Z- TRANSFORM

Definition:

Let $x(n)$ be a discrete time signal. Then

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} \qquad (1)$$

is called the z-transform of $x(n)$ if the series converges for some values of the complex variable z. Then we write,

$$x(n) \longleftrightarrow X(z).$$

The region of the complex z-plane in which $X(z)$ is finite (i.e. converges) is called the region of convergence (ROC).

Z - TRANSFORM

Z- transform plays a role in discrete time signals & systems similar to what Laplace Transform plays in continuous time signals and systems.

The Z-transform has a relationship to the Fourier Transform (DTFT) as well.

The complex variable z can be written as

$$z = re^{j\omega}$$

using the above in ①, we get

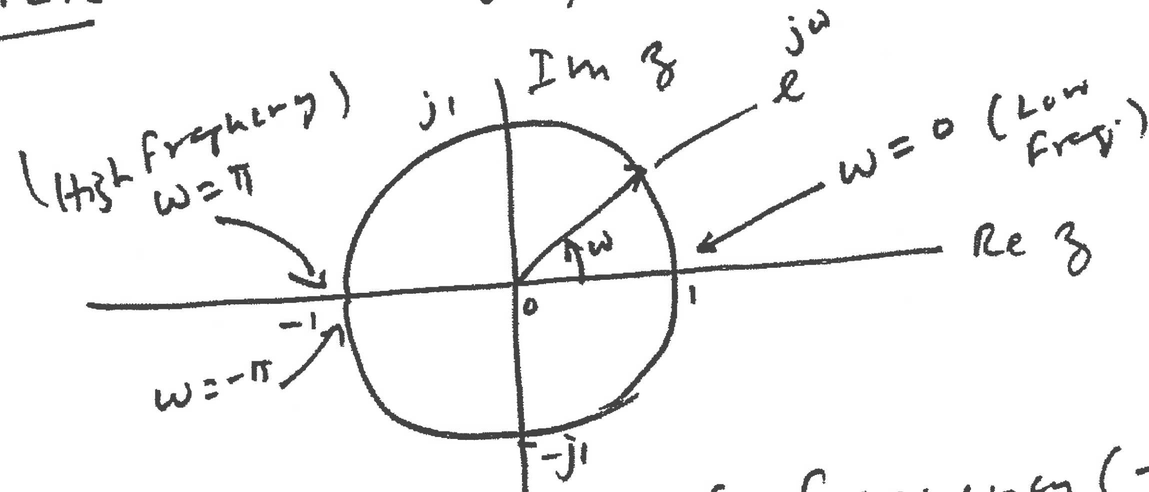
$$X(z)\Big|_{z=re^{j\omega}} = \sum_{n=-\infty}^{\infty} \left[x(n)r^{-n}\right]e^{-j\omega n} \quad \text{— ②}$$

or $X(re^{j\omega})$ is the FT of $\left[x(n)r^{-n}\right]$.

Z - Transform

In (2), if we set $r = 1$, we obtain the F.T. of $x(n)$, i.e. $X(\omega)$, provided it exists[†].

$r = 1$ in $z = re^{j\omega}$ describes the unit circle in the z-plane.



The principal range of frequency $(-\pi, \pi)$ is mapped onto the unit circle.

† $\sum_{n=-\infty}^{\infty} |x(n)| < \infty$ guarantees the existence of $X(\omega)$.

Z - Transform

<u>Example:</u> (Right sided Sequence)

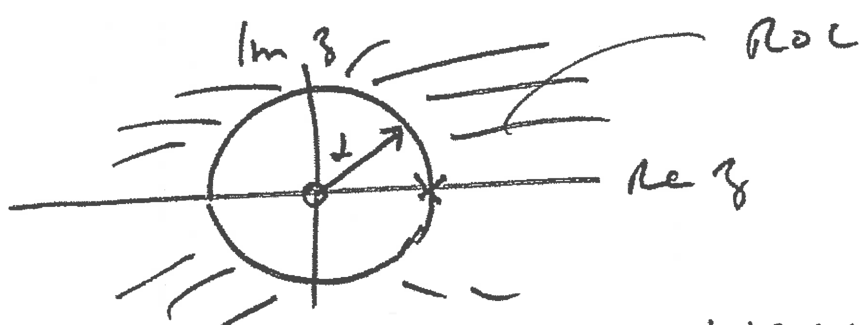
$$x(n) = \alpha^n u(n) \longleftarrow \text{Causal signal (Right sided)}$$

$$X(z) = \sum_{n=0}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{\alpha}{z}\right)^n$$

The above will converge only for $\left|\frac{\alpha}{z}\right| < 1$

or $\boxed{|z| > |\alpha|} \Longleftarrow$ ROC

$$X(z) = \frac{1}{1-(\alpha/z)} = \frac{1}{1-\alpha z^{-1}}, \quad \text{R.O.C. } |z| > |\alpha|$$



$$X(z) = \frac{z}{z-\alpha}$$

$\boxed{\begin{array}{l}\text{Pole at } z = \alpha \\ \text{Zero at } z = 0\end{array}}$

(assuming α is +ve).

$X(\omega) = X(z)\big|_{z=e^{j\omega}}$ will exist only for $|\alpha| < 1$. (why?)

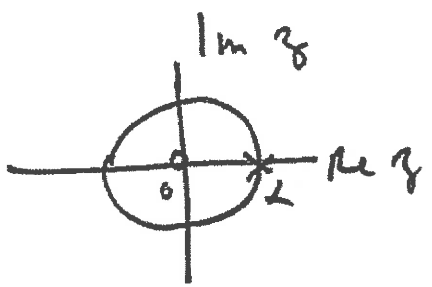
Z - TRANSFORM

<u>Example:</u> (Left sided Sequence)

$$x(n) = -\alpha^n u(-n-1)$$

$$X(z) = \sum_{n=-\infty}^{-1} -\alpha^n z^{-n} = -\sum_{n=-\infty}^{-1} (\alpha z^{-1})^n$$

$$= -\sum_{n=1}^{\infty} (\alpha^{-1} z)^n = 1 - \sum_{n=0}^{\infty} (\alpha^{-1} z)^n$$



$$= 1 - \frac{1}{1-\alpha^{-1} z} \quad \text{for } |\alpha^{-1} z| < 1$$

$$\Rightarrow \boxed{|z| < |\alpha|}$$

ROC

$$= \frac{z}{z-\alpha} = \frac{1}{1-\alpha z^{-1}}$$

Note that we obtained the same $X(z)$ as in previous example. But what is different is ROC

Z - TRANSFORM

<u>Example:</u> (Two sided sequence)

$$x(n) = \alpha^n u(n) - \beta^n u(-n-1)$$

$$X(z) = \sum_{n=0}^{\infty} \alpha^n z^{-n} - \sum_{n=-\infty}^{-1} \beta^n z^{-n}$$

The first series will converge for $|z| > |\alpha|$, and the second series will converge for $|z| < |\beta|$. For what values of z will $X(z)$ converge? The two R.O.C.s must <u>overlap</u>.

