SPRING 2004

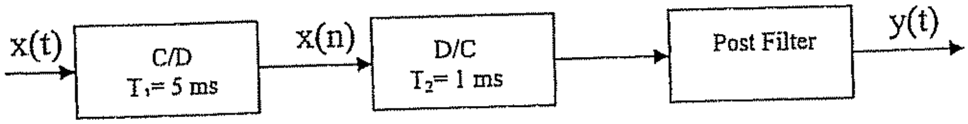
DSP EXAM 1 SOLUTION

1. (20 points)

Consider the simple signal processing system shown in the figure below. The sampling periods of the C/D (continuous-to-digital) and D/C converters are $T_1 = 5$ ms and $T_2 = 1$ ms, respectively. Determine the output y(t) of the system, if the input is given by

$$x(t) = 3\cos 100\pi t + 2\sin 250\pi t$$

The post-filter is an ideal low pass filter with a cutoff frequency of 500 Hz.



$$x_a(t) = 3\cos 100\pi t + 2\sin(250\pi t)$$

$$T = 5\,\text{ms} \Rightarrow F_s = 200\,\text{Hz}.$$

$$x(n) = 3\cos\left(2\pi\cdot\frac{50}{200}n\right) + 2\sin\left(2\pi\cdot\frac{125}{200}n\right)$$

$$= 3\cos\left(2\pi\cdot\frac{1}{4}n\right) + 2\sin\left(2\pi\cdot\frac{5}{8}n\right)$$

$$= 3\cos\left(2\pi\cdot\frac{1}{4}n\right) + 2\sin\left(2\pi\left(1-\frac{3}{8}\right)n\right)$$

$$= 3\cos\left(2\pi\cdot\frac{1}{4}n\right) - 2\sin\left(2\pi\cdot\frac{3}{8}n\right) \quad \text{—(1)}$$

For reconstruction, however, a different sampling rate is used, i.e. $F_s' = 1$ kHz, but $y(n) = x(n)$. Only the time interval between the samples is changed (decreased in this case).

$$y(n) = 3\cos\left(2\pi\cdot\frac{1}{4}n\right) - 2\sin\left(2\pi\cdot\frac{3}{8}n\right)$$

$$y_a(t) = 3\cos\left(2\pi\cdot\frac{1}{4}\cdot 1000t\right) - 2\sin\left(2\pi\cdot\frac{3}{8}\cdot 1000t\right)$$

$$= \boxed{3\cos(2\pi\cdot 250t) - 2\sin(2\pi\cdot 375t)}$$

2. (20 points)
The input $x(n) = \{1,2,-1,\gamma\}$ to a LTI system with an impulse response $\{1,\alpha,\beta,2\}$ yields an output $y(n) = \{1,4,2,0,9,-4,4\}$. What is the output $y_1(n)$ of the LTI system due to an input $x_1(n) = \{1, \underset{\wedge}{0}, -1\}$?

$$y(n) = x(n) * h(n)$$

$$y(0) = x(0)h(0) = 1.1 = 1$$

$$y(1) = x(0)h(1) + x(1).h(0) = 1.\alpha + 2.1 = 4 \Rightarrow \boxed{\alpha = 2}$$

$$y(2) = x(0)h(2) + x(1)h(1) + x(2)h(0)$$
$$= 1\,\beta + 2.\alpha + (-1).1 = \beta + 2\alpha - 1 = 2 \Rightarrow \boxed{\beta = -1}$$

$$y(3) = x(0)h(3) + x(1)\,h(2) + x(2)\,h(1) + x(3)h(0)$$
$$= 1.2 + 2.\beta + (-1).\alpha + \gamma.1 = 0$$
$$= 2 + (-2) - 2 + \gamma = 0 \Longrightarrow \boxed{\gamma = 2}$$

Thus,
$$h(n) = \{1, 2, -1, 2\}$$

$$y_1(n) = h(n) * x_1(n) = h(n) * \{1, \underset{\wedge}{0}, -1\}$$
$$= h(n) * \delta(n+1) - h(n) * \delta(n-1)$$
$$= \{1, \underset{\uparrow}{2}, -1, 2\} - \{\underset{\uparrow}{0}, 1, 2, -1, 2\}$$
$$= \boxed{\{1, \underset{\uparrow}{2}, -2, 0, 1, -2\}}$$

3. **(20 points)**
The following LCCDE describes the input-output relationship of a causal, LTI system; determine the impulse response, h (n) of the LTI system.

$$y(n) - \tfrac{1}{2}\, y(n-1) + \tfrac{1}{4}\, y(n-2) = x(n) - \tfrac{1}{4}\, x(n-1)$$

<u>(Note: Your answer for h(n) should contain only real coefficients to receive credit.)</u>

$$y(n) - \frac{1}{2} y(n-1) + \frac{1}{4} y(n-2) = x(n) - \frac{1}{4} x(n-1)$$

Taking z-transform on both sides, we have:

$$Y(z) - \frac{1}{2} z^{-1} Y(z) + \frac{1}{4} z^{-2} Y(z) = X(z) - \frac{1}{4} z^{-1} X(z)$$

or

$$H(z) = \frac{Y(z)}{X(z)} = \frac{(1 - \frac{1}{4} z^{-1})}{(1 - \frac{1}{2} z^{-1} + \frac{1}{4} z^{-2})} \quad \text{—— } (1)$$

