \ldots, x_n)$ represents an n-dimensional vector $\mathbf{x}$. A vector may be represented as a row (*row vector*):

$$\mathbf{x} = [x_1 \quad x_2 \quad \cdots \quad x_n]$$

or as a column (*column vector*):

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Simultaneous linear equations can be viewed as the transformation of one vector into another. Consider, for example, the n simultaneous linear equations

$$\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ y_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ &\vdots \\ y_m &= a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{aligned} \tag{B.48}$$

If we define two column vectors $\mathbf{x}$ and $\mathbf{y}$ as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \tag{B.49}$$

then Eqs. (B.48) may be viewed as the relationship or the function that transforms vector $\mathbf{x}$ into vector $\mathbf{y}$. Such a transformation is called the *linear transformation* of vectors. To perform a linear transformation, we need to define the array of coefficients a_{ij} appearing in Eqs. (B.48). This array is called a *matrix* and is denoted by $\mathbf{A}$ for convenience:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \tag{B.50}$$

A matrix with m rows and n columns is called a matrix of the order (m, n) or an $(m \times n)$ matrix. For the special case of $m = n$, the matrix is called a *square matrix* of order n.