$\Rightarrow$ There must be a common region between

$$|z| > |\alpha| \text{ and } |z| < |\beta|$$

$\Rightarrow$ $X(z)$ will converge only if $|\beta| > |\alpha|$.

R.O.C then is given by

$$\boxed{|\alpha| < |z| < |\beta|}$$

Z- TRANSFORM

<u>Example (cont'd)</u>

For $|\beta| > |\alpha|$, the z-transform is then given by

$$X(z) = \frac{1}{1-\alpha z^{-1}} + \frac{1}{1-\beta z^{-1}}$$

$$= \frac{z}{z-\alpha} + \frac{z}{z-\beta} = \frac{2z\left(z - \frac{(\alpha+\beta)}{2}\right)}{(z-\alpha)\,(z-\beta)},$$

ROC → $\boxed{|\alpha| < |z| < |\beta|}$

<u>Poles</u> at

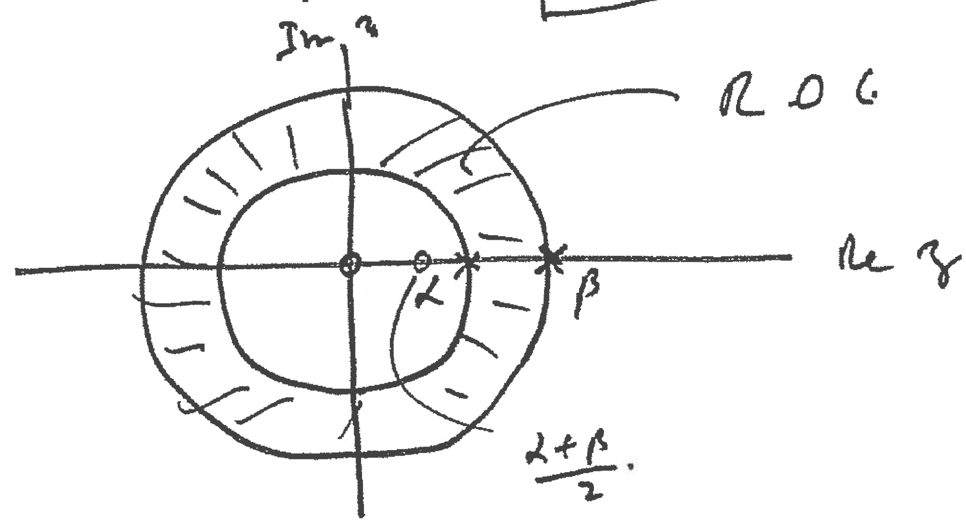
$z = \alpha$

$z = \beta$

<u>Zeros</u> at

$z = 0$

$z = \left(\frac{\alpha+\beta}{2}\right).$



Z - TRANSFORM

EXAMPLE: Finite Length Sequence

$$x(n) = \begin{cases} \alpha^n & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

$$X(z) = \sum_{n=0}^{N-1} \alpha^n z^{-n} = \sum_{n=0}^{N-1} (\alpha z^{-1})^n$$

Always converges. (why?)

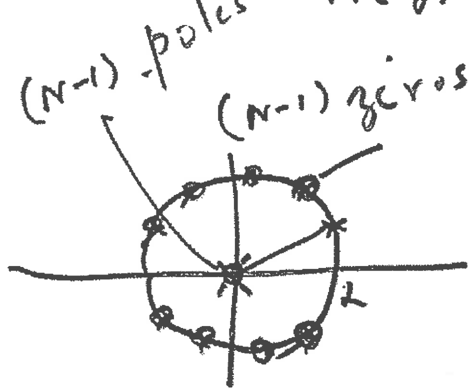
as long as $|\alpha z^{-1}|$ is finite.

$$= \frac{1-(\alpha z^{-1})^N}{1-\alpha z^{-1}} = \frac{1-\alpha^N z^{-N}}{1-\alpha z^{-1}}$$

$$= \frac{(z^N - \alpha^N)}{(z^{N-1})(z-\alpha)}.$$

N - poles
N - zeros

?



ROC - Entire plane except $z = 0$.

Pole/zero Cancelation.

Poles (N)

$N-1 \rightarrow$ at $z = 0$

$1 \rightarrow z = \alpha$.

Zeros (N)

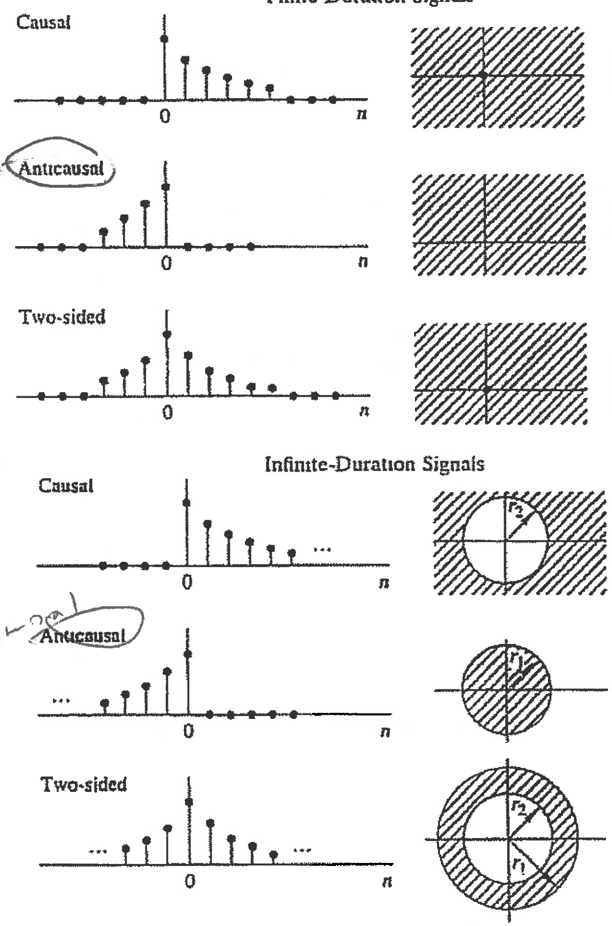
$$z = \alpha e^{j\frac{2\pi k}{N}}$$

$k = 0, 1, \ldots N-1$

Z- TRANSFORM

ROC

CHARACTERISTIC FAMILIES OF SIGNALS WITH THEIR CORRESPONDING ROC



Signal	ROC
Finite-Duration Signals	
Causal	Entire z-plane except $z = 0$
Non-causal Anticausal	Entire z-plane except $z = \infty$
Two-sided	Entire z-plane except $z = 0$ and $z = \infty$
Infinite-Duration Signals	
Causal	$\lvert z\rvert > r_2$
Non-causal Anticausal	$\lvert z\rvert < r_1$
Two-sided	$r_2 < \lvert z\rvert < r_1$