From z-transform tables, we have:

$$[r^n \cos w_0 n]\, u(n) \longleftrightarrow \frac{[1 - (r\cos w_0) z^{-1}]}{(1 - (2 r \cos w_0) z^{-1} + r^2 z^{-2})} \quad \text{—— } (2)$$

Comparing (1) & (2), we have:

$$r \cos w_0 = \frac{1}{4} \quad \text{and} \quad r^2 = 1/4 \Rightarrow r = \frac{1}{2}, \; \cos w_0 = \frac{1}{2}$$

$$\Rightarrow w_0 = \frac{\pi}{3}.$$

Thus,

$$\boxed{h(n) = \left[\left(\frac{1}{2}\right)^n \cos \frac{\pi}{3} n\right] u(n)}$$

4. (20 points)
When the input to an LTI system is x(n) = $(1/3)^n$ u(n) + $(2)^n$ u(-n-1), the corresponding output is given by y(n) = 5 $(1/3)^n$ u(n) – 5 $(2/3)^n$ u(n). (a) Find the system transfer function H (z), and indicate the region of convergence, (b) Sketch the Direct Form II (canonical or minimum delays) implementation of the system, and (c) Is the system stable? Is it causal?

$$x(n) = \left(\tfrac{1}{3}\right)^n u(n) + (2)^n u(-n-1)$$

$$X(z) = \frac{z}{z - 1/3} - \frac{z}{z-2}, \quad ROC_x : \tfrac{1}{3} < |z| < 2$$

$$= \frac{-5/3\, z}{(z - 1/3)(z-2)} \qquad ROC_x : \uparrow \quad \text{— 1}$$

$$y(n) = 5 \cdot (1/3)^n u(n) - 5 \cdot (2/3)^n u(n)$$

$$Y(z) = \frac{5z}{(z - 1/3)} - \frac{5z}{(z - 2/3)}, \quad ROC_y : |z| > 2/3$$

$$= \frac{-5/3\, z}{(z - 1/3)(z - 2/3)} \qquad ROC_y : \uparrow \quad \text{— 2}$$

$$Y(z) = H(z) \cdot X(z) \implies ROC_H : |z| > 2/3$$

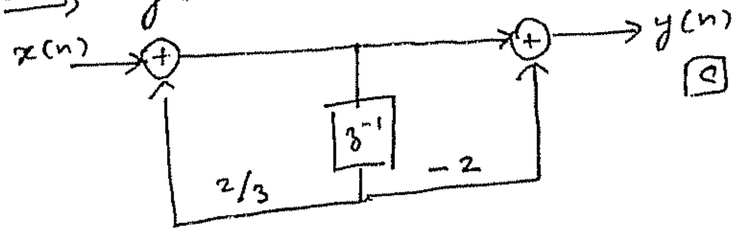
a. $$\boxed{H(z) = \frac{Y(z)}{X(z)} = \frac{(z-2)}{(z - 2/3)}} \quad \text{— 3}$$

[note: $ROC_y \geq ROC_H \cap ROC_x$ ↑ at least]

b. $$(z - 2/3)\, Y(z) = (z-2)\, X(z)$$

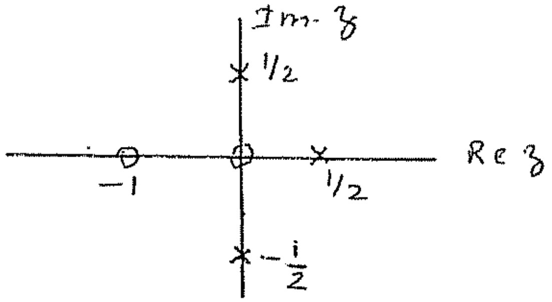
or $$(1 - 2/3\, z^{-1})\, Y(z) = (1 - 2z^{-1})\, X(z)$$

$$\implies y(n) - 2/3\, y(n-1) = x(n) - 2\, x(n-1) \quad \text{— 4}$$



c. Stable, because unit cir. is in the ROC.
Causal because ROC_H is $|z| > 2/3$.

5. (20 points)
The z-transform of the sequence x (n) is denoted by X (z). The pole-zero plot of X (z) is shown in the figure below. There are 3 finite poles (z=1/2, +/-1/2 j), and 2 finite zeros (z=0,-1). Sketch the pole-zero plots of (a) $X_1(z)$, where $x_1(n) = x(1-n)$, and (b) $X_2(z)$, where $x_2(n) = (1/2)^n x(n)$. Gain factor is unity.



