4. (20 points)
When the input to an LTI system is $x(n) = (1/3)^n u(n) + (2)^n u(-n-1)$, the corresponding output is given by $y(n) = 5\,(1/3)^n u(n) - 5\,(2/3)^n u(n)$. (a) Find the system transfer function H (z), and indicate the region of convergence, (b) Sketch the Direct Form II (canonical or minimum delays) implementation of the system, and (c) Is the system stable? Is it causal?

$$x(n) = \left(\tfrac{1}{3}\right)^n u(n) + (2)^n u(-n-1)$$

$$X(z) = \frac{z}{z-1/3} - \frac{z}{z-2}, \quad ROC_x: \tfrac{1}{3} < |z| < 2$$

$$= \frac{-5/3\, z}{(z-1/3)(z-2)} \qquad ROC_x: \uparrow \quad \text{— (1)}$$

$$y(n) = 5\cdot(1/3)^n u(n) - 5\cdot(2/3)^n u(n)$$

$$Y(z) = \frac{5z}{(z-1/3)} - \frac{5z}{(z-2/3)}, \quad ROC_y: |z| > 2/3$$

$$= \frac{-5/3\, z}{(z-1/3)(z-2/3)} \qquad ROC_y: \uparrow \quad \text{— (2)}$$

$$Y(z) = H(z)\cdot X(z) \implies ROC_H: |z| > 2/3$$

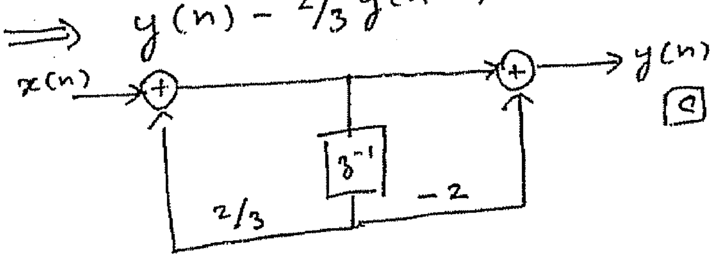
a.
$$\boxed{H(z) = \frac{Y(z)}{X(z)} = \frac{(z-2)}{(z-2/3)}} \quad \text{— (3)}$$

[note: $ROC_y \supseteq ROC_H \cap ROC_x$ at least]

b.
$$(z-2/3)\,Y(z) = (z-2)\,X(z)$$

or
$$(1-2/3\, z^{-1})\,Y(z) = (1-2z^{-1})\,X(z) \quad \text{— (4)}$$

$$\implies y(n) - 2/3\, y(n-1) = x(n) - 2\,x(n-1)$$

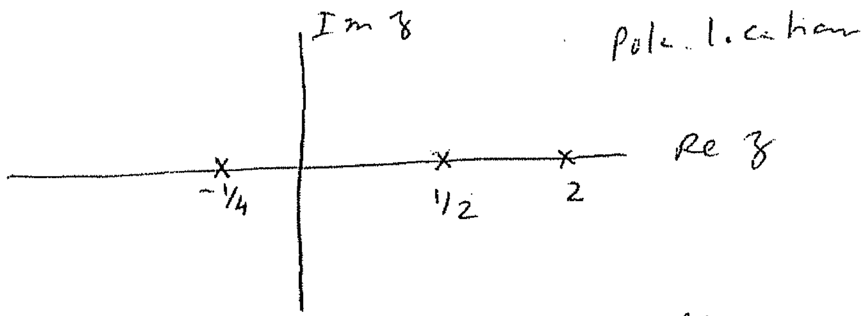


c. Stable, because unit cir. is in the ROC.
Causal because ROC_H is $|z| > 2/3$.

4. Find all possible discrete time sequences that correspond to each of the following z-transforms; comment on the stability of each of the resulting sequences.

$$H(z) = \frac{[3 - 9/2\, z^{-1} + 3/8\, z^{-2}]}{(1 + \tfrac{1}{4} z^{-1})(1 - \tfrac{1}{2} z^{-1})(1 - 2 z^{-1})} = \frac{A}{(1 + \tfrac{1}{4} z^{-1})} + \frac{B}{(1 - \tfrac{1}{2} z^{-1})} + \frac{C}{(1 - 2 z^{-1})}$$

$$= \frac{1}{(1 + \tfrac{1}{4} z^{-1})} + \frac{1}{(1 - \tfrac{1}{2} z^{-1})} + \frac{1}{(1 - 2 z^{-1})}$$



There are __four__ possible sequence

(i) $|z| > 2$, __causal, unstable__ sequence

$$h_1(n) = \left(-\tfrac{1}{4}\right)^n u(n) + \left(\tfrac{1}{2}\right)^n u(n) + (2)^n u(n)$$

(ii) $|z| < \frac{1}{4}$, __anticausal, unstable__ sequence

$$h_2(n) = -\left(-\tfrac{1}{4}\right)^n u(-n-1) - \left(\tfrac{1}{2}\right)^n u(-n-1) - (2)^n u(-n-1)$$

(iii) $\frac{1}{2} < |z| < 2$, __Two-sided, stable__ sequence

$$h_3(n) = \left(-\tfrac{1}{4}\right)^n u(n) + \left(\tfrac{1}{2}\right)^n u(n) - (2)^n u(-n-1)$$

(iv) $\frac{1}{4} < |z| < \frac{1}{2}$, __Two-sided, unstable__ sequence

$$h_4(n) = \left(-\tfrac{1}{4}\right)^n u(n) - \left(\tfrac{1}{2}\right)^n u(-n-1) - (2)^n u(-n-1)$$

4. (20 points)
Determine all possible signals associated with the z-transform

$$X(z) = \frac{5z^{-1}}{(1-2z^{-1})\ (3-z^{-1})}.$$ Indicate whether each of the signals is stable or not.