(From Proakis)

Z-TRANSFORM
ROC PROPERTIES

The examples of the previous section suggest that the properties of the region of convergence depend on the nature of the signal. These properties are summarized next, followed by some discussion and intuitive justification. We assume specifically that the algebraic expression for the z-transform is a rational function and that $x[n]$ has finite amplitude, except possibly at $n = \infty$ or $n = -\infty$.

PROPERTY 1: The ROC is a ring or disk in the z-plane centered at the origin; i.e., $0 \le r_R < |z| < r_L \le \infty$.

PROPERTY 2: The Fourier transform of $x[n]$ converges absolutely if and only if the ROC of the z-transform of $x[n]$ includes the unit circle.

PROPERTY 3: The ROC cannot contain any poles.

PROPERTY 4: If $x[n]$ is a *finite-duration sequence*, i.e., a sequence that is zero except in a finite interval $-\infty < N_1 \le n \le N_2 < \infty$, then the ROC is the entire z-plane, except possibly $z = 0$ or $z = \infty$.

PROPERTY 5: If $x[n]$ is a *right-sided sequence*, i.e., a sequence that is zero for $n < N_1 < \infty$, the ROC extends outward from the *outermost* (i.e., largest magnitude) finite pole in $X(z)$ to (and possibly including) $z = \infty$.

PROPERTY 6: If $x[n]$ is a *left-sided sequence*, i.e., a sequence that is zero for $n > N_2 > -\infty$, the ROC extends inward from the *innermost* (smallest magnitude) nonzero pole in $X(z)$ to (and possibly including) $z = 0$.

PROPERTY 7: A *two-sided sequence* is an infinite-duration sequence that is neither right sided nor left sided. If $x[n]$ is a two-sided sequence, the ROC will consist of a ring in the z-plane, bounded on the interior and exterior by a pole and, consistent with property 3, not containing any poles.

PROPERTY 8: The ROC must be a connected region.

(From Oppenheim)

Z- TRANSFORM

The z-Transform and Its Application to the Analysis of LTI Systems

TABLE 3.3 SOME COMMON Z-TRANSFORM PAIRS

	Signal, $x(n)$	z-Transform, $X(z)$	ROC
1	$\delta(n)$	1	All z
2	$u(n)$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
3	$a^n u(n)$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
4	$na^n u(n)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
5	$-a^n u(-n-1)$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
6	$-na^n u(-n-1)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
7	$(\cos\omega_0 n)u(n)$	$\frac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\|z\| > 1$
8	$(\sin\omega_0 n)u(n)$	$\frac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\|z\| > 1$
9	$(a^n\cos\omega_0 n)u(n)$	$\frac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\|z\| > \|a\|$
10	$(a^n\sin\omega_0 n)u(n)$	$\frac{az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\|z\| > \|a\|$

(From Proakis)

PROPERTIES OF Z-TRANSFORMS

* Linearity

$x_1(n) \longleftrightarrow X_1(z)$ $\quad ROC_1$

$x_2(n) \longleftrightarrow X_2(z)$ $\quad ROC_2$

$\alpha x_1(n) + \beta x_2(n) \longleftrightarrow \alpha X_1(z) + \beta X_2(z)$

ROC includes $ROC_1 \cap ROC_2$

Note: If $ROC_1 \cap ROC_2$ is null, the transform of the sum does not exist.

See example on pages Z-6 & Z-7

Z-TRANSFORM: PROPERTIES

* TIME SHIFTING

$$x(n) \longleftrightarrow X(z) \qquad \text{ROC}$$

$$x(n-n_0) \longleftrightarrow z^{-n_0} X(z) \qquad \text{ROC.}$$

Let $x_1(n) = x(n-n_0)$

$$X_1(z) = \sum_{n=-\infty}^{\infty} x(n-n_0) z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x(k) z^{-(k+n_0)} = z^{-n_0} \cdot \sum_{k=-\infty}^{\infty} x(k) z^{-k}$$

$$= z^{-n_0} \cdot X(z).$$

~~Note that time shift does not result in any change of ROC.~~ Time shift may result in modification of ROC. These modifications result in one of the following ways:

The modification of ROC

1. $|z| < \infty$ in addition, or 2. $|z| > 0$ in addition

or 3. no modification

Z-TRANSFORM: PROPERTIES

* SCALING IN Z-DOMAIN (more correctly: Multiplication by an exponential sequence)

$$\alpha^n x(n) \longleftrightarrow X(\alpha^{-1} z), \quad \text{ROC: } |\alpha| r_1 < |z| < |\alpha| r_2$$

if

$$x(n) \longleftrightarrow X(z), \quad \text{ROC: } r_1 < |z| < r_2$$

<u>Proof:</u>

$$Z[\alpha^n x(n)] = \sum_{n=-\infty}^{\infty} \alpha^n x(n) z^{-n} = \sum_{n=-\infty}^{\infty} x(n) (\alpha^{-1} z)^{-n}$$

$$= X(\alpha^{-1} z)$$

which converges if

$r_1 < |\alpha^{-1} z| < r_2 \Rightarrow |\alpha| \cdot r_1 < |z| < |\alpha| r_2.$