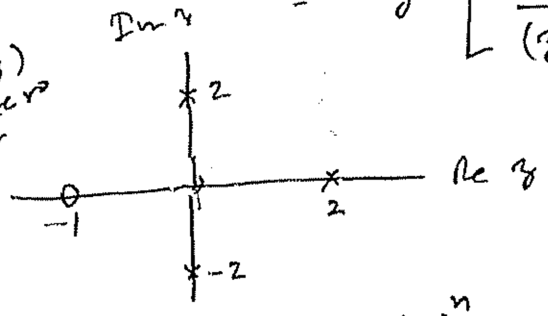
$$X(z) = \frac{z(z+1)}{(z-1/2)(z^2+1/4)} \quad \text{—(1)}$$

a. $x_1(n) = x(1-n) = x(-(n-1))$

$\Rightarrow X_1(z) = z^{-1} \cdot Z[x(-n)] = z^{-1} X(z^{-1})$

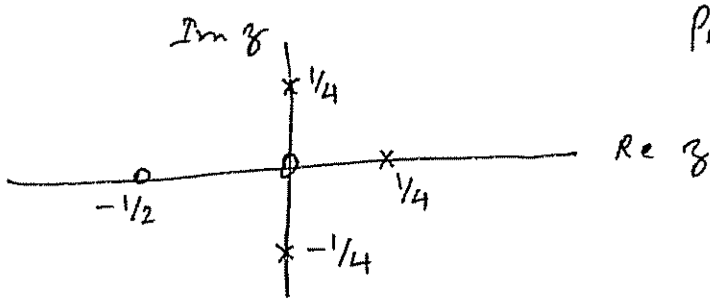
$$= z^{-1}\left[\frac{z^{-1}(z^{-1}+1)}{(z^{-1}-1/2)(z^{-2}+1/4)}\right] = \frac{(1+z)}{(1-1/2 z)(1+1/4 z^2)} \quad \text{—(2)}$$

$X_1(z)$ pole-zero plot



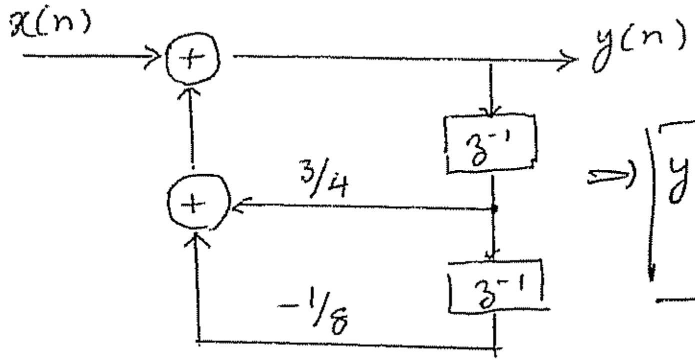
b. $x_2(n) = (1/2)^n x(n) \rightarrow X_2(z) = X\left(\frac{z}{1/2}\right) = X(2z)$

$$X_2(z) = \frac{2z(2z+1)}{(2z-1/2)(4z^2+1/4)}$$



Pole-zero plot of $X_2(z)$

6. **(20 points)**
Determine the impulse response h(n) of the causal LTI system implemented as shown below, <u>**without using z-transforms.**</u>



$$\Rightarrow \boxed{y(n) - \frac{3}{4} y(n-1) + \frac{1}{8} y(n-2) = x(n) \quad (1)}$$

Characteristic Equation:

$$1 - 3/4\, \alpha^{-1} + \frac{1}{8} \alpha^{-2} = 0 \quad \Rightarrow \quad \boxed{\alpha_1 = 1/2, \ \alpha_2 = 1/4}$$

The form of the solution is:

$$h(n) = \left[A \cdot (1/2)^n + B (1/4)^n\right] u(n) \quad (2)$$

h(n) should satisfy (1); we compute h(0) and h(1) with $x(n) = \delta(n)$.

$$h(0) - 3/4\, \cancel{h(-1)}^{\,0} + \frac{1}{8} \cancel{h(-2)}^{\,0} = 1 \quad \Rightarrow \quad \boxed{h(0) = 1}$$

$$h(1) - 3/4\, h(0) + 1/8\, \cancel{h(-1)}^{\,0} = 0 \quad \Rightarrow \quad \boxed{h(1) = 3/4}$$

$$\left.\begin{matrix} h(0) = 1 \\ h(1) = 3/4 \end{matrix}\right\} \quad (3)$$

From (2) and (3), we have:

$$h(0) = \left[A \cdot (1/2)^0 + B (1/4)^0\right] u(0) = 1 \quad \Rightarrow \quad \boxed{A + B = 1} \quad (4)$$

$$h(1) = \left[A \cdot (1/2)^1 + B (1/4)^1\right] u(1) = 3/4 \quad \Rightarrow \quad \boxed{\frac{A}{2} + \frac{B}{4} = 3/4} \quad (5)$$

From (4) and (5), $\boxed{A = 2 \text{ and } B = -1}$.

Thus,

$$\boxed{h(n) = \left[2 \cdot \left(\frac{1}{2}\right)^n - (1/4)^n\right] u(n)}$$