$$X(z) = \frac{5/3\ z^{-1}}{(1-2z^{-1})(1-1/3\,z^{-1})} = \frac{A}{(1-2z^{-1})} + \frac{B}{(1-1/3\,z^{-1})}$$

$$A = \left.\frac{5/3\ z^{-1}}{(1-1/3\,z^{-1})}\right|_{z^{-1}=\frac{1}{2}} = \frac{5/6}{5/6} = \boxed{1}$$

$$B = \left.\frac{5/3\ z^{-1}}{(1-2z^{-1})}\right|_{z^{-1}=3} = \frac{5}{-5} = \boxed{-1}$$

Three possible sequences

ROC 1: $|z| > 2$, causal sequence, NOT stable

$$\boxed{x(n) = \left[(2)^n - \left(\tfrac{1}{3}\right)^n\right]u(n)}$$

ROC 2: $|z| < \frac{1}{3}$, Anticausal sequence, NOT stable

$$\boxed{x(n) = \left[-(2)^n + \left(\tfrac{1}{3}\right)^n\right]u(-n-1)}$$

ROC 3: $\frac{1}{3} < |z| < 2$, Two-sided sequence, STABLE

$$\boxed{x(n) = -\left(\tfrac{1}{3}\right)^n u(n) - (2)^n u(-n-1)}$$

6. (20 points)
A causal, LTI system is described by the following LCCDE: —— (1)

$$y(n) - 0.5\, y(n-1) = x(n), \quad n \geq 0 \text{ with } y(-1) = 1, \text{ and } x(n) = (1/3)^n u(n).$$

Determine $y_{zi}(n)$, the zero input response and $y_{zs}(n)$, the zero state response of the system. What is the response of the system if the initial condition is changed to $y(-1) = 2$?

Taking 1-sided z-transform of (1), we get

$$Y(z) - 0.5\left[z^{-1}Y(z) + y(-1)\right] = X(z)$$

$$Y(z) = \underbrace{\frac{X(z)}{(1-\frac{1}{2}z^{-1})}}_{Z^+[y_{zs}(n)]} + \underbrace{\frac{\frac{1}{2}y(-1)}{(1-\frac{1}{2}z^{-1})}}_{Z^+[y_{zi}(n)]} \quad \text{—— (2)}$$

$$X(z) = Z^+\left[\left(\tfrac{1}{3}\right)^n u(n)\right] = \frac{1}{(1-\frac{1}{3}z^{-1})} \quad \text{—— (3)}$$

using (3) in (2) we get

$$Y(z) = \frac{1}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{3}z^{-1})} + \frac{\frac{1}{2}y(-1)}{(1-\frac{1}{2}z^{-1})} \quad \text{—— (4)}$$

from (4)

$$Z^+[y_{zs}(n)] = \frac{3}{(1-\frac{1}{2}z^{-1})} + \frac{-2}{(1-\frac{1}{3}z^{-1})} \quad \text{—— (5)}$$

$$\Rightarrow \boxed{y_{zs}(n) = \left[3\left(\tfrac{1}{2}\right)^n - 2\left(\tfrac{1}{3}\right)^n\right]u(n)}$$

$$\boxed{y_{zi}(n) = \tfrac{1}{2}\cdot y(-1)\cdot\left(\tfrac{1}{2}\right)^n u(n)} \Rightarrow \boxed{y_{zi}(n) = \tfrac{1}{2}\left(\tfrac{1}{2}\right)^n u(n)}$$

$$y(n) = y_{zs}(n) + y_{zi}(n).$$

Only $y_{zi}(n)$ is changed for part(ii). The new

$$y_{zi_{new}}(n) = \tfrac{1}{\cancel{2}}\cdot\cancel{2}\cdot\left(\tfrac{1}{2}\right)^n u(n) = \left(\tfrac{1}{2}\right)^n u(n)$$

$$\boxed{y_{new}(n) = y_{zs}(n) + y_{zi_{new}}(n)}$$

3. **(20 points)**
The following LCCDE describes the input-output relationship of a causal, LTI system; determine the impulse response, h (n) of the LTI system.

$$y(n) - \tfrac{1}{2} y(n-1) + \tfrac{1}{4} y(n-2) = x(n) - \tfrac{1}{4} x(n-1)$$

<u>(Note: Your answer for h(n) should contain only real coefficients to receive credit.)</u>

$$y(n) - \frac{1}{2} y(n-1) + \frac{1}{4} y(n-2) = x(n) - \frac{1}{4} x(n-1)$$

Taking z-transform on both sides, we have:

$$Y(z) - \frac{1}{2} z^{-1} Y(z) + \frac{1}{4} z^{-2} Y(z) = X(z) - \frac{1}{4} z^{-1} X(z)$$

or

$$H(z) = \frac{Y(z)}{X(z)} = \frac{(1 - \frac{1}{4} z^{-1})}{(1 - \frac{1}{2} z^{-1} + \frac{1}{4} z^{-2})} \quad \text{— (1)}$$

From z-transform tables, we have:

$$[r^n \cos \omega_0 n] u(n) \longleftrightarrow \frac{[1 - (r\cos\omega_0) z^{-1}]}{(1 - (2r\cos\omega_0) z^{-1} + r^2 z^{-2})} \quad \text{— (2)}$$

Comparing (1) & (2), we have:

$r\cos\omega_0 = \frac{1}{4}$ and $r^2 = 1/4 \Rightarrow r = \frac{1}{2}$, $\cos\omega_0 = \frac{1}{2}$

$\Rightarrow \omega_0 = \frac{\pi}{3}$.

Thus,

$$\boxed{h(n) = \left[\left(\frac{1}{2}\right)^n \cos \frac{\pi}{3} n\right] u(n)}$$