Z - TRANSFORM PROPERTIES

EXAMPLE:

$$u(n) \longleftrightarrow \frac{1}{1-z^{-1}} \qquad |z|>1$$

$$x(n) = \left[r^n \cos w_0 n\right] u(n)$$

$$= \left[\frac{\left(re^{jw_0}\right)^n + \left(re^{-jw_0}\right)^n}{2}\right] u(n)$$

$$X(z) = \frac{1}{2} \cdot \frac{1}{1 - re^{jw_0} z^{-1}} + \frac{1}{2} \frac{1}{1 - re^{-jw_0} z^{-1}}, \qquad |z|>r$$

$$= \frac{1}{2} \frac{\left(2 - 2r\cos w_0 z^{-1}\right)}{\left(1 - re^{jw_0} z^{-1}\right)\left(1 - re^{-jw_0} z^{-1}\right)}$$

$$\boxed{X(z) = \frac{\left(1 - r\cos w_0 z^{-1}\right)}{1 - 2r\cos w_0 z^{-1} + z^{-2}}, \qquad |z| > r}$$

Z-TRANSFORM: PROPERTIES

* **CONJUGATION**

$$x^*(n) \longleftrightarrow X^*(z^*) \qquad \text{ROC remains same.}$$

Proof:

$$Z[x^*(n)] = \sum_{n=-\infty}^{\infty} x^*(n) z^{-n} = \left[\sum_{n=-\infty}^{\infty} x(n)(z^*)^{-n}\right]^*$$

$$= X^*(z^*).$$

* **Time Reversal**

$$x(-n) \longleftrightarrow X(z^{-1}) \qquad \frac{1}{r_2} < |z| < \frac{1}{r_1}$$

if $\quad x(n) \longleftrightarrow X(z) \qquad r_1 < |z| < r_2.$

Proof:

$$Z[x(-n)] = \sum_{n=-\infty}^{\infty} x(-n) z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x(k) z^{k} = \sum_{k=-\infty}^{\infty} x(k)\left(\frac{1}{z}\right)^{-k}$$

$$= X(z^{-1}), \qquad \frac{1}{r_2} < |z| < \frac{1}{r_1}$$

Z-TRANSFORM: PROPERTIES

EXAMPLE

$$x(n) = \alpha^n u(-n).$$

$$= \left(\frac{1}{\alpha}\right)^{-n} u(-n)$$

But $\left(\frac{1}{\alpha}\right)^n u(n) = \frac{1}{1-(1/\alpha)z^{-1}}$. $\quad |z| > \left|\frac{1}{\alpha}\right|$

Using the time reversal property, we have:

$$X(z) = \left[\frac{1}{1-(1/\alpha)z^{-1}}\right]\Bigg|_{z\to z^{-1}} = \frac{1}{1-\frac{1}{\alpha}z} = \frac{\alpha}{\alpha - z},$$

$$|z| < \alpha.$$

<u>Direct application of the definition:</u>

$$Z[x(n)] = Z[\alpha^n u(-n)] = \sum_{n=-\infty}^{0} \alpha^n z^{-n} = \sum_{k=0}^{\infty} \left(\frac{z}{\alpha}\right)^k$$

$$= \frac{1}{1 - z/\alpha}, \quad \left|\frac{z}{\alpha}\right| < 1$$

$$\Longrightarrow |z| < |\alpha|.$$

Z- Transform Properties

* Multiplication by "n" / Differentiation in z-domain

$$n\,x(n) \longleftrightarrow -z\,\frac{dX(z)}{dz}$$

ROC remains the same.

Proof:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} (-n)\cdot x(n)\, z^{-(n-1)}$$

Multiplying both sides by $-z$, we get

$$-z\,\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} n\,x(n)\, z^{-n} = Z\left[n\,x(n)\right].$$

Example:
(Continued)

$$x(n) = n\alpha^n u(-n)$$

$$Z\left[\alpha^n u(-n)\right] = \frac{1}{1-\alpha^{-1}z}.$$

$$X(z) = -z\cdot\frac{d}{dz}\left[\frac{1}{1-\alpha^{-1}z}\right] = -\frac{\alpha^{-1}z}{(1-\alpha^{-1}z)^2}, \quad |z|<\alpha.$$

Z-TRANSFORM: PROPERTIES

CONVOLUTION (VERY IMPORTANT)

$$x_1(n) * x_2(n) \longleftrightarrow X_1(z)\,X_2(z),$$

ROC includes the intersection of ROC_{x_1} & ROC_{x_2}.

Proof:

$$Z[x_1(n) * x_2(n)] = \sum_{n=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty} x_1(k)\,x_2(n-k)\right] z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x_1(k) \underbrace{\sum_{n=-\infty}^{\infty} x_2(n-k)\, z^{-n}}$$

$$= \sum_{k=-\infty}^{\infty} x_1(k)\, z^{-k} \cdot X_2(z) = X_1(z)\,X_2(z),$$

EXAMPLE:

$$x(n) = \alpha^n u(n) \qquad h(n) = \beta^n u(n), \qquad \alpha \neq \beta$$

$$y(n) = x(n) * h(n)$$

$$Y(z) = X(z) \cdot H(z) = \frac{1}{1-\alpha z^{-1}} \cdot \frac{1}{(1-\beta z^{-1})} \qquad \boxed{|z| > \max(|\alpha|, |\beta|)}$$

$$= \frac{z^2}{(z-\alpha)(z-\beta)}.$$

$$\frac{Y(z)}{z} = \frac{(\alpha/\alpha-\beta)}{(z-\alpha)} + \frac{(\beta/\beta-\alpha)}{(z-\beta)}.$$

$$Y(z) = \frac{\alpha}{(\alpha-\beta)} \cdot \frac{z}{z-\alpha} - \frac{\beta}{(\alpha-\beta)} \cdot \frac{z}{z-\beta}$$

or

$$\boxed{y(n) = \frac{1}{(\alpha-\beta)} \left[\alpha^{n+1} - \beta^{n+1} \right] u(n)}$$

Z-TRANSFORM: SUMMARY OF PROPERTIES

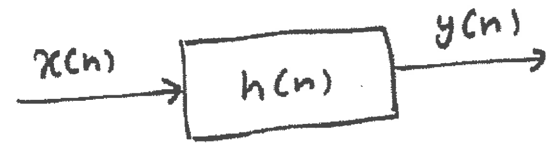
TABLE 3.2 PROPERTIES OF THE Z-TRANSFORM

Property	Time Domain	z-Domain	ROC
Notation	$x(n)$ $x_1(n)$ $x_2(n)$	$X(z)$ $X_1(z)$ $X_2(z)$	ROC: $r_2 < \lvert z\rvert < r_1$ ROC_1 ROC_2
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(z) + a_2X_2(z)$	At least the intersection of ROC_1 and ROC_2
Time shifting	$x(n-k)$	$z^{-k}X(z)$	That of $X(z)$, except $z = 0$ if $k > 0$ and $z = \infty$ if $k < 0$
Scaling in the z-domain	$a^n x(n)$	$X(a^{-1}z)$	$\lvert a\rvert r_2 < \lvert z\rvert < \lvert a\rvert r_1$
Time reversal	$x(-n)$	$X(z^{-1})$	$\frac{1}{r_1} < \lvert z\rvert < \frac{1}{r_2}$
Conjugation	$x^*(n)$	$X^*(z^*)$	ROC
Real part	$\mathrm{Re}\{x(n)\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Includes ROC
Imaginary part	$\mathrm{Im}\{x(n)\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	Includes ROC
Differentiation in the z-domain	$nx(n)$	$-z\frac{dX(z)}{dz}$	$r_2 < \lvert z\rvert < r_1$
Convolution	$x_1(n) * x_2(n)$	$X_1(z)X_2(z)$	At least, the intersection of ROC_1 and ROC_2
Correlation	$r_{x_1x_2}(l) = x_1(l) * x_2(-l)$	$R_{x_1x_2}(z) = X_1(z)X_2(z^{-1})$	At least, the intersection of ROC of $X_1(z)$ and $X_2(z^{-1})$
Initial value theorem	If $x(n)$ causal	$x(0) = \lim_{z\to\infty} X(z)$	
Multiplication	$x_1(n)x_2(n)$	$\frac{1}{2\pi j}\oint_C X_1(v)X_2\left(\frac{z}{v}\right)v^{-1}dv$	At least $r_{1l}r_{2l} < \lvert z\rvert < r_{1u}r_{2u}$
Parseval's relation	$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi j}\oint_C X_1(v)X_2^*(1/v^*)v^{-1}dv$		

(from Proakis)

z - Transform

LTI System Analysis



$$y(n) = h(n) * x(n)$$

$$Y(z) = H(z) \cdot X(z)$$

$$H(z) = \frac{Y(z)}{X(z)}$$ ← System Function or Transfer Function.

Note: The system is relaxed, i.e. initial conditions are zero.

LTI system described by LCCDE: with $a_0 = 1$

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{l=0}^{M} b_l x(n-l)$$

$$\Longrightarrow \quad y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{l=0}^{M} b_l x(n-l)$$

Z-TRANSFORMS: LTI SYSTEMS

Taking z-transform on both sides, we have

$$Y(z) = \left[-\sum_{k=1}^{N} a_k z^{-k} Y(z)\right] + \left[\sum_{\ell=0}^{M} b_\ell z^{-\ell} X(z)\right]$$

$$Y(z)\left[1 + \sum_{k=1}^{N} a_k z^{-k}\right] = X(z)\left[\sum_{\ell=0}^{M} b_\ell z^{-\ell}\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\left(\sum_{\ell=0}^{M} b_\ell z^{-\ell}\right)}{\left(1 + \sum_{k=1}^{N} a_k z^{-k}\right)}$$ ← System function / Transfer function

All a_k's are zero, $k=1,2..N \Rightarrow$ $\boxed{\text{ALL ZERO SYSTEM}} \Rightarrow$ FIR

All b_ℓ, $\ell = 1,2..M$ are zero (<u>NOT</u> b_0) $\Rightarrow$ $\boxed{\text{ALL POLE SYSTEM}}$ $\Rightarrow$ IIR

General form ⟶ Pole-zero system.

A Digital Filter design specifies a_k's and b_ℓ's such that $H(z)\big|_{z=e^{j\omega}}$ meets a frequency response requirement.

Z- TRANSFORMS

STABILITY & CAUSALITY

For an LTI system, for stability (BIBO)

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

$$H(z)\big|_{z=e^{jw}} = \sum_{n=-\infty}^{\infty} h(n) e^{-jwn}$$

$$\sum_{n=-\infty}^{\infty} \left| h(n) e^{-jwn} \right| = \sum_{n=-\infty}^{\infty} |h(n)|$$

If $\sum_{n=-\infty}^{\infty} |h(n)| < \infty$, then ROC includes the unit circle, and FT exists.

For H(z) to be stable, ROC must contain the unit circle.

z- Transforms
Causality & Stability

For causal system, the ROC of $H(z)$ must be outside of the outermost pole, <u>including $z = \infty$</u>.

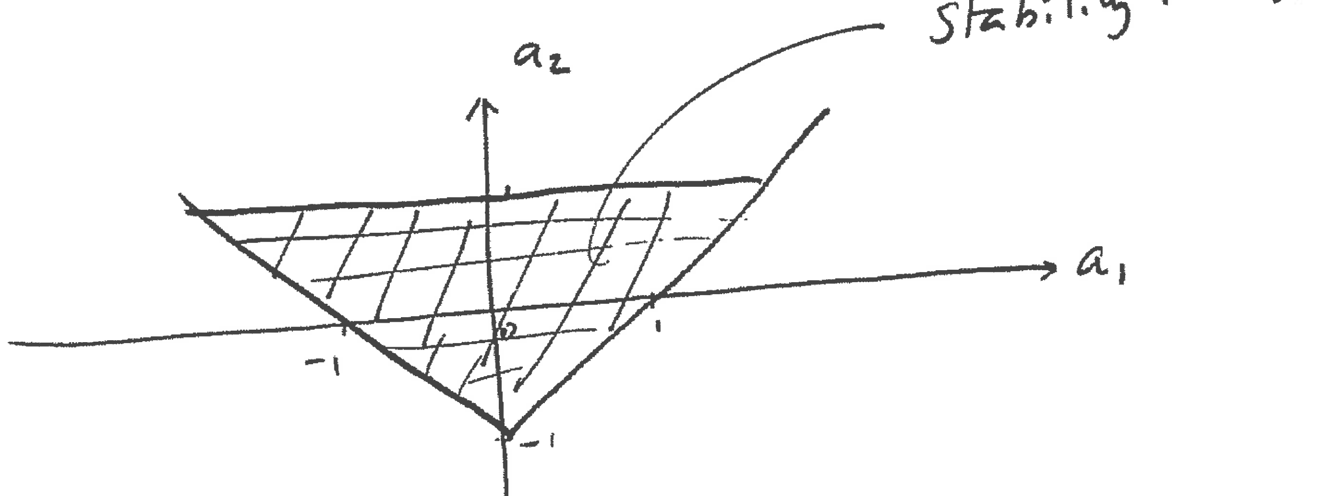
But a <u>causal</u>, <u>stable</u> system must have $H(z)$ that includes the unit circle.

$\Longrightarrow$ For a <u>causal</u>, <u>stable</u> system, the poles of $H(z)$ must be inside the unit circle.

Z-TRANSFORM: STABILITY

Causal, LTI System:

$$H(z) = \frac{b_0}{1 + a_1 z^{-1} + a_2 z^{-2}}$$



(Reading Assignment)

Z-TRANSFORMS: STABILITY & CAUSALITY

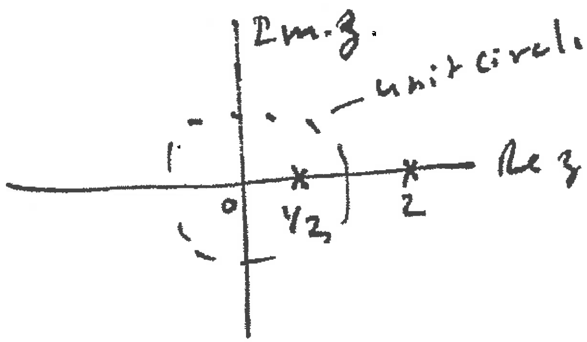
Given $H(z)$, ROC tells us about $h(n)$.

$H(z)$ + ROC $\longrightarrow$ unique $h(n)$.

Example:

$$y(n) - \tfrac{5}{2}\,y(n-1) + y(n-2) = x(n)$$

$$H(z) = \frac{1}{1 - \tfrac{5}{2}z^{-1} + z^{-2}} = \frac{1}{(1-\tfrac{1}{2}z^{-1})(1-2z^{-1})}$$



For the above $H(z)$, there are 3 possible choices of ROC

1. ROC_1: System is causal. Then ROC_1 is $|z| > 2$. System is not stable. $h(n)$ is right sided
2. ROC_2: System is stable. ROC_2 includes unit circle. $h(n)$ is 2-sided. $z = 1/2$ is causal pole, and $z = 2$ is anticausal pole.
3. ROC_3: ROC_3 is $|z| < 1/2$. System is not stable; not causal.

Z-TRANSFORMS: INVERSION

* Contour Integration

$$x(n) = \frac{1}{2\pi j} \oint_c X(z) z^{n-1} \, dz$$

(Rarely used.)

* Long Division (Power Series Expansion):

$$X(z) = \sum_{n=-\infty}^{\infty} c_n z^{-n} \quad \text{in ROC}$$

$$\Longrightarrow \quad x(n) = c_n.$$

<u>Example:</u>

$$X(z) = \frac{1}{1-az^{-1}}, \quad |z| > |a| \Longrightarrow \text{Right sided } x(n).$$

$$\begin{array}{r|l} & 1 + az^{-1} + a^2z^{-2} \\ \hline 1-az^{-1} & 1 \\ & 1 - az^{-1} \\ \hline & az^{-1} \\ & az^{-1} - a^2z^{-2} \end{array} \quad \cdots \rightarrow \quad x(n) = \{\underset{\uparrow}{1}, a, a^2, \ldots\}$$

or $x(n) = a^n u(n)$.

Z-TRANSFORM :: INVERSION

Example:

$$X(z) = \frac{1}{1 - az^{-1}} \qquad |z| < |a|$$

$$= \frac{z}{z-a}$$

$$\begin{array}{r|l} & -a^{-1}z - a^{-2}z^2 - a^{-3}z^3 \cdots \\ \hline -a+z & z \\ & \underline{z - a^{-1}z^2} \\ & \quad a^{-1}z^2 \\ & \quad \underline{a^{-1}z^2 - a^{-2}z^3} \\ & \qquad -a^{-2}z^3 \end{array}$$

$$X(z) = -a^{-1}z - a^{-2}z^2 - a^{-3}z^3 - \cdots$$

$$x(n) = \{ \cdots -a^{-3}, -a^{-2}, -a^{-1}, \underset{\uparrow}{0} \}$$

$$= -a^{n}u(-n-1).$$

z-Transform: Inversion

* Partial Fractions (commonly used)

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}.$$

If $N > M$, $X(z)$ is a proper fraction.

If $N \leq M$, convert $X(z)$:

$$X(z) = \text{Polynomial in } z + X_1(z)$$

↑ Proper fraction.

When $X(z)$ is a proper fraction, we can write

$$\frac{X(z)}{z} = \sum_{i=1}^{N} \frac{A_i}{(z - p_i)}.$$

[Assuming that the poles are distinct].

or $$X(z) = \sum_{i=1}^{N} \frac{A_i z}{(z - p_i)}.$$

z-Transform: Inversion

Partial fraction (continued)

Depending on the ROC, we can associate the poles, p_i to the causal or anti-causal part, and using the tables, we can write $x(n)$.

$$Z^{-1}\left[\frac{z}{z-p_i}\right] = \begin{cases} (p_i)^n u(n) & \text{if ROC: } |z| > |p_i| \text{ — causal} \\ -(p_i)^n u(-n-1) & \text{if ROC: } |z| < |p_i| \text{ — anti-causal} \end{cases}$$

Example:
(Book example 3.4.8)

$$X(z) = \frac{1}{1 - 3/2\, z^{-1} + 1/2\, z^{-2}} = \frac{2z}{z-1} - \frac{z}{z-1/2}$$

$$= \frac{2}{1-z^{-1}} - \frac{1}{1-1/2\, z^{-1}}$$

Z - Transform : Inversion

Case (i): ROC: $|z| > 1$.

$x(n)$ is causal.

$$\Rightarrow \boxed{x(n) = 2u(n) - \left(\tfrac{1}{2}\right)^n u(n)}$$

Case (ii): ROC: $|z| < 1/2 \Rightarrow$ anticausal $x(n)$

$$\boxed{x(n) = -2u(-n-1) + \left(\tfrac{1}{2}\right)^n u(-n-1)}$$

Case (iii) ROC: $1/2 < |z| < 1$

$$\boxed{x(n) = -2u(-n-1) - \left(\tfrac{1}{2}\right)^n u(n)}$$

z - Transform: Inversion

Complex roots occur as conjugate pairs for real coeffs in $X(z)$.

$$X(z) = \left[\frac{A_k z}{z - p_k} + \frac{A_k^* z}{z - p_k^*} \right] \qquad |z| > |p_k|$$

$$p_k = r_k e^{j\beta_k}, \qquad A_k = |A_k| \cdot e^{j\alpha_k}.$$

$$\boxed{x(n) = 2\,|A_k|\, r_k^n \cdot \cos(\beta_k n + \alpha_k) \cdot u(n)}$$

↑ Response of a complex conjugate pole pair.

Z - TRANSFORM:
One-sided

The one sided z-transform of a sequence $x(n)$ is given by

$$X^{+}(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

provided it converges for some values of z. (ROC)

* **Unique** only for causal signals
* Ignores $x(n)$ for $n<0$.
* $X^{+}(z) = X(z)$ for the signal $x(n)u(n)$
* ROC is as for causal signals.
* General use in solving difference equations (LCCDE) with nonzero initial conditions.

One-sided z-transform

Shifting Property: (very useful)

Delay:

$$x(n) \xleftrightarrow{z^+} X^+(z)$$

then

$$x(n-k) \xleftrightarrow{z^+} z^{-k}\left[X^+(z) + \sum_{n=1}^{k} x(-n) z^{n}\right], \quad \boxed{k > 0}$$

Proof:

$$Z_+\left[x(n-k)\right] = \sum_{n=0}^{\infty} x(n-k) z^{-k}$$

$$= \sum_{m=-k}^{\infty} x(m) z^{-(m+k)}$$

$$= \left[\sum_{m=-k}^{-1} x(m) z^{-(m+k)}\right] + z^{-k} \underbrace{\sum_{m=0}^{\infty} x(m) z^{-m}}_{X^+(z)}$$

$$= \left[z^{-k} \sum_{m=1}^{k} x(-m) z^{m}\right] + z^{-k} X^+(z)$$

QED.

One-sided z-Transform

<u>Advance:</u>

$$x(n) \xleftrightarrow{z^+} X^+(z)$$

then

$$x(n+k) \xleftrightarrow{z^+} z^k \left[X^+(z) - \sum_{n=0}^{k-1} x(n) z^{-n} \right]$$

Proof is left as exercise.

<u>Final value Theorem:</u>

$$x(n) \xleftrightarrow{z^+} X^+(z)$$

then

$$\lim_{n\to\infty} x(n) = \lim_{z\to 1} (z-1) \cdot X^+(z).$$

one-sided z-Transform

initial conditions:

LCCDE with <u>nonzero</u> initial conditions:

$$\sum_{k=1}^{N} a_k y(n-k) = \sum_{l=1}^{M} b_l x(n-l), \quad n \geq 0, \quad a_0 = 1$$

with $y(-1), y(-2), \ldots y(-N)$ specified

<u>and</u> input is applied at $n=0$, with $x(n)=0$ for $n<0$.

Taking the one-sided z-transform of LCCDE, we have:

$$Y^+(z) = Z^+\left[-\sum_{k=1}^{N} a_k y(n-k)\right] + Z^+\left[\sum_{l=1}^{M} b_l x(n-l)\right]$$

$$= -\sum_{k=1}^{M} a_k z^{-k}\left[Y^+(z) + \sum_{n=1}^{k} y(-n) z^{n}\right] + \sum_{l=1}^{M} b_l z^{-l} X^+(z).$$

$$Y^+(z)\cdot\left[1+\sum_{k=1}^{N} a_k z^{-k}\right] = -\sum_{k=1}^{N} a_k z^{-k}\left[\sum_{n=1}^{k} y(-n) z^{n}\right] + \sum_{l=1}^{M} b_l z^{-l} X^+(z)$$

$$Y^+(z) = \frac{-\sum_{k=1}^{N} a_k z^{-k}\left[\sum_{n=1}^{k} y(-n) z^{n}\right]}{\left[1+\sum_{k=1}^{N} a_k z^{-k}\right]} + \frac{\sum_{l=1}^{M} b_l z^{-l} X^+(z)}{\left(1+\sum_{k=1}^{N} a_k z^{-k}\right)}$$

Natural Response (Zero-Input)

Forced Response (Zero State)

IMPULSE RESPONSE USING Z-TRANSFORM

Example:

Find the impulse response of the causal, stable LSI system described by the following difference equation

$$y(n) - y(n-1) + \frac{2}{9} y(n-2) = x(n) \quad — (1)$$

Calculation of impulse response implies that the initial conditions are zero (i.e. relaxed system, $y(-1) = y(-2) = 0$), and the input $x(n) = \delta(n)$.

Taking the z-transform of (1), we get

$$Y(z) - z^{-1} Y(z) + \frac{2}{9} z^{-2} Y(z) = X(z) \quad — (2)$$

which can be rearranged to obtain the system transfer function $H(z)$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{(1 - z^{-1} + \frac{2}{9} z^{-2})} = \frac{z^2}{(z^2 - z + 2/9)} \quad — (3)$$

$$\frac{H(z)}{z} = \frac{z}{(z^2 - z + 2/9)} = \frac{A}{(z - 1/3)} + \frac{B}{(z - 2/3)}$$

$$= \frac{-1}{z - 1/3} + \frac{2}{z - 2/3} \quad — (4)$$

Impulse Response using z-Transform (continued)

From (4), we write:

$$H(z) = \frac{-z}{(z - 1/3)} + \frac{2z}{(z - 2/3)} \quad \text{—(5)}$$

We are given that the system is causal; therefore, we can find the inverse z-transform of $H(z)$ [because the ROC is known] to obtain the impulse response $h(n)$.

$$\boxed{h(n) = z^{-1}\{H(z)\} = \left[-\left(\tfrac{1}{3}\right)^n + 2\left(\tfrac{2}{3}\right)^n\right] u(n)} \quad \text{—(6)}$$

where we used the z-transform tables to find the inverses

SOLVING DIFFERENCE EQUATIONS WITH ZERO INITIAL CONDITIONS USING Z-TRANSFORMS

Example: LSI, stable, causal system

$$y(n) - y(n-1) + \frac{2}{9} y(n-2) = x(n), \qquad x(n) = \left(\tfrac{1}{4}\right)^n u(n)$$

$$y(-1) = y(-2) = 0$$

$$H(z) = \frac{z^2}{(z^2 - z + 2/9)}$$ (from the previous example)

$$H(z) = \frac{Y(z)}{X(z)} \implies Y(z) = H(z)X(z) \quad — (1)$$

For our problem,

$$X(z) = \frac{z}{(z - 1/4)} \quad — (2)$$

Substituting for $H(z)$ and $X(z)$ in (1), we get

$$Y(z) = \frac{z^3}{(z^2 - z + 2/9)(z - 1/4)} = \frac{z^3}{(z - 1/3)(z - 2/3)(z - 1/4)}$$

or

$$\frac{Y(z)}{z} = \frac{z^2}{(z - 1/3)(z - 2/3)(z - 1/4)} = \frac{A}{z - 1/3} + \frac{B}{z - 2/3} + \frac{C}{z - 1/4} \quad — (3)$$

SOLVING DIFF. EQN USING Z-Transforms
(Zero Initial Condition)

In ③, we solve for A, B & C to get

$$A = -4, \quad B = -\frac{16}{5} \quad \text{and} \quad C = 9/5$$

Thus

$$Y(z) = \frac{-4z}{(z - 1/3)} + \frac{-\frac{16}{5}z}{(z - 2/3)} + \frac{9/5}{(z - 1/4)}$$

or

$$\boxed{y(n) = \left[-4\left(\frac{1}{3}\right)^n - \frac{16}{5}\left(\frac{2}{3}\right)^n + \frac{9}{5}\left(\frac{1}{4}\right)^n\right] u(n)}$$

What if the initial conditions are nonzero?
Next example illustrates this situation.

SOLVING DIFFERENCE EQUATIONS WITH NON-ZERO INITIAL CONDITIONS USING Z-TRANSFORM

In the case of most difference equations, we solve for $y(n)$, $n \geq 0$ with non-zero initial conditions, $y(-1), y(-2), \ldots y(-N)$ where N is the order of the difference equation. We use the unilateral z-transform in this situation to account for the non-zero initial conditions.

Recall:

Bilateral z-transform – Time shifting property

$$Z[y(n-N)] = z^{-N} Y(z).$$

Unilateral z-transform – Time shifting Property

$$\boxed{Z[y(n-N)] = z^{-N}\left[Y(z) + \sum_{k=1}^{N} y(-k) z^{k}\right]}$$

For example,

$$Z[y(n-2)] = z^{-2}\left[Y(z) + z\, y(-1) + z^{2} y(-2)\right]$$

$$= z^{-2} Y(z) + z^{-1} y(-1) + y(-2).$$

SOLVING DIFF. EQUATIONS WITH Non-zero Initial Conditions (Continued)

<u>Example:</u> This is the same example that we considered in Ch. 6 - pg. 6.38

$$x(n) = 4\,u(n)$$

$$y(n) - \frac{1}{6}\,y(n-1) - \frac{1}{6}\,y(n) = x(n), \quad (1)$$

$$y(-1) = 0, \quad y(-2) = 12$$

Taking the unilateral transform of (1) on both sides, we get

$$Y(z) - \frac{1}{6}z^{-1}\left[Y(z) + y(-1)\cdot z\right] - \frac{1}{6}z^{-2}\left[Y(z) + y(-1)z + y(-2)z^{2}\right] = X(z) \quad (2)$$

or

$$Y(z)\left[1 - \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}\right] - \frac{1}{6}\,y(-1) - \frac{1}{6}\,y(-1)z^{-1} - \frac{1}{6}\,y(-2) = X(z) \quad (3)$$

or

$$Y(z) = \frac{X(z)}{\left(1 - \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}\right)} + \frac{\left[\frac{1}{6}\,y(-1) + \frac{1}{6}\,y(-1)z^{-1} + \frac{1}{6}\,y(-2)\right]}{\left(1 - \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}\right)} \quad (4).$$

Thus,

Note that $H(z) = \dfrac{1}{1 - \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}}$

(4) Can also be written as

$$Y(z) = \underbrace{H(z)\,X(z)}_{\substack{\text{yields} \\ \text{"zero state"} \\ \text{response } [Y_{zs}(z)]}} + \underbrace{H(z)\cdot\left[\tfrac{1}{6}\,y(-1)\,z^{-1} + \tfrac{1}{6}\,y(-1) + \tfrac{1}{6}\,y(-2)\right]}_{\substack{\text{yields} \\ \text{"zero input"} \\ \text{response } [Y_{zi}(z)]}} \quad \text{— (5)}$$

Each term on the RHS of (5) can be expanded into partial fractions, and the inverse-transform obtained to finally obtain $y(n)$.

"Zero state" Response

$$Y_{zs}(z) = H(z)\,X(z)$$

$$= \frac{1}{(1 - \tfrac{1}{6}z^{-1} - \tfrac{1}{6}z^{-2})}\cdot\left(4\,\frac{z}{z-1}\right) = \frac{4z^3}{(z-1)\,(z^2 - \tfrac{1}{6}z - \tfrac{1}{6})}$$

$$= \frac{4z^3}{(z-1)(z-\tfrac{1}{2})(z+\tfrac{1}{3})} \quad \text{— (6)}$$

Solving Diff Eqn with Nonzero I.C. using z-transform (continued)

(6) Can be re-expressed as

$$\frac{Y_{zs}(z)}{z} = \frac{A}{(z-1)} + \frac{B}{(z-1/2)} + \frac{C}{(z+1/3)}$$

where $A = 6$ $\quad B = -2.4$ $\quad C = 0.4$

or $y_{zs}(n) = 6\,u(n) - 2.4\left(\frac{1}{2}\right)^n u(n) + 0.4\left(-1/3\right)^n u(n)$ — (7)

"Zero-input" Response:

$$Y_{ZI}(z) = H(z) \cdot \frac{1}{6}\, y(-2) \qquad [\text{because } y(-1) = 0$$

$$= H(z)\, \frac{1}{6} \cdot 12 = 2\,H(z) = \frac{2z^2}{(z+1/3)(z-1/2)}$$ — (8)

$$\frac{Y_{ZI}(z)}{z} = \frac{2}{(z+1/3)(z-1/2)} = \frac{A_1}{(z+1/3)} + \frac{B_1}{(z-1/2)}$$ — (9)

where $A_1 = 0.8$ $\quad B_1 = 1.2$

SOLVING DIFF EQN with nonzero IC using z-transform (continued)

$$Y_{ZI}(z) = \frac{0.8\,z}{(z+1/3)} + \frac{1.2\,z}{(z+1/2)}$$

$$\Rightarrow y_{zi}(n) = \left[0.8\left(-\tfrac{1}{3}\right)^n + 1.2\left(\tfrac{1}{2}\right)^n\right] u(n) \quad — (8)$$

The total solution is given by $y(n) = y_{zs}(n) + y_{zi}(n)$, i.e. adding eqn. (7) and (8).

$$\boxed{y(n) = 6\,u(n) - 1.2\,(1/2)^n\,u(n) + 1.2\,(-1/3)^n\,u(n)}$$

which is the same answer that we got in ch. 6, pg. 6.35 of notes